Text Book for

INTERMEDIATE

Second Year

Mathematics

Paper - IIB

Coordinate Geometry, Calculus

Telugu and Sanskrit Akademi
Andhra Pradesh

Intermediate
Second Year

Mathematics
Paper - IIB
Text Book

Pages : xvi + 364 + iv + iv

Reprint 2023

Copies : 24000

Published, Printed & Distributed by
Telugu and Sanskrit Akademi, A.P.

Price: Rs. **188.00**

Laser Typeset by **Pavan Graphics, Hyderabad**

Published and Printed by
M/s GBR Offset Printers & Publishers
Surampalli, NTR Dist.

on behalf of **Telugu and Sanskrit Akademi**

Y.S. JAGAN MOHAN REDDY

CHIEF MINISTER
ANDHRA PRADESH

AMARAVATI

MESSAGE

I congratulate Akademi for starting its activities with printing of textbooks from the academic year 2021 – 22.

Education is a real asset which cannot be stolen by anyone and it is the foundation on which children build their future. As the world has become a global village, children will have to compete with the world as they grow up. For this there is every need for good books and good education.

Our government has brought in many changes in the education system and more are to come. The government has been taking care to provide education to the poor and needy through various measures, like developing infrastructure, upgrading the skills of teachers, providing incentives to the children and parents to pursue education. Nutritious mid-day meal and converting Anganwadis into pre-primary schools with English as medium of instruction are the steps taken to initiate children into education from a young age. Besides introducing CBSE syllabus and Telugu as a compulsory subject, the government has taken up numerous innovative programmes.

The revival of the Akademi also took place during the tenure of our government as it was neglected after the State was bifurcated. The Akademi, which was started on August 6, 1968 in the undivided state of Andhra Pradesh, was printing text books, works of popular writers and books for competitive exams and personality development.

Our government has decided to make available all kinds of books required for students and employees through Akademi, with headquarters at Tirupati.

I extend my best wishes to the Akademi and hope it will regain its past glory.

Y.S. JAGAN MOHAN REDDY

Dr. Nandamuri Lakshmiparvathi
M.A., M.Phil., Ph.D.
Chairperson, (Cabinet Minister Rank)
Telugu and Sanskrit Akademi, A.P.

Message of Chairperson, Telugu and Sanskrit Akademi, A.P.

In accordance with the syllabus developed by the Board of Intermediate, State Council for Higher Education, SCERT etc., we design high quality Text books by recruiting efficient Professors, department heads and faculty members from various Universities and Colleges as writers and editors. We are taking steps to print the required number of these books in a timely manner and distribute through the Akademi's Regional Centers present across the Andhra Pradesh.

In addition to text books, we strive to keep monographs, dictionaries, dialect texts, question banks, contact texts, popular texts, essays, linguistics texts, school level dictionaries, glossaries, etc., updated and printed and made available to students from time to time.

For competitive examinations conducted by the Andhra Pradesh Public Service Commission and for Entrance examinations conducted by various Universities, the contents of the Akademi publications are taken as standard. So, I want all the students and Employees to make use of Akademi books of high standards for their golden future.

Congratulations and best wishes to all of you.

N. Lakshmiparvathi

Nandamuri Lakshmiparvathi
Chairperson, Telugu and Sanskrit Akademi, A.P.

J. SYAMALA RAO, I.A.S.,
Principal Secretary to Government

Higher Education Department
Government of Andhra Pradesh

MESSAGE

I Congratulate Telugu and Sanskrit Akademi for taking up the initiative of printing and distributing textbooks in both Telugu and English media within a short span of establishing Telugu and Sanskrit Akademi.

Number of students of Andhra Pradesh are competing of National Level for admissions into Medicine and Engineering courses. In order to help these students Telugu and Sanskrit Akademi consultation with NCERT redesigned their Textbooks to suit the requirement of National Level Examinations in a lucid language.

As the content in Telugu and Sanskrit Akademi books is highly informative and authentic, printed in multi-color on high quality paper and will be made available to the students in a time bound manner. I hope all the students in Andhra Pradesh will utilize the Akademi textbooks for better understanding of the subjects to compete of state and national levels.

(J. SYAMALA RAO)

THE CONSTITUTION OF INDIA

PREAMBLE

WE, THE PEOPLE OF INDIA, having solemnly resolved to constitute India into a [SOVEREIGN SOCIALIST SECULAR DEMOCRATIC REPUBLIC] and to secure to all its citizens:

JUSTICE, social, economic and political;

LIBERTY of thought, expression, belief, faith and worship;

EQUALITY of status and of opportunity; and to promote among them all

FRATERNITY assuring the dignity of the individual and the [unity and integrity of the Nation];

IN OUR CONSTITUENT ASSEMBLY this twenty-sixth day of November, 1949 do HEREBY ADOPT, ENACT AND GIVE TO OURSELVES THIS CONSTITUTION.

Textbook Development Committee

Chief Coordinator

Prof. P.V. Arunachalam, Founder Vice-Chancellor, Dravidian University, Kuppam

Editors

Prof. K. Pattabhi Rama Sastry, Professor of Mathematics (Retd), Andhra University, Visakhapatnam

Prof. V. Siva Rama Prasad, Professor of Mathematics (Retd.), Osmania University, Hyderabad

Prof. T. Ram Reddy, Professorof Mathematics(Retd.), Kakatiya University, Warangal

Sri A. Padmanabham, Head of the Department of Mathematics (Retd.), Maharani College, Peddapuram

Prof, K. Rama Mohana Rao, Professor of Applied Mathematics (Retd.), Andhra University. Visakhapatnam

Authors

Prof. K. Sambaiah, Chairman, BOS, Department of Mathematics, Kakatiya University. Warangal

Sri K.V.S. Prasad, Lecturer in Mathematics, Govt. Degree College (Women), Srikakulam

Sri Potukuchi Rajamouli, Sr. Lecturer in Mathematics (Retd.), P.B. Siddhardha College of Arts and Science(A), Vijayawada

Prof. G.V. Ravindranath Babu, Professor of Mathematics, Andhra University. Visakhapatnam

Sri A. Padmanabham, Head of the Department of Mathematics (Retd.), Maharani College, Peddapuram

Dr. P.V. Satyanarayana Murthy, Reader & Head (Retd.), Department of Mathematics, S.K.B.R. College, Amalapuram.

Subject Committee Members of BIE

Prof. D. Rama Murthy, Professor of Mathematics (Retd.), Osmania University, Hyderabad

Prof. S. Raj Reddy, Professor of Mathematics (Retd.), Kakatiya University, Warangal

Prof. D.R.V. Prasada Rao, Professor of Mathema tics (Retd.), S.K.University, Ananthapur

Dr. D. Chitti Babu, Reader, Govt. College (Autonomous), Rajahmundry

Dr.C.T. Suryanarayana Chari, Reader, Silver Jubilee Govt. College, Kurnool

Dr. Y. Bhaskar Reddy, Reader, Govt. Degree College, Rangasaipet, Warangal

Dr. M.V.N. Patrudu, Principal, Govt. Junior College, Veeraghattam, Srikakulam (Dist)

Sri T. Markandeya Naidu, Lecturer, P.V.K.N. Govt. Degree College, Chittore

Sri C. Linga Reddy, Principal, Govt. Junior College (Boys), Nirmal, Adilabad (Dist)

Sri Kodi Nageswara Rao, School Assistant (Retd.), Chagallu, West Godavari (Dist)

Sri C. Sadasiva Sastry, Lecturer in Mathematics (Retd.), Sri Ramabhadra Jr. College, Hyderabad

Smt. S.V. Sailaja, Junior Lecturer, New Govt. Junior College, Kukatpally, Hyderabad

Sri K. Chandra Sekhara Rao, Junior Lecturer, GJC, Uppugundur, Prakasam (Dist)

Sri P. Harinadha Chary, Junior Lecturer, Sri Srinivasa Jr. College, Tiruchanur, Tirupati

Panel of Experts of BIE

Prof. D. Rama Murthy, Professor of Mathematics (Retdl. Osmanla Unlversity. Hyderabad

Prof. D.R.V. Prasad a Rao, Professor of Mathematics (Retdl. S.K. University. Ananthapur

Dr, C.T. Suryanarayana Chari, Reader in Mathematics, Silver Jubilee Government College (Autonomous) , Kurnool

Sri C. Sadasiva Sastry, Lecturer in Mathematics (Retd). Sri Ramabhadra Junlor College. Hyderabad

Dr. M.V.N. Patrudu, Principal, Govt. Junior College, Veeraghattam. Srikakulam (Dist)

Smt. S.V. Sailaja, Junior Lecturer, New Govt. Junior College, Kukatpally. Hyderabad

Text Book Review Committee

Editors

Prof. K. Rama Mohan Rao
Professor of Applied Mathematics (Retd.)
Andhra University, Visakhapatnam.

Prof. G. Chakradhara Rao
Professor of Mathematics (Retd.)
Andhra University, Visakhapatnam.

Reviewed by

Prof. S. Kalesha Vali
Department of Engineering Mathematics
Andhra University, Visakhapatnam.

Dr. Y. Purushotama Reddy
Principal (Retd.)
Government Degree Collge, Bantumilli,
Krishna Dist.

Sri K.V.S. Prasad
Lecturer in Mathematics (Retd.)
Government College for Women,
Srikakulam.

Dr. K. Ravi Babu
Lecturer in Mathematics
Dr. V.S. Krishna Govt. Degree & P.G. Collge(A),
Visakhapatnam.

Coordinating Committee of Board of Intermediate Education, A.P.

Sri M.V. Seshagiri Babu, I.A.S.
Secretary
Board of Intermediate Education,
Andhra Pradesh

Educational Research & Training Wing (Text Books)

Dr. A. Srinivasulu
Professor

Sri. M. Ravi Sankar Naik
Assistant Professor

Dr. M. Ramana Reddy
Assistant Professor

Sri J.V. Ramana Gupta
Assistant Professor

Telugu and Sanskrit Akademi, Andhra Pradesh Coordinating Committee

Sri V. Ramakrishna, I.R.S.
Director

Dr. M. Koteswaramma, M.Com., Ph.D.
Research Officer

Dr. S.A.T. Rajyalakshmi M.Sc., B.Ed., M.A., Ph.D.
Research Assistant

Dr. K. Glory Sathyavani, M.Sc., Ph.D., M.Ed.
Research Assistant

Foreword

The Government of India vowed to remove the educational disparities and adopt a common core curriculum across the country especially at the Intermediate level. Ever since the Government of Andhra Pradesh and the Board of Intermediate Education (BIE) swung into action with the task of evolving a revised syllabus in all the Science subjects on par with that of CBSE, approved by NCERT, its chief intention being enabling the students from Andhra Pradesh to prepare for the National Level Common Entrance tests like NEET, ISEET etc for admission into Institutions of professional courses in our Country.

For the first time BIE AP has decided to prepare the Science textbooks. Accordingly an Academic Review Committee was constituted with the Commissioner of Intermediate Education, AP as Chairman and the Secretary, BIE AP; the Director SCERT and the Director Telugu Akademi as members. The National and State Level Educational luminaries were involved in the textbook preparation, who did it with meticulous care. The textbooks are printed on the lines of NCERT maintaining National Level Standards.

The Education Department of Government of Andhra Pradesh has taken a decision to publish and to supply all the text books with free of cost for the students of all Government and Aided Junior Colleges of newly formed state of Andhra Pradesh.

We express our sincere gratitude to the Director, NCERT for according permission to adopt its syllabi and curriculum of Science textbooks. We have been permitted to make use of their textbooks which will be of great advantage to our student community. I also express my gratitude to the Chairman, BIE and the honorable Minister for HRD and Vice Chairman, BIE and Secretary (SE) for their dedicated sincere guidance and help.

I sincerely hope that the assorted methods of innovation that are adopted in the preparation of these textbooks will be of great help and guidance to the students.

I wholeheartedly appreciate the sincere endeavors of the Textbook Development Committee which has accomplished this noble task.

Constructive suggestions are solicited for the improvement of this textbook from the students, teachers and general public in the subjects concerned so that next edition will be revised duly incorporating these suggestions.

It is very much commendable that Intermediate text books are being printed for the first time by the Akademi from the 2021-22 academic year.

Sri. V. Ramakrishna I.R.S.
Director
Telugu and Sanskrit Akademi,
Andhra Pradesh

Preface

The Board of Intermediate Education, has recently revised the syllabus in Mathematics for the Intermediate Course with effect from the Akademic year 2012-13. Accordingly, Telugu Akademi has prepared the necessary Text Books in Mathematics.

In accordance with the current syllabus, the topics relating to paper II-B; **Coordinate Geometry** and **Calculus** are dealt with in this book. They are presented in eight chapters. Coordinate Geometry consists of five chapters: **Circle, System of Circles, Parabola, Ellipse, Hyperbola** and Calculus is presented in three chapters, **Integration, Definite Integrals** and **Differential Equations.**

Every chapter herein is divided into various sections and subsections. depending on the contents discussed. These contents are strictly in accordance with the prescribed syllabus and they reflect faithfully the scope and spirit of the same. Necessary definitions, theorems, corollaries. proofs and notes are given in detail. Key concepts are given at the end of each chapter, Illustrative examples and solved problems are in plenty. and these shall help the students in understanding the subject matter.

Every chapter contains exercises in a graded manner which enable the students to solve them by applying the knowledge acquired. All these problems are classified according to the nature of their answers as **I - very short, II - short and III - long.** Answers are provided for all the exercises at the end of each chapter.

Keeping in view the National level competitives examinations, some concepts and notions are highlighled for the benefit of the students. Care has been taken regarding rigor and logical consistency in the presentation of concepts and in proving theorems. Alt the end of the text Book, a lisl of some **Reference Books** in the subject matter is furnished.

The Members of the Mathematics Subject Committee, constituted by Board of Intermediate Education, were invited to interact with the team of the Authors and Editors. They pursued the contents chapter wise. and gave some useful suggestions and comments which are duly incorporated. The special feature of this Book, brought out in a new formal, is that each chapter begins with a thought mostly on Mathematics through a quotation from a famous thinker. It carries a portrait of a noted mathematician with a brief write-up.

In the concluding part of each chapter some relevant historical notes are appended. Wherever found appropriate, references are also made of the contributions of ancient Indian scientists to the advancement of Mathematics. The purpose is to enable the students to have a glimpse into the history of Mathematics in general and the contributions of Indian mathematicians in particular.

Inspite of enough care taken in the scrutiny at various stages in the preparation of the book, errors might have crept in. The readers are therefore, requested to identify and bring them to the notice of the Akademi. We will appreciate if suggetions to enhance the quality of the book are given. Efforts will be made to incorporate them in the subsequent editions.

Prof. P.V. Arunachalam
Chief Coordinator

Preface to the Reviewed Edition

Telugu Akademi is publishing Text books for Two year Intermediate in English and Telugu medium since its inception, periodical review and revision of these publications has been undertaken as and when there was an updation of Intermediate syllabus.

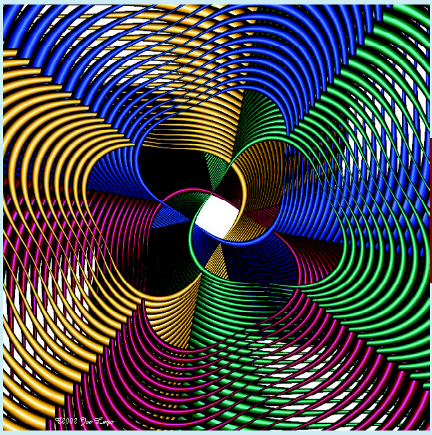
In this reviewed Edition, now being undertaken by the Telugu Akademi, Andhra Pradesh the basic content of its earlier Edition is considered and it is reviewed by a team of experienced teachers. Modification by way correcting errors, print mistakes, incorporating additional content where necessary to elucidate a concept and / or a definition, modification of existing content to remove obscurities for releasing the concept more easily are carried out mainly in this review.

Not withstanding the effort and time spent by the review team in this endeavour, still a few aspects that still need modification or change might have been left unnoticed.

Constructive suggestions from the academic fraternity are welcome and the Akademi will take necessary steps to incorporate them in the forth coming edition.

We appreciate the encouragement and support extended by the Academic and Administrative staff of the Telugu Akademi in fulfilling our assignment with satisfaction.

Editors
(Reviewed Edition)



Contents

1. Circle 1 - 78

Introduction 1

1.1 Equation of a circle, standard form, centre and radius 2

1.2 Position of a point in the plane of a circle - Definition of a tangent 24

1.3 Position of a straight line in the plane of a circle Condition for a line to be tangent 31

1.4 Chord of contact and polar 44

1.5 Relative Positions of two circles 60

2. System of Circles 79 - 102

Introduction 79

2.1 Angle between two intersecting circles 79

2.2 Radical axis of two circles 89

3. Parabola 103 - 130

Introduction 103

3.1 Conic Sections 103

3.2 Equation of tangent and normal at a point on the parabola 117

4. Ellipse 131 - 160

	Introduction	....	131
4.1	Equation of ellipse in standard form, Parametric equations	....	131
4.2	Equation of tangent and normal at a point on the ellipse	....	148

5. Hyperbola 161 - 176

	Introduction	....	161
5.1	Equation of hyperbola in standard form- Parametric equations	....	161
5.2	Equation of Tangent and Normal at a point on the hyperbola	....	168

6. Integration 177 - 260

	Introduction	....	177
6.1	Integration as the inverse process of differentiation, standard forms and properties of integrals	....	178
6.2	Method of substitution - Integration of algebraic, exponential, logarithmic, trigonometric and inverse trigonometric functions - Integration by parts	....	187
6.2(A)	Integration by the method of substitution - Integration of algebraic and trigonometric functions	....	187
6.2(B)	Integration by parts - Integration of exponential, logarithmic and inverse trigonometric functions	....	209
6.3	Integration - Partial fractions method	....	233
6.4	Reduction formulae	....	240

7. Definite Integrals 261 - 314

	Introduction	....	261
7.1	Definite Integral as the limit of sum	....	262
7.2	Interpretation of definite integral as an area	....	263

7.3	The Fundamental Theorem of Integral Calculus	268
7.4	Properties	269
7.5	Reduction Formulae	287
7.6	Applications of definite integral to areas	296
8.	**Differential Equations**	**315 - 356**
	Introduction	315
8.1	Formation of differential equations - Degree and order of an ordinary differential equation	315
8.2	Solving Differential Equations	323
8.2(a)	Variables separable method	323
8.2(b)	Homogeneous Differential Equation	331
8.2(c)	Non - Homogeneous Differential Equations	341
8.2(d)	Linear Differential Equations	345
	Reference Books	**357**
	Syllabus	**358 - 360**
	Model Question Paper	**361 - 363**

Coordinate Geometry

Chapter 1
Circle

"Learn to be silent. Let your quiet mind listen and absorb"

- Pythagoras

Introduction

Geometry has probably originated in ancient Egypt and flourished in Greece, India and China. In the sixth century B.C., the systematic development of geometry has begun.

Great mathematicians such as Thales, Menachmus and Archimedes worked on the circle and a tangent to it during the fifth century B.C. Thirty or forty years after the work of Aristotle, Euclid (a teacher of mathematics of Alexandria in Egypt) collected all the known works and arranged them in his famous book called **"The Elements"**.

Rene Descartes introduced a very important branch of mathematics known as coordinate geometry which is a fusion of geometry and algebra. In honour of Descartes the subject is named as Cartesian Geometry.

The shape of a wheel of a bicycle, a wheel of bullock cart, bangle and some coins are of circular shape (see Fig. 1.1). In this chapter, we deal with the circle and obtain its equation. We derive the equation of a chord, tangent and normal. Further we obtain the parametric equations of a circle and study some important topics related to circles.

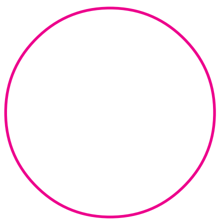
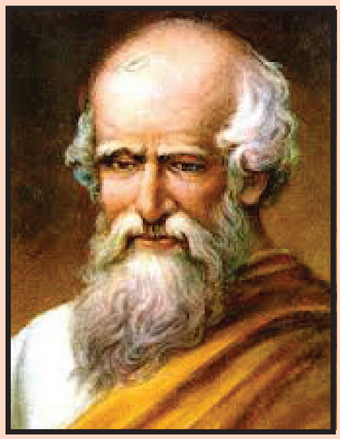


Fig. 1.1



Archimedes
(287 - 212 B.C.)

Archimedes of Syracuse *was an ancient Greek mathematician, physicist and engineer. Although little is known of his life, he is regarded as one of the leading scientists in classical antiquity. He made several discoveries in the fields of mathematics and geometry.*

1.1 Equation of a circle, standard form, centre and radius

1.1.1 Definitions

A circle is the set of points in a plane such that they are equidistant from a fixed point lying in the plane (see Fig. 1.2).

The fixed point is called the centre and the distance from the centre to a point on the circle is called the radius of the circle. Further, twice of the radius of the circle is called its diameter. In the Fig. 1.2, C is the centre of the circle and CP is its radius.

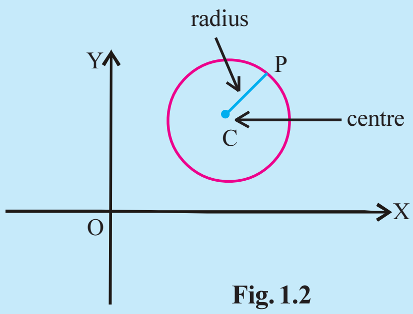


Fig. 1.2

1.1.2 Standard form

Now, we proceed to find the equation of circle in standard form and its other forms.

1.1.3 Theorem : *The equation of the circle with centre O(0, 0) and radius r is* $x^2 + y^2 = r^2$.

Proof : A point $P(x, y)$ is on the circle if and only if the distance between P and O is r (see Fig. 1.3).

$\therefore \quad PO = r$

i.e., $\quad x^2 + y^2 = r^2 \quad \ldots(1)$

which is the required equation of circle. The equation (1) is called standard form of the circle.

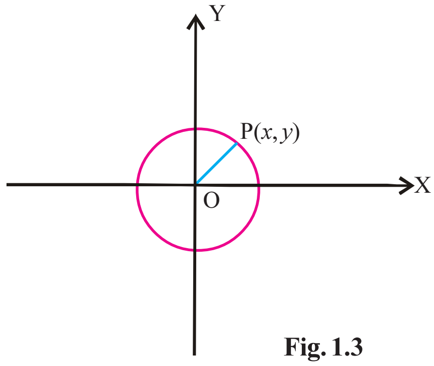


Fig. 1.3

1.1.4 Theorem : *The equation of the circle with centre at C(h, k) and radius r is*

$$(x-h)^2 + (y-k)^2 = r^2.$$

Proof: A point $P(x, y)$ is on the circle if and only if the distance between P and C is r (see Fig. 1.4).

i.e., $\quad \sqrt{(x-h)^2 + (y-k)^2} = r$

i.e., $\quad (x-h)^2 + (y-k)^2 = r^2$

which is the required equation of the circle.

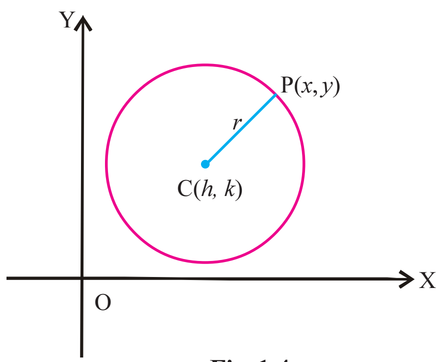


Fig. 1.4

In the following, we obtain a necessary and sufficient condition for a second degree equation in x and y to represent a circle. This facilitates us to decide by just looking at the coefficients whether the equation represents a circle.

1.1.5 Theorem : *The general equation of second degree*

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \qquad \text{... (1)}$$

where the coefficients a, h, b, g, f and c are real numbers, represents a circle if and only if

(i) $a = b \neq 0$ *(ii)* $h = 0$ *and* *(iii)* $g^2 + f^2 - ac \geq 0$

Proof : Suppose that the equation (1) represents a circle. We shall prove

(i) $a = b \neq 0$ (ii) $h = 0$ and (iii) $g^2 + f^2 - ac \geq 0$.

Let (α, β) be the centre and r be the radius of the circle (1). Then by Theorem 1.1.4, the equation of the circle is

$$(x - \alpha)^2 + (y - \beta)^2 = r^2.$$

i.e., $$x^2 + y^2 - 2\alpha x - 2\beta y + \alpha^2 + \beta^2 - r^2 = 0 \qquad \text{... (2)}$$

The equations (1) and (2) represent the same circle. Comparing the coefficients in (1) and (2) we get $h = 0$ and

$$\frac{a}{1} = \frac{b}{1} = \frac{2g}{-2\alpha} = \frac{2f}{-2\beta} = \frac{c}{\alpha^2 + \beta^2 - r^2} \qquad \text{... (3)}$$

$a = b$ follows from equation (3).

Since, equation (1) is a second degree and $h = 0$

$$a \neq 0, \; b \neq 0, \; a = b \neq 0$$

Further, from equation (3), we have

$$\alpha \quad \frac{g}{a}, \beta \quad \frac{f}{a} \text{ and}$$

$$\alpha^2 + \beta^2 - r^2 = \frac{c}{a} \qquad \text{... (4)}$$

Thus, $$\frac{g^2}{a^2} \quad \frac{f^2}{b^2} \quad r^2 \quad \frac{c}{a}$$

$$g^2 \quad f^2 \quad ac \quad a^2r^2 \quad 0$$

i.e., $g^2+f^2-ac \geq 0$ $\quad (\because a^2>0)$

Conversely, suppose that (i) $a=b \neq 0$ (ii) $h=0$ (iii) $g^2+f^2-ac \geq 0$, we shall prove that $ax^2+2hxy+by^2+2gx+2fy+c=0$ represents a circle.

Since $a=b$ and $h=0$, the general equation (1) of second degree becomes

$$ax^2+ay^2+2gx+2fy+c=0$$

$$\therefore\ x^2+y^2+\frac{2g}{a}x+\frac{2f}{a}y+\frac{c}{a}=0 \qquad (\because a \neq 0)$$

$$\therefore\ \left(x+\frac{g}{a}\right)^2+\left(y+\frac{f}{a}\right)^2=\frac{g^2}{a^2}+\frac{f^2}{a^2}-\frac{c}{a}$$

$$\therefore\ \left(x+\frac{g}{a}\right)^2+\left(y+\frac{f}{a}\right)^2=\left(\sqrt{\frac{g^2+f^2-ac}{a^2}}\right)^2 \qquad \text{... (5)}$$

Since $g^2+f^2-ac \geq 0$, the equation (5) represents a circle whose centre is $\left(-\frac{g}{a}, -\frac{f}{a}\right)$ and radius is $\frac{\sqrt{g^2+f^2-ac}}{a}$.

1.1.6 Note

(i) *$x^2+y^2+2gx+2fy+c=0$ is considered as general equation of the circle.*

(ii) *The centre of the circle $x^2+y^2+2gx+2fy+c=0$ is $(-g, -f)$.*

(iii) *The radius of the circle $x^2+y^2+2gx+2fy+c=0$ is $\sqrt{g^2+f^2-c}$.*

(iv) *If $g^2+f^2-c=0$ then $x^2+y^2+2gx+2fy+c=0$ represents a point circle. In this case the centre itself is the point circle. The equation of a point circle having the centre at the origin is $x^2+y^2=0$.*

(v) *The equation of a circle through (0, 0) will be in the form $x^2+y^2+2gx+2fy=0$.*

(vi) *The equation of a circle having the centre on the X-axis will be in the form of $x^2+y^2+2gx+c=0$ ($\because$ y-coordinate of the centre is zero).*

(vii) *The equation of a circle having the centre on the Y-axis will be in the form of $x^2+y^2+2fy+c=0$ ($\because$ x-coordinate of the centre is zero).*

(viii) *Two or more circles are said to be concentric if their centres are same.*

(ix) *The equation of a circle concentric with the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ will be in the form of $x^2 + y^2 + 2gx + 2fy + c' = 0$ where c' is any constant.*

(x) *If the radius of a circle is 1 then it is called a unit circle.*

1.1.7 Solved Problems

1. Problem: *Find the equation of circle with centre (1, 4) and radius 5.*

Solution : Here $(h, k) = (1, 4)$ and $r = 5$. Therefore, by Theorem 1.1.4, we have

$$(x-1)^2 + (y-4)^2 = 5^2$$

i.e., $$x^2 + y^2 - 2x - 8y - 8 = 0.$$

2. Problem: *Find the centre and radius of the circle $x^2 + y^2 + 2x - 4y - 4 = 0$.*

Solution : Here $2g = 2;\quad 2f = -4;\quad c = -4$

$\therefore\quad g = 1,\quad f = -2,\quad c = -4$

$\therefore$ Centre $(-g, -f) = (-1, 2)$ and

radius $= \sqrt{g^2 + f^2 - c} = \sqrt{1 + 4 - (-4)} = 3.$

3. Problem: *Find the centre and radius of the circle $3x^2 + 3y^2 - 6x + 4y - 4 = 0$.*

Solution: First we reduce the given equation to a circle in general form. Dividing the given equation of circle by 3, we get

$$x^2 + y^2 - 2x + \frac{4}{3}y - \frac{4}{3} = 0$$

Hence $2g = -2;\ 2f = \frac{4}{3};\ c = -\frac{4}{3}$

i.e., $g = -1;\ f = \frac{2}{3};\ c = -\frac{4}{3}$.

$\therefore$ Centre $= (-g, -f) = (1, -\frac{2}{3})$ and radius $\sqrt{g^2 + f^2 - c} = \sqrt{1 + \frac{4}{9} + \frac{4}{3}} = \frac{5}{3}$.

4. Problem : *Find the equation of the circle whose centre is (–1, 2) and which passes through (5, 6).*

Solution: Let C (=(–1, 2)) be the centre of the circle (see Fig. 1.5).

Since (5, 6) is a point on the circle, the radius of the circle is

$\sqrt{(5+1)^2 + (6-2)^2} = \sqrt{52}$.

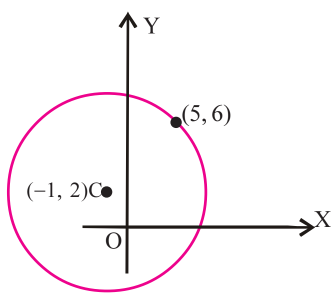


Fig. 1.5

Hence the equation of the required circle is

$$(x+1)^2+(y-2)^2=\left(\sqrt{52}\right)^2$$

i.e., $x^2+y^2+2x-4y-47=0.$

5. Problem : *Find the equation of the circle passing through (2, 3) and concentric with the circle*

$$x^2+y^2+8x+12y+15=0. \qquad \dots(1)$$

Solution: Let the equation of required concentric circle be $x^2+y^2+8x+12y+c'=0$ (By Note 1.1.6(ix)). If it passes through (2, 3) (see Fig. 1.6) we have

$$4+9+16+36+c'=0$$

$$65+c'=0$$

$$\therefore \quad c'=-65.$$

Hence the required circle is $x^2+y^2+8x+12y-65=0.$

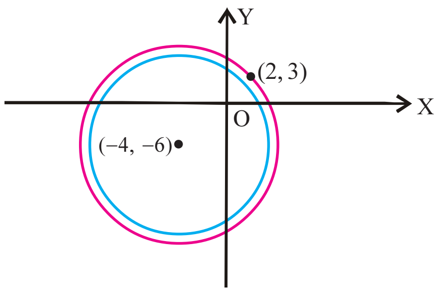


Fig. 1.6

6. Problem: *From the point A(0, 3) on the circle $x^2+4x+(y-3)^2=0$ a chord AB is drawn and extended to a point M such that AM = 2AB. Find the equation of the locus of M.*

Solution: Let $\mathrm{M}=(x',y')$

Given that $\mathrm{AM}=2\mathrm{AB}$

$\therefore \quad \mathrm{AB}+\mathrm{BM}=\mathrm{AB}+\mathrm{AB}$

$\therefore \quad \mathrm{AB}=\mathrm{BM}$

i.e., B is mid point of AM.

$$\therefore \quad \mathrm{B}=\left(\frac{x'}{2},\frac{y'+3}{2}\right)$$

B is a point on the given circle (see Fig. 1.7)

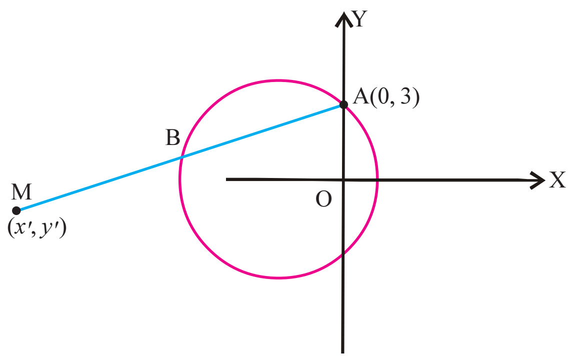


Fig. 1.7

$$\therefore \left(\frac{x'}{2}\right)^2+4\left(\frac{x'}{2}\right)+\left(\frac{y'+3}{2}-3\right)^2=0$$

$$\frac{x'^2}{4}+2x'+\frac{y'^2-6y'+9}{4}=0$$

i.e., $x'^2+y'^2+8x'-6y'+9=0.$

Hence the locus of M is $x^2+y^2+8x-6y+9=0$, which is a circle.

7. Problem: *If the circle $x^2 + y^2 + ax + by - 12 = 0$ has the centre at (2, 3) then find a, b and the radius of the circle.*

Solution : The equation of the circle is

$$x^2 + y^2 + ax + by - 12 = 0. \quad \text{... (1)}$$

The centre of (1) is $\left(-\frac{a}{2}, -\frac{b}{2}\right)$

i.e., $\left(-\frac{a}{2}, -\frac{b}{2}\right) = (2, 3)$

$\Rightarrow \quad a = -4, \ b = -6.$

The equation (1) becomes $x^2 + y^2 - 4x - 6y - 12 = 0$, hence $g = -2, f = -3$ and $c = -12$.

Therefore, the radius of the circle is $\sqrt{g^2 + f^2 - c} = \sqrt{4 + 9 - (-12)} = 5.$

8. Problem : *If the circle $x^2 + y^2 - 4x + 6y + a = 0$ has radius 4 then find a.*

Solution: Comparing the given equation of circle with the general form of equation of a circle, we have

$2g = -4; \quad 2f = 6; \quad c = a.$

i.e., $g = -2; \quad f = 3; \quad c = a.$

Given that the radius of the circle is 4.

$\therefore \quad \sqrt{g^2 + f^2 - c} = 4$

i.e., $\sqrt{4 + 9 - c} = 4$

i.e., $\sqrt{13 - a} = 4$

i.e., $a = -3.$

9. Problem : *Find the equation of the circle passing through (4, 1), (6, 5) and having the centre on the line $4x + y - 16 = 0$.*

Solution: Let the equation of the required circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \text{... (1)}$$

Since it passes through (4, 1), we have

$16 + 1 + 8g + 2f + c = 0$

i.e., $17 + 8g + 2f + c = 0 \quad \text{... (2)}$

Similarly since (6, 5) lies on (1), we obtain

$$36 + 25 + 12g + 10f + c = 0$$

i.e., $61 + 12g + 10f + c = 0$ ... (3)

Given that the centre of (1) lies on $4x + y - 16 = 0$

$\therefore$ $4(-g) + (-f) - 16 = 0$

$4g + f + 16 = 0$ ... (4)

Solving the equations (2), (3) and (4) for g, f and c we get

$$g = -3, \quad f = -4 \quad \text{and} \quad c = 15$$

Thus the equation of the required circle is

$$x^2 + y^2 - 6x - 8y + 15 = 0.$$

10. Problem : *Suppose a point (x_1, y_1) satisfies*

$x^2 + y^2 + 2gx + 2fy + c = 0$ ... (1)

then show that it represents a circle whenever g, f and c are real.

Solution : Comparing the given equation with general equation of second degree we have x^2 coefficient = y^2 coefficient and the coefficient of $xy = 0$. The given equation represents a circle if $g^2 + f^2 - c \geq 0$.

Since (x_1, y_1) is a point on (1), we have

$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0$ ... (2)

Now $g^2 + f^2 - c = g^2 + f^2 + x_1^2 + y_1^2 + 2gx_1 + 2fy_1 = (x_1 + g)^2 + (y_1 + f)^2 \geq 0.$

Since g, f and c are real by Theorem 1.1.5 equation (1) represents a circle.

1.1.8 Theorem

(i) If $g^2 - c > 0$ then the intercept made on the X-axis by the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is $2\sqrt{g^2 - c}$.

(ii) If $f^2 - c > 0$ then the intercept made on the Y-axis by the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ is $2\sqrt{f^2 - c}$.

Proof

(i) The points of intersection of the given circle

$x^2 + y^2 + 2gx + 2fy + c = 0$ ... (1)

and $y = 0$ (i.e., X-axis equation) ... (2)

are the common points of (1) and (2).

Put $y = 0$ in (1) to get the abscissae of the points of intersection. The abscissae of common points are the roots of

$x^2 + 2gx + c = 0$ ... (3)

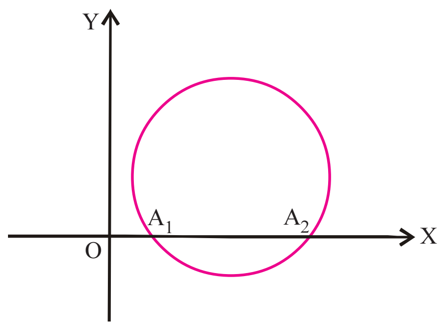


Fig. 1.8(*a*)

The discriminant of this equation is $4(g^2 - c)$. Since $g^2 - c > 0$, the equation (3) has two real and distinct roots, say x_1 and x_2. Suppose the points of intersection are $A_1(x_1, 0)$ and $A_2(x_2, 0)$ (see Fig. 1.8(*a*)). We have to prove that $A_1A_2 = 2\sqrt{g^2 - c}$.

Since x_1 and x_2 are the roots of (3), we have

$$x_1 + x_2 = -2g,$$
$$x_1 x_2 = c.$$

Consider $(x_1 - x_2)^2 = (x_1 + x_2)^2 - 4x_1 x_2$
$$= (-2g)^2 - 4c$$
$$= 4(g^2 - c)$$

Taking the square root, we get

$$|x_1 - x_2| = 2\sqrt{g^2 - c}$$

i.e., $A_1A_2 = 2\sqrt{g^2 - c}$

Thus the intercept made by (1) on X-axis is $2\sqrt{g^2 - c}$.

(ii) The points of intersection of the given circle

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \text{... (1)}$$

and $x = 0$ (the equation of Y-axis) ... (4)

are the common points of (1) and (4).

Put $x = 0$ in (1) to get the ordinates of the points of intersection. The ordinates of common points are the roots of

$$y^2 + 2fy + c = 0 \quad \text{... (5)}$$

The discriminant of this equation is $4(f^2 - c)$. Since $f^2 - c > 0$ the equation (5) has two real and distinct roots say y_1 and y_2. Suppose the points of intersection are $B_1(0, y_1)$ and $B_2(0, y_2)$ (see Fig. 1.8(*b*)). We have to prove that $B_1B_2 = 2\sqrt{f^2 - c}$.

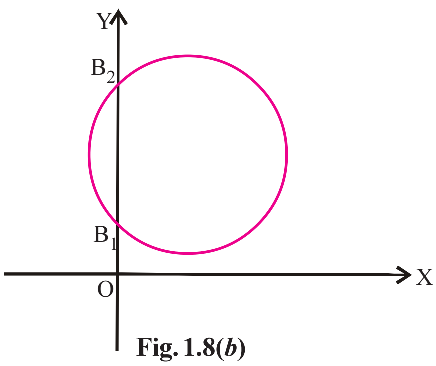


Fig. 1.8(*b*)

Since y_1 and y_2 are the roots of (5) we have

$$y_1 + y_2 = -2f,$$
$$y_1 y_2 = c.$$

Consider $(y_1 - y_2)^2 = (y_1 + y_2)^2 - 4y_1y_2$
$$= (2f)^2 - 4c$$
$$= 4(f^2 - c).$$

$$\therefore \quad |y_1 - y_2| = 2\sqrt{f^2 - c}\,.$$

$$\text{i.e.,} \quad B_1B_2 = 2\sqrt{f^2 - c}\,.$$

Thus the intercept made by (1) on Y-axis is $2\sqrt{f^2 - c}$.

1.1.9 Note

(i) $g^2 - c = 0 \Rightarrow A_1A_2 = 0 \Rightarrow A_1, A_2$ *are coincident i.e., the X-axis touches the circle in two coincident points. Thus the X-axis touches the circle at the point of coincidence (see Fig. 1.9)*

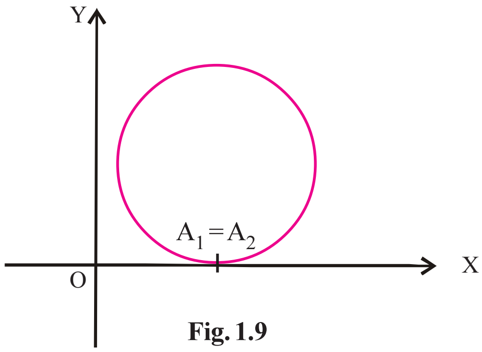


Fig. 1.9

(ii) $f^2 - c = 0 \Rightarrow B_1B_2 = 0 \Rightarrow B_1$ *and* B_2 *are coincident i.e., the Y-axis touches the circle in two coincident points. Thus Y-axis touches the circle at the point of coincidence (see Fig. 1.10)*

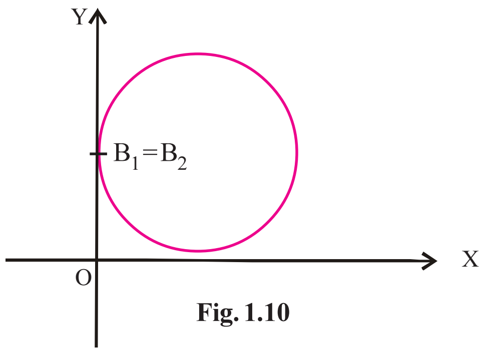


Fig. 1.10

(iii) *If* $g^2 - c < 0$ *then the circle (1) does not meet the X-axis.*

(iv) *If* $f^2 - c < 0$ *then the circle (1) does not meet the Y-axis.*

1.1.10 Example

Let us find the equation of the circle which touches the X-axis at a distance of 3 from the origin and making intercept of length 6 on the Y-axis.

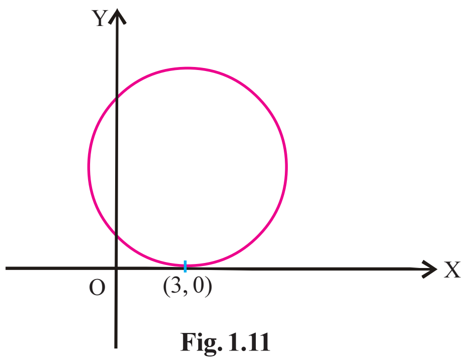


Fig. 1.11

Let the equation of the required circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0 \qquad \text{... (1)}$$

This meets the X-axis at (3, 0) (see Fig. 1.11).

(3, 0) is a point on (1)

$$\therefore \quad 9 + 0 + 6g + 0 + c = 0$$

$$\text{i.e.,} \quad 6g + c = -9 \qquad \text{... (2)}$$

By Note 1.1.9(i), we have

$$g^2 - c = 0 \qquad \text{... (3)}$$

Adding (2) and (3) we get

$$g^2 + 6g + 9 = 0$$

i.e., $(g+3)^2 = 0$

i.e., $g = -3$ ... (4)

From (3) and (4), we get

$$c = 9 \qquad \text{... (5)}$$

Given that the intercept on Y-axis made by (1) is 6.

Therefore by Theorem 1.1.8(ii) we have

$$2\sqrt{f^2 - c} = 6$$

i.e., $2\sqrt{f^2 - 9} = 6$

i.e., $\sqrt{f^2 - 9} = 3$

i.e., $f^2 - 9 = 9$

i.e., $f^2 = 18.$

Hence $f = \pm 3\sqrt{2}$.

Since $g = -3$, $f = \pm 3\sqrt{2}$ and $c = 9$, we have two circles satisfying the hypothesis, these circles are $x^2 + y^2 - 6x + 6\sqrt{2}\,y + 9 = 0$ and $x^2 + y^2 - 6x - 6\sqrt{2}\,y + 9 = 0$.

1.1.11 Definition

If A and B are two distinct points on a circle then

(i) *the line* $\overleftrightarrow{AB}$ *through A and B is called a secant (see Fig. 1.12)*

(ii) *The segment* $\overline{AB}$, *the join of A and B is called a chord and the length of the chord is denoted by AB (see Fig. 1.13)*

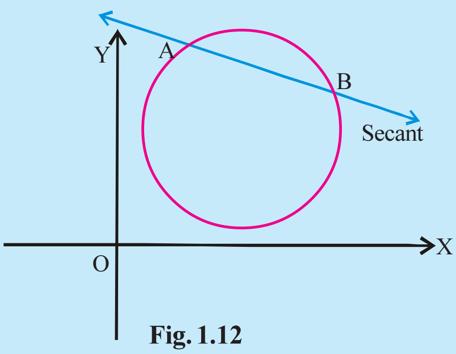


Fig. 1.12

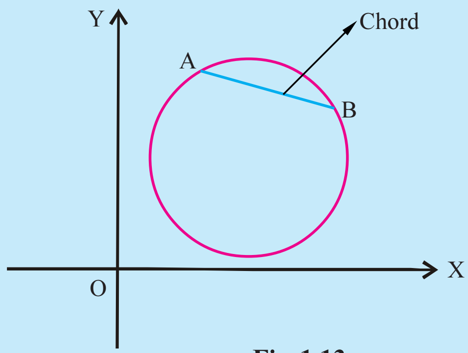


Fig. 1.13

1.1.12 Equation of a circle with a given line segment as diameter

In this section, we derive the equation of circle whose diameter extremities are given.

1.1.13 Theorem : *The equation of the circle whose diameter extremities are* (x_1, y_1) *and* (x_2, y_2) *is*

$$(x - x_1)\,(x - x_2) + (y - y_1)\,(y - y_2) = 0.$$

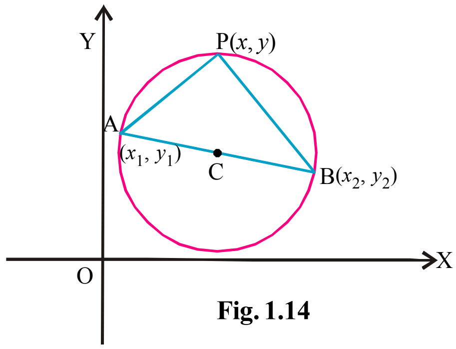


Fig. 1.14

Proof : Let $A = (x_1, y_1)$, $B = (x_2, y_2)$ and C be the centre of the circle (see Fig. 1.14).

Let $P(x, y)$ be any point on it other than A and B. Join A and B, A and P and also P and B. We know that

$$A\hat{P}B = 90^o.$$

i.e., the lines AP and BP are perpendicular

$\therefore$ (slope of AP) (slope of BP) = –1.

$$\text{i.e.,}\quad \frac{(y - y_1)}{(x - x_1)} \times \frac{(y - y_2)}{(x - x_2)} = -1$$

$$\text{i.e.,}\quad (x - x_1)\,(x - x_2) + (y - y_1)\,(y - y_2) = 0. \qquad \dots (1)$$

Also clearly A and B satisfy (1). Therefore any point $P(x, y)$ on the circle satisfies equation (1). Conversely if a point $P(x, y)$ satisfies (1) then $A\hat{P}B = 90^o$ and hence P lies on the circle.

Thus (1) is the equation of the required circle.

1.1.14 Solved Problems

1. Problem : *Find the equation of the circle whose extremities of a diameter are (1, 2) and (4, 5).*

Solution : Here $(x_1, y_1) = (1, 2)$ and $(x_2, y_2) = (4, 5)$.

By Theorem 1.1.13, the equation of the required circle is

$$(x - 1)\,(x - 4) + (y - 2)(y - 5) = 0.$$

$$\text{i.e.,}\quad x^2 + y^2 - 5x - 7y + 14 = 0.$$

2. Problem : *Find the other end of the diameter of the circle* $x^2 + y^2 - 8x - 8y + 27 = 0$ *if one end of it is (2, 3).*

Solution : Let A(2, 3) and AB be the diameter (see Fig. 1.15) of the circle

$$x^2 + y^2 - 8x - 8y + 27 = 0.$$

The centre of the circle C is (4, 4). Let the other end B of the diameter be (α, β). Then, C is the mid point of AB.

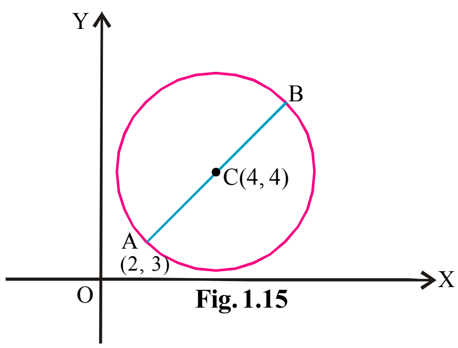


Fig. 1.15

$$\therefore \left(\frac{2+\alpha}{2}, \frac{3+\beta}{2}\right) = (4, 4)\cdot$$

$\therefore \alpha = 6$ and $\beta = 5$.

$\therefore$ The other end of the diameter is (6, 5).

1.1.15 Equation of circle through 3 non-collinear points

We derive a formula to find the equation of a circle through three given points in the next section.

1.1.16 Theorem : *The equation of the circle passing through three non-collinear points $P(x_1, y_1)$, $Q(x_2, y_2)$ and $R(x_3, y_3)$ is*

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}(x^2+y^2)+\begin{vmatrix} c_1 & y_1 & 1 \\ c_2 & y_2 & 1 \\ c_3 & y_3 & 1 \end{vmatrix}x$$

$$+\begin{vmatrix} x_1 & c_1 & 1 \\ x_2 & c_2 & 1 \\ x_3 & c_3 & 1 \end{vmatrix}y+\begin{vmatrix} x_1 & y_1 & c_1 \\ x_2 & y_2 & c_2 \\ x_3 & y_3 & c_3 \end{vmatrix}=0$$

where $c_i = -(x_i^2+y_i^2) \qquad (i=1,2,3)$.

Proof : Let the equation of the circle passing through the points P, Q and R be

$$x^2+y^2+2gx+2fy+c=0. \qquad \text{... (1)}$$

Since the points P, Q and R are lying on (1), we have

$$x_1^2+y_1^2+2gx_1+2fy_1+c=0 \qquad \text{... (2)}$$

$$x_2^2+y_2^2+2gx_2+2fy_2+c=0 \qquad \text{... (3)}$$

$$x_3^2+y_3^2+2gx_3+2fy_3+c=0 \qquad \text{... (4)}$$

Let $2g=a,\ \ 2f=b$ and $c_i=-(x_i^2+y_i^2)\ \ (i=1,2,3)$. ... (5)

The equation (2), (3) and (4) can be written as

$$ax_1+by_1+c=c_1 \qquad \text{... (6)}$$

$$ax_2+by_2+c=c_2 \qquad \text{... (7)}$$

$$ax_3+by_3+c=c_3 \qquad \text{... (8)}$$

Let $\Delta = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$. Then $\Delta \neq 0$ since P, Q, R are non-collinear.

Consider $\begin{vmatrix} c_1 & y_1 & 1 \\ c_2 & y_2 & 1 \\ c_3 & y_3 & 1 \end{vmatrix} = \begin{vmatrix} ax_1+by_1+c & y_1 & 1 \\ ax_2+by_2+c & y_2 & 1 \\ ax_3+by_3+c & y_3 & 1 \end{vmatrix}$

$$= \begin{vmatrix} ax_1 & y_1 & 1 \\ ax_2 & y_2 & 1 \\ ax_3 & y_3 & 1 \end{vmatrix} + \begin{vmatrix} by_1 & y_1 & 1 \\ by_2 & y_2 & 1 \\ by_3 & y_3 & 1 \end{vmatrix} + \begin{vmatrix} c & y_1 & 1 \\ c & y_2 & 1 \\ c & y_3 & 1 \end{vmatrix}$$

$= a\Delta + 0 + 0$ ($\because$ two column elements are proportional)

$$\therefore \quad 2g = a = \frac{\begin{vmatrix} c_1 & y_1 & 1 \\ c_2 & y_2 & 1 \\ c_3 & y_3 & 1 \end{vmatrix}}{\Delta} \qquad \ldots (9)$$

Similarly

$$2f = b = \frac{\begin{vmatrix} x_1 & c_1 & 1 \\ x_2 & c_2 & 1 \\ x_3 & c_3 & 1 \end{vmatrix}}{\Delta} \qquad \ldots (10)$$

and

$$c = \frac{\begin{vmatrix} x_1 & y_1 & c_1 \\ x_2 & y_2 & c_2 \\ x_3 & y_3 & c_3 \end{vmatrix}}{\Delta}. \qquad \ldots (11)$$

Substituting the values of g, f and c in (1), we get the equation of the circle passing through the points P, Q and R as

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}(x^2+y^2) + \begin{vmatrix} c_1 & y_1 & 1 \\ c_2 & y_2 & 1 \\ c_3 & y_3 & 1 \end{vmatrix} x + \begin{vmatrix} x_1 & c_1 & 1 \\ x_2 & c_2 & 1 \\ x_3 & c_3 & 1 \end{vmatrix} y + \begin{vmatrix} x_1 & y_1 & c_1 \\ x_2 & y_2 & c_2 \\ x_3 & y_3 & c_3 \end{vmatrix} = 0$$

1.1.17 Note

(i) *The centre of the circle passing through three non-collinear points $P(x_1, y_1)$, $Q(x_2, y_2)$ and $R(x_3, y_3)$*

$$is \left(\frac{\begin{vmatrix} c_1 & y_1 & 1 \\ c_2 & y_2 & 1 \\ c_3 & y_3 & 1 \end{vmatrix}}{-2 \times \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}}, \frac{\begin{vmatrix} x_1 & c_1 & 1 \\ x_2 & c_2 & 1 \\ x_3 & c_3 & 1 \end{vmatrix}}{-2 \times \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}} \right)$$

(from equations (9) and (10) of Theorem 1.1.16) where $c_i = -(x_i^2 + y_i^2)$ $(i = 1, 2, 3)$.

(ii) *We can also find the equation of the circle passing through three non-collinear points in the following ways.*

(a) *First we suppose that the equation of the circle passing through the given three points P, Q and R in general form. Substitute the coordinates of P, Q and R in this equation. We get three equations involving three unknowns g, f and c. Solve them for g, f and c. Substitute these values in the supposed equation, we get the required circle.*

(b) *In this method, we suppose that the centre of the circle passing through the points P, Q and R as C(a, b). Construct the equations from CP = CQ and CP = CR. These two equations yield two simultaneous equations in a and b. Solve them for a and b. Thus the centre of the required circle is known. Next find the radius of the circle (i.e., CP). Now we can write the equation of required circle using*

$(x-h)^2 + (y-k)^2 = r^2$ *where (h, k) is the centre and r is the radius of the circle.*

(c) *In this method, first we find the equations of any two sides of PQ, QR and RP. Next find the intersection of perpendicular bisector of two sides. It is the centre of required circle. The distance from the centre to any point of three given points is the radius. We compute this radius. Using* $(x-h)^2 + (y-k)^2 = r^2$, *we can find the equation of required circle.*

(iii) $P(x_1, y_1)$, $Q(x_2, y_2)$, $R(x_3, y_3)$ *and* $S(x_4, y_4)$ *are said to be concyclic if these points lie on the same circle.*

1.1.18 Example

Let us find the equation of the circle passing through P(1, 1), Q(2, −1) and R(3, 2).

We find the equation of the required circle using Theorem 1.1.16.

Here $(x_1, y_1) = (1, 1)$; $(x_2, y_2) = (2, -1)$; $(x_3, y_3) = (3, 2)$ and

$$c_1 = -(x_1^2 + y_1^2) = -(1+1) = -2$$

$$c_2 = -(x_2^2 + y_2^2) = -(4+1) = -5$$

$$c_3 = -(x_3^2 + y_3^2) = -(9+4) = -13$$

By Theorem 1.1.16, the equation of circle is

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & 1 \\ 3 & 2 & 1 \end{vmatrix}(x^2+y^2) + \begin{vmatrix} -2 & 1 & 1 \\ -5 & -1 & 1 \\ -13 & 2 & 1 \end{vmatrix}x$$

$$+\begin{vmatrix} 1 & -2 & 1 \\ 2 & -5 & 1 \\ 3 & -13 & 1 \end{vmatrix}y + \begin{vmatrix} 1 & 1 & -2 \\ 2 & -1 & -5 \\ 3 & 2 & -13 \end{vmatrix} = 0$$

i.e., $5(x^2+y^2) - 25x - 5y + 20 = 0$

i.e., $x^2 + y^2 - 5x - y + 4 = 0.$

Other methods

This problem can also be solved in the following ways.

Method 1 (Using Note 1.1.17 ii(a))

Let the equation of the circle through P, Q and R be

$$x^2 + y^2 + 2gx + 2fy + c = 0. \quad \text{... (1)}$$

Since it passes through P(1, 1), Q(2, –1) and R(3, 2), we have

$$2 + 2g + 2f + c = 0$$

$$5 + 4g - 2f + c = 0$$

$$13 + 6g + 4f + c = 0$$

Solving the above three equations for g, f and c we get $g = -\frac{5}{2}$, $f = -\frac{1}{2}$ and $c = 4$. Substituting these values in (1), we get the equation of the required circle as

$$x^2 + y^2 - 5x - y + 4 = 0.$$

Method 2 (Using Note 1.1.17 ii(b))

Let C = (a, b) be the centre of the required circle. Then CP = CQ and CP = CR.

CP = CQ $\Rightarrow$ $CP^2 = CQ^2$

$\Rightarrow$ $(a-1)^2 + (b-1)^2 = (a-2)^2 + (b+1)^2$

$\Rightarrow$ $2a - 4b = 3$ ... (2)

CP = CR $\Rightarrow$ $CP^2 = CR^2$

$\Rightarrow$ $(a-1)^2 + (b-1)^2 = (a-3)^2 + (b-2)^2$

$\Rightarrow$ $4a + 2b = 11$ ... (3)

Solving (2) and (3) we get $a = \frac{5}{2}$ and $b = \frac{1}{2}$. Now the radius of the required circle is

$$CP = \sqrt{\left(1-\frac{5}{2}\right)^2 + \left(1-\frac{1}{2}\right)^2} = \sqrt{\frac{5}{2}}.$$

Hence the equation of the required circle is

$$\left(x-\frac{5}{2}\right)^2 + \left(y-\frac{1}{2}\right)^2 = \left(\sqrt{\frac{5}{2}}\right)^2$$

i.e, $x^2 + y^2 - 5x - y + 4 = 0.$

Method 3 (Using Note 1.1.17 ii(c))

The perpendicular bisector of $\overline{QR}$ is

$$x + 3y - 4 = 0 \qquad \text{... (4)}$$

Similarly the perpendicular bisector of $\overline{PR}$ is

$$4x + 2y - 11 = 0 \qquad \text{... (5)}$$

The point of intersection of (4) and (5) is the centre of the required circle. Hence the centre is $\left(\frac{5}{2}, \frac{1}{2}\right)$ say C.

The radius of the required circle is CP or CQ or CR.

$$CP = \sqrt{\frac{5}{2}}.$$

Hence the required circle is

$$\left(x-\frac{5}{2}\right)^2 + \left(y-\frac{1}{2}\right)^2 = \left(\sqrt{\frac{5}{2}}\right)^2$$

i.e., $x^2 + y^2 - 5x - y + 4 = 0.$

1.1.19 Solved Problems

1. Problem : *Find the equation of the circum-circle of the triangle formed by the line $ax + by + c = 0$ $(abc \neq 0)$, and the coordinate axes.*

Solution : Let the line $ax + by + c = 0$ cut the X, Y axes at A and B respectively (see Fig. 1.16), the figure is drawn for $-\frac{c}{a} > 0$ and $-\frac{c}{b} > 0$.

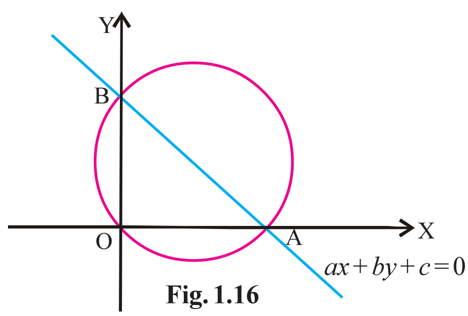


Fig. 1.16

We have to find circle passing through A, B and the origin (0, 0).

Clearly $A = \left(-\frac{c}{a}, 0\right)$ and $B = \left(0, -\frac{c}{b}\right)$.

Let the equation of required circle be

$$x^2 + y^2 + 2gx + 2fy + c' = 0 \quad \text{... (1)}$$

Since (0, 0) is a point lying on (1), we have

$$c' = 0 \quad \text{... (2)}$$

Since the points A and B are also lying on (1), we have

$$\frac{c^2}{a^2} + 0 + 2g\left(-\frac{c}{a}\right) + 0 + 0 = 0$$

and $$0 + \frac{c^2}{b^2} + 0 + 2f\left(-\frac{c}{b}\right) + 0 = 0$$

i.e., $$g = \frac{c}{2a} \text{ and } f = \frac{c}{2b}.$$

Substituting g, f and c' values in (1) we get

$$x^2 + y^2 + \frac{c}{a}x + \frac{c}{b}y = 0$$

i.e., $ab(x^2 + y^2) + c(bx + ay) = 0$, which is the required circle.

2. Problem : *Find the equation of the circle which passes through the vertices of the triangle formed by* $L_1 = x + y + 1 = 0$, $L_2 = 3x + y - 5 = 0$ *and* $L_3 = 2x + y - 5 = 0$.

Solution : Suppose L_1, L_2; L_2, L_3 and L_3, L_1 intersect at A, B and C respectively.

Consider a curve whose equation is

$$k(x + y + 1)(3x + y - 5) + l(3x + y - 5)(2x + y - 5) + m(2x + y - 5)(x + y + 1) = 0 \quad \text{... (1)}$$

We can verify the fact that this curve passes through A, B and C.

Hence we find k, l and m such that the equation (1) represents a circle. If the equation (1) represents a circle we have (by Theorem 1.1.5).

(i) coefficient of x^2 = coefficient of y^2

$$3k + 6l + 2m = k + l + m$$

i.e., $$2k + 5l + m = 0. \quad \text{... (2)}$$

(ii) coefficient of $xy = 0$

$$4k + 5l + 3m = 0. \quad \text{... (3)}$$

Applying cross multiplication rule for (2) and (3) we get

$$\begin{array}{cccccc} 5 & 1 & & 2 & 5 \\ 5 & 3 & & 4 & 5 \end{array}$$

$$\frac{}{15-5} = \frac{}{4-6} = \frac{m}{10-20}$$

i.e., $\dfrac{k}{10} = \dfrac{l}{-2} = \dfrac{m}{-10}$

i.e., $\dfrac{}{5} = \dfrac{}{-1} = \dfrac{m}{-5}.$

Hence the required equation is

$$5(x+y+1)(3x+y-5) - 1(3x+y-5)(2x+y-5) - 5(2x+y-5)(x+y+1) = 0$$

i.e., $x^2 + y^2 - 30x - 10y + 25 = 0.$

3. Problem : *Find the equation of the circle which passes through the vertices of the triangle formed by $x = 0, y = 0$ and $\frac{x}{a} + \frac{y}{b} = 1.$*

Solution : Observe that the vertices of the triangle are (0, 0), (a, 0) and (0, b)

Let the equation of the circle be $x^2 + y^2 + 2gx + 2fy + c = 0$ ... (1)

Since, the circle (1) passes through (0, 0) (a, 0) and (0, b),

we have $c = 0$, $a^2 + 2ga = 0$ and $b^2 + 2fb = 0$

Therefore, $c = 0$, $g \quad \frac{a}{2}$ and $f \quad \frac{b}{2}$

Hence, the required circle equation is $x^2 + y^2 - ax - by = 0$

1.1.20 Parametric equations of a circle

Parametric equations of a circle describe the coordinates of a point on the circle in terms of a single variable θ (say). We call this single variable as parameter. Now we derive the parametric equations of a circle.

1.1.21 Theorem : *The parametric equations of a circle with centre (h, k) and radius r (> 0) are given by*

$$x = h + r\cos\theta$$

$$y = k + r\sin\theta$$

where $0 \le \theta < 2\pi.$

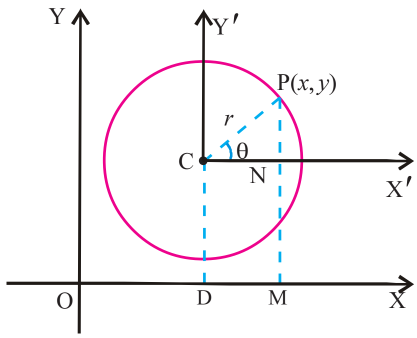


Fig. 1.17

Proof : Let the centre of the circle be C.

Then $C = (h, k)$. Let $P(x, y)$ be any point on the circle with the centre C and radius r. Draw $\overrightarrow{CX'}$ parallel to $\overrightarrow{OX}$ and CY' parallel to $\overrightarrow{OY}$. Join C and P. Note that $CP = r$.

Let $P\hat{C}X' = \theta$.

Draw a line from P parallel to Y-axis meeting CX' at N and meeting X-axis at M.

Then $OM = x$; $PM = y$ (see Fig. 1.17, it is drawn for the case $h > 0,\ k > 0$).

Now draw a line parallel to Y-axis from C meeting the X-axis at D. Then $OD = h$; $CD = k$. The triangle CPN is a right angled triangle.

$$\therefore\ \cos\theta = \frac{CN}{CP} = \frac{DM}{CP} = \frac{OM - OD}{CP} = \frac{x - h}{r}$$

i.e., $x - h = r\cos\theta.$

$$\therefore\ \boxed{x = h + r\cos\theta} \qquad \ldots (1)$$

Consider $\sin\theta = \dfrac{PN}{CP} = \dfrac{PM - MN}{CP} = \dfrac{PM - CD}{CP} = \dfrac{y - k}{r}$

i.e., $y - k = r\sin\theta$

$$\therefore\ \boxed{y = k + r\sin\theta} \qquad \ldots (2)$$

Hence the equations (1) and (2) constitute the parametric equations of a circle where $0 \le \theta < 2\pi$.

Conversely if $x = h + r\cos\theta,\quad y = k + r\sin\theta$ where $0 \le \theta < 2\pi$ then $(x - h)^2 + (y - k)^2 = r^2$. Therefore the point (x, y) lies on the circle. Hence equations (1) and (2) are the parametric equations of the circle where $0 \le \theta < 2\pi$.

1.1.22 Note

(i) *If the centre of the circle is the origin, then parametric equations of the circle having radius r is* $x = r\cos\theta,\ y = r\sin\theta$ *where* $0 \le \theta < 2\pi$.

(ii) *The point* $(h + r\cos\theta_1,\ k + r\sin\theta_1)$ *is referred as the point* θ_1 *(a particular value of the parameter* θ*) on the circle having the centre (h, k) and radius r.*

1.1.23 Solved Problems

1. Problem : *Obtain the parametric equations of the circle* $x^2 + y^2 = 1$.

Solution : Here the centre of the circle is (0, 0) and radius is $r = 1$ (see Fig. 1.18)

$\therefore$ The parametric equations of the circle $x^2 + y^2 = 1$ are

$$x = 1 \,.\, \cos\theta = \cos\theta$$
$$y = 1 \,.\, \sin\theta = \sin\theta, \qquad 0 \le \theta < 2\pi$$

(by Note 1.1.22(i))

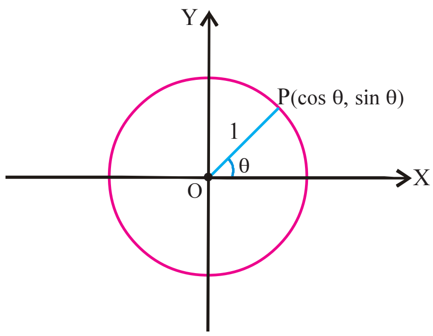


Fig. 1.18

Note that every point on this circle can be expressed as $(\cos\theta, \sin\theta)$.

2. Problem : *Obtain the parametric equation of the circle represented by*

$$x^2 + y^2 + 6x + 8y - 96 = 0.$$

Solution : Here the centre (h, k) of the circle is $(-3, -4)$ and radius

$$r = \sqrt{9 + 16 - (-96)} = 11.$$

$\therefore$ By Theorem 1.1.21, the parametric equation of the given circle are

$$x = -3 + 11\cos\theta$$
$$y = -4 + 11\sin\theta$$

where $0 \le \theta < 2\pi$.

Exercise 1(a)

I. 1. Find the equations of circles with centre C and radius r where

(i) $C = (2, -3),\ r = 4$ (ii) $C = (-1, 2),\ r = 5$

(iii) $C = (a, -b),\ r = a + b$

(iv) $C = (-a, -b),\ r = \sqrt{a^2 - b^2}$ $(|a| > |b|)$

(v) $C = (\cos\alpha, \sin\alpha),\ r = 1.$ (vi) $C = (-7, -3),\ r = 4$

(vii) $C = \left(-\frac{1}{2}, -9\right),\ r = 5$ (viii) $C = \left(\frac{5}{2}, -\frac{4}{3}\right),\ r = 6$

(ix) $C = (1, 7),\ r = \frac{5}{2}$ (x) $C = (0, 0),\ r = 9.$

2. Find the equation of the circle passing through the origin and having the centre at $(-4, -3)$.
3. Find the equation of the circle passing through $(2, -1)$ having the centre at $(2, 3)$.
4. Find the equation of the circle passing through $(-2, 3)$, having the centre at $(0, 0)$.
5. Find the equation of the circle passing through $(3, 4)$ and having the centre at $(-3, 4)$.
6. Find the value of a if $2x^2 + ay^2 - 3x + 2y - 1 = 0$ represents a circle and also find its radius.
7. Find the values of a, b if $ax^2 + bxy + 3y^2 - 5x + 2y - 3 = 0$ represents a circle. Also find the radius and centre of the circle.
8. If $x^2 + y^2 + 2gx + 2fy - 12 = 0$ represents a circle with centre $(2, 3)$ find g, f and its radius.
9. If $x^2 + y^2 + 2gx + 2fy = 0$ represents a circle with centre $(-4, -3)$ then find g, f and the radius of the circle.
10. If $x^2 + y^2 - 4x + 6y + c = 0$ represents a circle with radius 6 then find the value of c.
11. Find the centre and radius of each of the circles whose equations are given below :

(i) $x^2 + y^2 - 4x - 8y - 41 = 0$ (ii) $3x^2 + 3y^2 - 5x - 6y + 4 = 0$

(iii) $3x^2 + 3y^2 + 6x - 12y - 1 = 0$ (iv) $x^2 + y^2 + 6x + 8y - 96 = 0$

(v) $2x^2 + 2y^2 - 4x + 6y - 3 = 0$ (vi) $2x^2 + 2y^2 - 3x + 2y - 1 = 0$

(vii) $\sqrt{1+m^2}\ (x^2 + y^2) - 2cx - 2mcy = 0$

(viii) $x^2 + y^2 + 2ax - 2by + b^2 = 0.$

12. Find the equations of the circles for which the points given below are the end points of a diameter.

(i) $(1, 2), (4, 6)$ (ii) $(-4, 3), (3, -4)$

(iii) $(1, 2), (8, 6)$ (iv) $(4, 2), (1, 5)$

(v) $(7, -3), (3, 5)$ (vi) $(1, 1), (2, -1)$

(vii) $(0, 0), (8, 5)$ (viii) $(3, 1)\ (2, 7)$

13. Obtain the parametric equation of each of the following circles.

(i) $x^2 + y^2 = 4$ (ii) $4(x^2 + y^2) = 9$

(iii) $2x^2 + 2y^2 = 7$ (iv) $(x - 3)^2 + (y - 4)^2 = 8^2.$

(v) $x^2 + y^2 - 4x - 6y - 12 = 0$ (vi) $x^2 + y^2 - 6x + 4y - 12 = 0$

II. 1. If the abscissae of points A, B are the roots of the equation $x^2+2ax-b^2=0$ and ordinates of A, B are roots of $y^2+2py-q^2=0$ then find the equation of a circle for which $\overline{AB}$ is a diameter.

2. (i) Show that A(3, −1) lies on the circle $x^2+y^2-2x+4y=0$. Also find the other end of the diameter through A.

(ii) Show that A(−3, 0) lies on $x^2+y^2+8x+12y+15=0$ and find the other end of diameter through A.

3. Find the equation of a circle which passes through (2, −3) and (−4, 5) and having the centre on $4x+3y+1=0$.

4. Find the equation of a circle which passes through (4, 1), (6, 5) and having the centre on $4x+3y-24=0$.

5. Find the equation of a circle which is concentric with $x^2+y^2-6x-4y-12=0$ and passing through (−2, 14)

6. Find the equation of the circle whose centre lies on the X-axis and passing through (−2, 3) and (4, 5).

7. If ABCD is a square then show that the points A, B, C and D are concyclic.

III. 1. Find the equation of circle passing through each of the following three points

(i) (3, 4), (3, 2), (1, 4) (ii) (1, 2), (3, −4), (5, −6),

(iii) (2, 1), (5, 5), (−6, 7), (iv) (5, 7), (8, 1), (1, 3),

(v) (0, 0), (2, 0), (0, 2).

2. (i) Find the equation of the circle passing through (0, 0) and making intercepts 4, 3 on X-axis and Y-axis respectively.

(ii) Find the equation of the circle passing through (0, 0) and making intercept 6 units on X-axis and intercept 4 units on Y-axis.

3. Show that the following four points in each of the following are concyclic and find the equation of the circle on which they lie.

(i) (1, 1), (−6, 0), (−2, 2), (−2, −8) (ii) (1, 2) (3, −4), (5, −6), (19, 8)

(iii) (1, −6) (5, 2), (7, 0), (−1, −4) (iv) (9, 1), (7, 9) (−2, 12), (6, 10)

4. If (2, 0), (0, 1) (4, 5) and (0, c) are concyclic then find c.

5. Find the equation of the circum–circle of the triangle formed by the straight lines given in each of the following:

(i) $2x+y=4,\ x+y=6,\ x+2y=5$

(ii) $x+3y-1=0,\ x+y+1=0,\ 2x+3y+4=0$

(iii) $5x-3y+4=0,\ 2x+3y-5=0,\ x+y=0$

(iv) $x-y-2=0,\ 2x-3y+4=0,\ 3x-y+6=0$

6. Show that the locus of the point of intersection of the lines $x\cos\alpha + y\sin\alpha = a$, $x\sin\alpha - y\cos\alpha = b$ (α is a parameter) is a circle.
7. Show that the locus of a point such that the ratio of distance of it from two given points is constant $k(\neq \pm 1)$ is a circle.

1.2 Position of a point in the plane of a circle-Definition of a tangent

In earlier classes, we have learnt that the tangent at any point of a circle is a straight line which meets the circle at that point only. The point is called the point of contact. This tangent is perpendicular to the radius drawn from the centre to the point of contact. In this section we give another definition of a tangent to the circle using the limit concept. Using this definition we find an equation of tangent at any point in section 1.3. We also learn the position of a point with respect to a circle and power of a point. Further, we define the length of a tangent from a point and obtain a formula for it.

1.2.1 Notation

Now we introduce certain notations that will be used in the rest of this section and subsequently.

(i) The expression $x^2 + y^2 + 2gx + 2fy + c$ is denoted by S

i.e., $\quad S \equiv x^2 + y^2 + 2gx + 2fy + c$.

(ii) The expression $xx_i + yy_i + g(x + x_i) + f(y + y_i) + c$ is denoted by S_i

Thus $\quad S_1 \equiv xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c,$

$\quad S_2 \equiv xx_2 + yy_2 + g(x + x_2) + f(y + y_2) + c.$

(iii) The expression $x_i x_j + y_i y_j + g(x_i + x_j) + f(y_i + y_j) + c$ is denoted by S_{ij} $(i, j = 1, 2, 3, ...)$ For example

$$S_{12} = x_1x_2 + y_1y_2 + g(x_1 + x_2) + f(y_1 + y_2) + c.$$
$$S_{11} = x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c.$$

1.2.2 Position of a point with respect to a circle

A circle in a plane divides the plane into three parts namely

(i) the interior of the circle (see Fig. 1.19(a))

(ii) the circumference which is the circular curve (see Fig.1.19(b))

(iii) the exterior of the circle (see Fig. 1.19(c)).

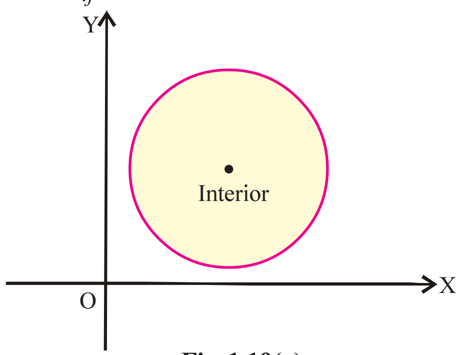


Fig. 1.19(a)

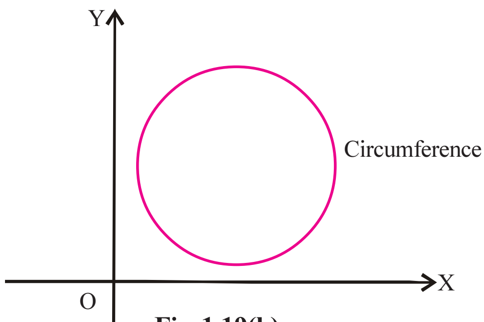


Fig. 1.19(b)

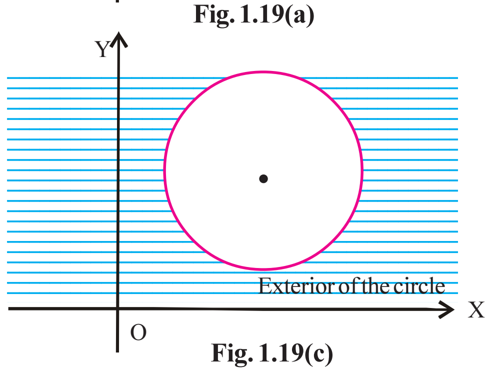


Fig. 1.19(c)

1.2.3 Theorem : *Let $S=0$ be a circle in a plane and $P(x_1, y_1)$ be any point in the same plane. Then*

(i) *P lies in the interior of the circle* $\Leftrightarrow S_{11} < 0$.

(ii) *P lies on the circle* $\Leftrightarrow S_{11} = 0$.

(iii) *P lies in the exterior of the circle* $\Leftrightarrow S_{11} > 0$.

Proof : Let $S \equiv x^2+y^2+2gx+2fy+c=0$ be the equation of the given circle and $P(x_1,y_1)$ be any point in the plane. Then $C(-g, -f)$ is the centre and $r=\sqrt{g^2+f^2-c}$ is the radius of the circle.

(i) P lies in the interior of the circle

$\Leftrightarrow$ $CP < r$ (see Fig. 1.20)

$\Leftrightarrow$ $CP^2 < r^2$

$\Leftrightarrow$ $(x_1+g)^2+(y_1+f)^2 < g^2+f^2-c$

$\Leftrightarrow$ $x_1^2+y_1^2+2gx_1+2fy_1+c<0$

$\Leftrightarrow$ $S_{11}<0$.

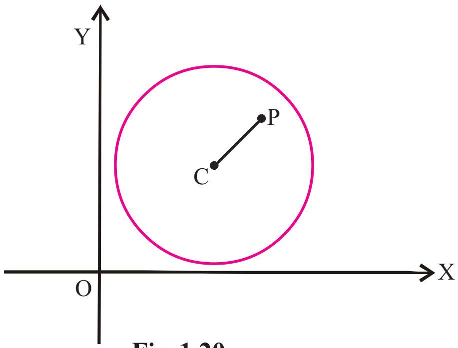


Fig. 1.20

(ii) P lies on the circle

$\Leftrightarrow$ $CP = r$ (see Fig. 1.21)

$\Leftrightarrow$ $CP^2 = r^2$

$\Leftrightarrow$ $(x_1+g)^2+(y_1+f)^2 = g^2+f^2-c$

$\Leftrightarrow$ $x_1^2+y_1^2+2gx_1+2fy_1+c=0$

$\Leftrightarrow$ $S_{11}=0$.

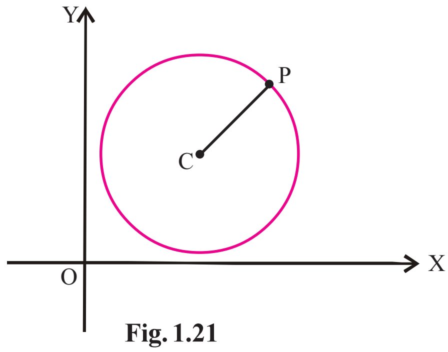


Fig. 1.21

(iii) P lies in the exterior of the circle

$\Leftrightarrow$ $CP > r$ (see Fig. 1.22)

$\Leftrightarrow$ $CP^2 > r^2$.

$\Leftrightarrow$ $(x_1+g)^2+(y_1+f)^2 > g^2+f^2-c$

$\Leftrightarrow$ $x_1^2+y_1^2+2gx_1+2fy_1+c>0$

$\Leftrightarrow$ $S_{11}>0$.

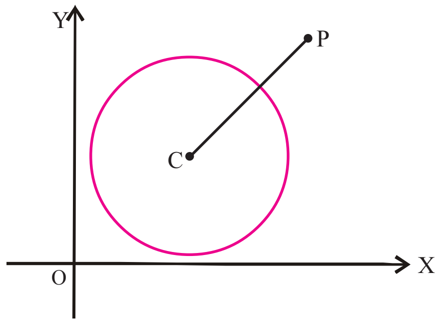


Fig. 1.22

1.2.4 Example

Let $S \equiv x^2+y^2+6x+8y-96=0$ be the equation of circle and P(1, 2) be a point in the plane. Here $(x_1, y_1)=(1, 2)$

$$S_{11} = 1^2+2^2+6(1)+8(2)-96=-69.$$

Since $S_{11} < 0$, by Theorem 1.2.3, the point $(1, 2)$ is in the interior of the circle. Note that the centre of the circle is $(-3, -4)$ and radius $r = 11$. The distance from the centre to the point $(1, 2)$ is $\sqrt{52}$ which is less than the radius 11. Hence, the point $(1, 2)$ is inside the circle.

1.2.5 Definition

Let P be any point on a given circle and Q be a neighbouring point of P lying on the circle. Join P and Q. Then $\overleftrightarrow{PQ}$ *is a secant (see Fig. 1.23(a)).*

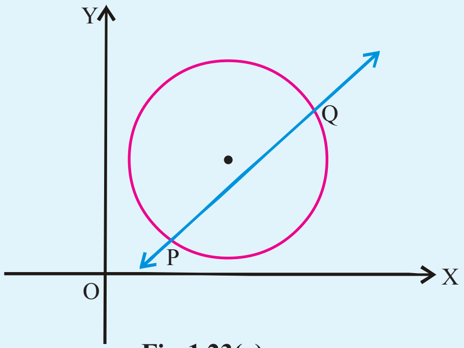


Fig. 1.23(a)

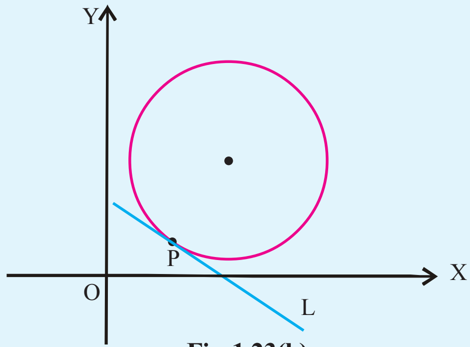


Fig. 1.23(b)

The limiting position of the line (secant) PQ when $Q \to P$ *along the circle, is called the tangent at P (see Fig. 1.23(b)).*

Explanation

Let the equation of $\overleftrightarrow{PQ}$ be $L_1 \equiv a_1x + b_1y + c_1 = 0$. Let Q_1 be another neighbouring point on the circle such that $PQ_1 < PQ$ (see Fig. 1.24). Let the equation of PQ_1 be $L_2 \equiv a_2x + b_2y + c_2 = 0$.

Similarly choose Q_2 on the circle such that $PQ_2 < PQ_1$. Let the equation of $\overleftrightarrow{PQ_2}$ be $L_3 \equiv a_3x + b_3y + c_3 = 0$. Let the limit of $L_1, L_2, L_3, \ldots,$ (straight line equations) be $L \equiv ax + by + c = 0$ as $Q \to P$ along the circle. Then L is called the tangent to the circle at P.

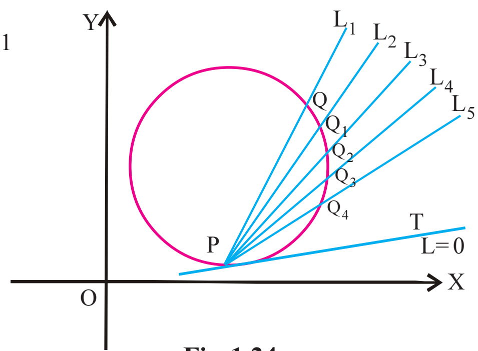


Fig. 1.24

1.2.6 Length of tangent

If P is an external point to the circle S = 0 and PT is the tangent from P to the circle S = 0 then $\overline{PT}$ *is called the length of the tangent from P to the circle (see Fig. 1.25)*

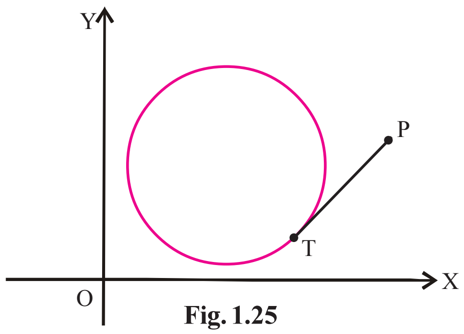


Fig. 1.25

1.2.7 Power of a Point

Suppose $S = 0$ is the equation of a circle with centre C and radius r. Let $P(x_1, y_1)$ be any point in the plane. Then $CP^2 - r^2$ is defined as the power of P with respect to $S = 0$ (see Fig. 1.26(a), (b), (c)).

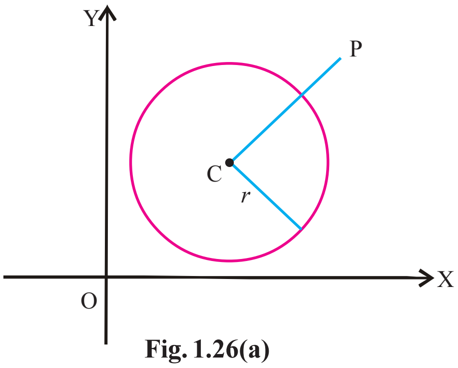


Fig. 1.26(a)

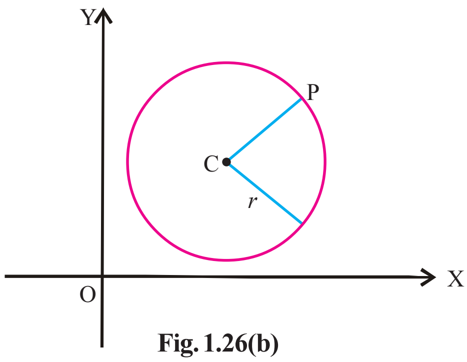


Fig. 1.26(b)

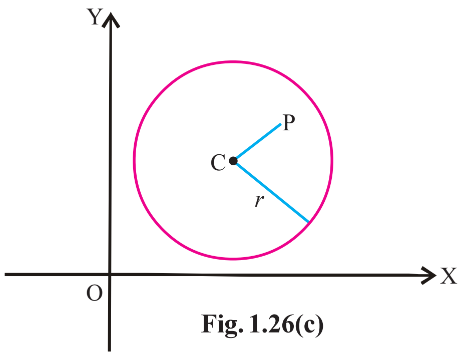


Fig. 1.26(c)

1.2.8 Note

A point $P(x_1, y_1)$ lies in the interior of the circle, on the circle or in the exterior of the circle according as the power of P with respect to the circle is negative, zero or positive respectively.

1.2.9 Theorem : *The power of a point $P(x_1, y_1)$ with respect to the circle $S = 0$ is S_{11}.*

Proof: As per the notation specified in 1.2.1, $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$.

The power of $P(x_1, y_1)$ is $CP^2 - r^2$ where C is the centre $(-g, -f)$ and r is the radius of the circle. Then

$$\begin{aligned} CP^2 - r^2 &= (x_1 + g)^2 + (y_1 + f)^2 - \left(\sqrt{g^2 + f^2 - c}\right)^2 \\ &= x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c \\ &= S_{11} \end{aligned}$$

Hence the power of $P(x_1, y_1)$ with respect to $S = 0$ is S_{11}.

1.2.10 Example

Let us find the power of $(1, 2)$ with respect to the circle $x^2 + y^2 + 6x + 8y - 96 = 0$.

Here $(x_1, y_1) = (1, 2)$. By Theorem 1.2.9 the power of $P(x_1, y_1)$ with respect to $S = 0$ is S_{11}.

$\therefore$ The power of $(1, 2)$ with respect to given circle is

$$1^2 + 2^2 + 6(1) + 8(2) - 96 = -69.$$

1.2.11 Theorem : *Let $S = 0$ be a circle and $P(x_1, y_1)$ be any point in the plane. If a line through P meets the circle at A and B then the power of P is equal to PA . PB.*

Proof : Let

$$S \equiv x^2 + y^2 + 2gx + 2fy + c = 0 \quad \text{... (1)}$$

be a circle. There will be infinitely many lines through the point P meeting the circle at two points (see Fig. 1.27). However the product PA. PB is the same, though the points A and B are different for different lines passing through P.

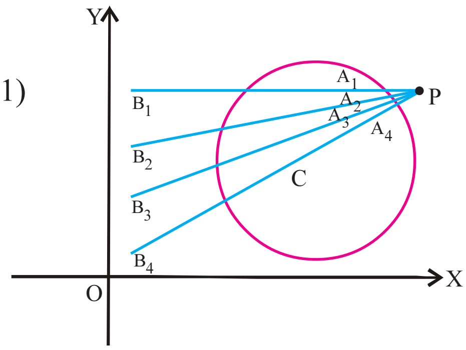


Fig. 1.27

Any point (x, y) on a straight line passing through (x_1, y_1) must satisfy the equations

$$x = x_1 + r\cos\theta$$

$$y = y_1 + r\sin\theta \quad \text{... (2)}$$

where r is the distance from (x_1, y_1) to (x, y) and θ is the angle made by the line with the positive X-axis.

To get the common points of the circle (1) and the line (2), we have to solve the equations (1) and (2). Therefore put $x = x_1 + r\cos\theta$, $y = y_1 + r\sin\theta$ in (1). Then

$$(x_1 + r\cos\theta)^2 + (y_1 + r\sin\theta)^2 + 2g(x_1 + r\cos\theta) + 2f(y_1 + r\sin\theta) + c = 0.$$

i.e., $x_1^2 + 2x_1 r\cos\theta + r^2\cos^2\theta + y_1^2 + 2ry_1\sin\theta + r^2\sin^2\theta$

$$+ 2gx_1 + 2gr\cos\theta + 2fy_1 + 2fr\sin\theta + c = 0$$

i.e., $r^2(\cos^2\theta + \sin^2\theta) + 2r[(x_1 + g)\cos\theta + (y_1 + f)\sin\theta] + x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0$

i.e., $r^2 + 2r[(x_1 + g)\cos\theta + (y_1 + f)\sin\theta] + S_{11} = 0$ ... (3)

Let r_1, r_2 be roots of (3). Then $r_1 . r_2 = S_{11}$

i.e., $PA . PB = S_{11}$

Since the power of P is S_{11}, we have

$$PA . PB = \text{Power of } P.$$

1.2.12 Corollary

If $S = 0$ is a circle and $P(x_1, y_1)$ is an exterior point with respect to $S = 0$ then the length of the tangent from $P(x_1, y_1)$ to $S = 0$ is $\sqrt{S_{11}}$ (see Fig. 1.28).

Proof : Let the tangent drawn from P touch the circle at A (Fig. 1.28). By Theorem 1.2.11, we have

$$PA . PA = S_{11}$$

$$PA^2 = S_{11}$$

$$\therefore \quad PA = \sqrt{S_{11}}.$$

i.e., the length of tangent from $P(x_1, y_1)$ to $S = 0$ is $\sqrt{S_{11}}$.

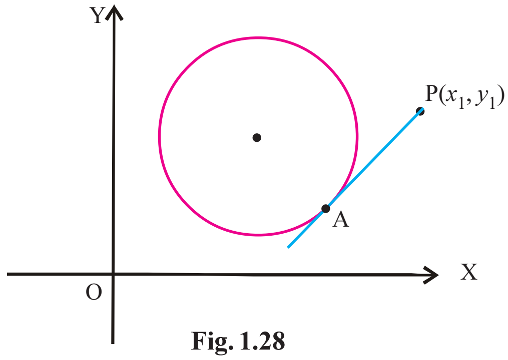


Fig. 1.28

1.2.13 Example

Let us find the length of tangent from (12, 17) to the circle $x^2 + y^2 - 6x - 8y - 25 = 0$.

By Corollary 1.2.12, the length of tangent from (12, 17) to the given circle is

$$\sqrt{(12)^2 + (17)^2 - 6(12) - 8(17) - 25} = \sqrt{100} = 10.$$

1.2.14 Solved Problems

1. Problem : *Locate the position of the point (2, 4) with respect to the circle*

$$x^2 + y^2 - 4x - 6y + 11 = 0.$$

Solution: Here $(x_1, y_1) = (2, 4)$ and $S \equiv x^2 + y^2 - 4x - 6y + 11 = 0$.

$$\therefore \quad S_{11} = (2)^2 + (4)^2 - 4(2) - 6(4) + 11 = -1.$$

Since $S_{11} < 0$, by Theorem 1.2.3, the point (2, 4) is inside the given circle.

2. Problem : *Find the length of the tangent from (1, 3) to the circle $x^2 + y^2 - 2x + 4y - 11 = 0$.*

Solution : Here $(x_1, y_1) = (1, 3)$ and $S \equiv x^2 + y^2 - 2x + 4y - 11 = 0$. By Corollary 1.2.12, the length of the tangent is $\sqrt{S_{11}}$.

Hence the required length of the tangent

$$= \sqrt{(1)^2 + (3)^2 - 2(1) + 4(3) - 11}.$$

$$= \sqrt{9}.$$

$$= 3.$$

3. Problem : *If a point P is moving such that the length of tangents drawn from P to*

$$x^2 + y^2 - 2x + 4y - 20 = 0 \quad \text{... (1)}$$

and $$x^2 + y^2 - 2x - 8y + 1 = 0 \quad \text{... (2)}$$

are in the ratio 2 : 1 then show that the equation of the locus of P is

$$x^2 + y^2 - 2x - 12y + 8 = 0.$$

Solution : Let $P(x_1, y_1)$ be any point on the locus and $\overline{PT}_1$, $\overline{PT}_2$ be the lengths of tangents from P to the circles (1) and (2) respectively. Then we have

$$\frac{\overline{PT}_1}{\overline{PT}_2} = \frac{2}{1}$$

i.e., $\sqrt{x_1^2 + y_1^2 - 2x_1 + 4y_1 - 20} = 2\sqrt{x_1^2 + y_1^2 - 2x_1 - 8y_1 + 1}$

i.e., $3(x_1^2 + y_1^2) - 6x_1 - 36y_1 + 24 = 0.$

The equation of the locus of P is

$$x^2 + y^2 - 2x - 12y + 8 = 0.$$

Exercise 1(b)

I. 1. Locate the position of the point P with respect to the circle S = 0 when

(i) P(3, 4) and $S \equiv x^2 + y^2 - 4x - 6y - 12 = 0$

(ii) P(1, 5) and $S \equiv x^2 + y^2 - 2x - 4y + 3 = 0$

(iii) P(4, 2) and $S \equiv 2x^2 + 2y^2 - 5x - 4y - 3 = 0$

(iv) P(2, −1) and $S \equiv x^2 + y^2 - 2x - 4y + 3 = 0$

2. Find the power of the point P with respect to the circle S = 0 when

(i) P = (5, −6) and $S \equiv x^2 + y^2 + 8x + 12y + 15.$

(ii) P = (−1, 1) and $S \equiv x^2 + y^2 - 6x + 4y - 12.$

(iii) P = (2, 3) and $S \equiv x^2 + y^2 - 2x + 8y - 23$

(iv) P = (2, 4) and $S \equiv x^2 + y^2 - 4x - 6y - 12$

3. Find the length of tangent from P to the circle S = 0 when

(i) P = (−2, 5) and $S \equiv x^2 + y^2 - 25$

(ii) P = (0, 0) and $S \equiv x^2 + y^2 - 14x + 2y + 25$

(iii) P = (2, 5) and $S \equiv x^2 + y^2 - 5x + 4y - 5$

II. 1. If the length of the tangent from (5, 4) to the circle $x^2 + y^2 + 2ky = 0$ is 1 then find k.

2. If the length of the tangent from (2, 5) to the circle $x^2 + y^2 - 5x + 4y + k = 0$ is $\sqrt{37}$ then find k.

III. 1. If a point P is moving such that the lengths of tangents drawn from P to the circles $x^2 + y^2 - 4x - 6y - 12 = 0$ and $x^2 + y^2 + 6x + 18y + 26 = 0$ are in the ratio 2 : 3 then find the equation of the locus of P.

2. If a point P is moving such that the lengths of tangents drawn from P to the circles $x^2 + y^2 + 8x + 12y + 15 = 0$ and $x^2 + y^2 - 4x - 6y - 12 = 0$ are equal then find the equation of the locus of P.

1.3 Position of a straight line in the plane of a circle Condition for a line to be tangent

In the earlier section we learnt the position of a point with respect to a circle. In this section we shall learn the position of a straight line in a plane with respect to a circle.

1.3.1 Different cases of position of a straight line with respect to a circle

Given a straight line L = 0 and a circle S = 0 we have three possibilities, namely :

(i) The line meets the circle in two distinct points (see Fig. 1.29).

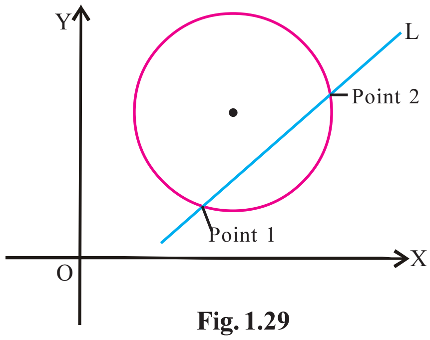


Fig. 1.29

(ii) The line meets the circle in one and only one point (i.e., touching the circle) (see Fig. 1.30).

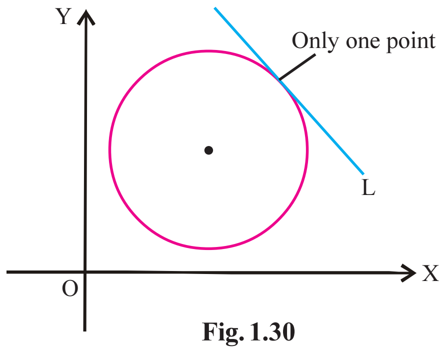


Fig. 1.30

(iii) The line L does not meet the circle i.e., L and the circle have no common points (see Fig. 1.31).

Now we examine under what conditions the above three situations arise.

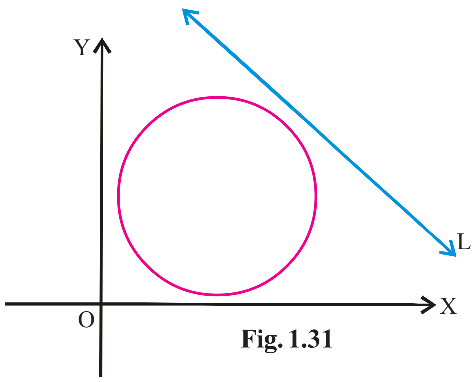


Fig. 1.31

1.3.2 Theorem : *A straight line $y = mx + c$*

(i) *meets the circle $x^2 + y^2 = r^2$ in two distinct points if* $\dfrac{c^2}{1+m^2} < r^2$.

(ii) *touches the circle $x^2 + y^2 = r^2$ if* $\dfrac{c^2}{1+m^2} = r^2$.

(iii) *has no points in common with the circle* $x^2 + y^2 = r^2$ *if* $\dfrac{c^2}{1+m^2} > r^2$.

Proof : The equation of the given circle is

$$x^2 + y^2 = r^2 \qquad \text{... (1)}$$

and the equation of the given straight line is

$$y = mx + c \quad (\text{i.e., } mx - y + c = 0) \qquad \text{... (2)}$$

If any point (x, y) is common to (1) and (2), the coordinates of the point satisfy both the equations (1) and (2). To solve them we eliminate y from (1) and (2). Substituting (2) in (1) we get

$$x^2 + (mx + c)^2 = r^2$$

i.e., $$x^2 (1 + m^2) + 2mcx + (c^2 - r^2) = 0 \qquad \text{... (3)}$$

The roots of (3) are real, coincident or imaginary according as

$$(2mc)^2 - 4(1 + m^2)(c^2 - r^2) \gtreqless 0$$

i.e., $$4m^2 c^2 - 4(c^2 + m^2c^2 - r^2 - r^2m^2)^2 \gtreqless 0$$

i.e., $$c^2 - r^2 (1 + m^2) \lesseqgtr 0$$

i.e., $$\frac{c^2}{(1+m^2)} \lesseqgtr r^2.$$

Case (i) : If $\dfrac{c^2}{(1+m^2)} < r^2$ then the straight line given by (2) meets the circle in two distinct points (see Fig. 1.29)

Case (ii) : If $\dfrac{c^2}{(1+m^2)} = r^2$, then the straight line given by (2) touches the circle (see Fig. 1.30)

Case (iii) : If $\dfrac{c^2}{(1+m^2)} > r^2$ then the straight line given by (2) does not cut or touch the circle (see Fig. 1.31). Hence they do not have common points.

1.3.3 Corollary

The condition that the straight line $y = mx + c$ *(i) intersects a circle, (ii) touches the circle, (iii) does not meet the circle is that the perpendicular distance from the centre of the circle to the line is less than or is equals to or greater than its radius respectively.*

Proof : By Theorem 1.3.2, the straight line $y = mx + c$ intersects or touches or does not meet the circle $x^2 + y^2 = r^2$ according as

$$\frac{c^2}{(1+m^2)} \lesseqgtr r^2$$

i.e., $$\frac{|c|}{\sqrt{1+m^2}} \lesseqgtr r. \qquad \text{... (1)}$$

The perpendicular distance from the centre (0, 0) to $y = mx + c$ is

$$\frac{|c|}{\sqrt{1+m^2}}. \qquad \text{... (2)}$$

From (1) and (2) the result follows.

1.3.4 Note

(i) *For all real values of m the straight line $y = mx \pm r\sqrt{1+m^2}$ is a tangent to the circle $x^2 + y^2 = r^2$ and the slope of the tangent is m.*

(ii) *A straight line $y = mx + c$ is a tangent to the circle $x^2 + y^2 = r^2$ if $c = \pm r\sqrt{1+m^2}$.*

(iii) *The equation of tangent to the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ having the slope m is $(y + f) = m(x + g) \pm r\sqrt{1+m^2}$ where r is the radius of the circle. For,*

the centre of the circle $C = (-g, -f)$ and $r = \sqrt{g^2 + f^2 - c}$. Shift the origin to $(-g, -f)$ without changing the direction of axes. Let the new axes be $\overleftrightarrow{CX'}$, $\overleftrightarrow{CY'}$.

If P is any point in the plane and (i) $P = (x, y)$ with respect to $\overleftrightarrow{OX}$, $\overleftrightarrow{OY}$

(ii) $P = (X, Y)$ with respect to $\overleftrightarrow{CX'}$, $\overleftrightarrow{CY'}$ (see 2.1.2 of Inter Mathematics - IB Text book) then

$$x = X - g \qquad \text{... (1)}$$

$$y = Y - f \qquad \text{... (2)}$$

The transformed equation of $S = 0$ is $X^2 + Y^2 = r^2$. By Note 1.3.4(i) the equation of tangent with slope m to the circle is

$$Y = mX \pm r\sqrt{1+m^2} \qquad \text{... (3)}$$

Equation (3) with respect to old axes $\overleftrightarrow{OX}$, $\overleftrightarrow{OY}$ is

$$\boxed{y + f = m(x + g) \pm r\sqrt{1+m^2}} \qquad \text{... (4)}$$

(from (1) and (2))

Thus the equation of tangent to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ having slope m is given by the equation (4).

1.3.5 Solved Problems

1. Problem: *If $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ represents a circle then show that the straight line $lx + my + n = 0$*

(i) *touches the circle $S = 0$ if $(g^2 + f^2 - c) = \dfrac{(gl + mf - n)^2}{(l^2 + m^2)}$*

(ii) *meets the circle $S = 0$ in two points if $g^2 + f^2 - c > \dfrac{(gl + mf - n)^2}{(l^2 + m^2)}$*

(iii) *will not meet the circle if $g^2 + f^2 - c < \dfrac{(gl + mf - n)^2}{(l^2 + m^2)}$.*

Solution : Let C be the centre and r be the radius of the circle S = 0. Then C = $(-g, -f)$ and $r = \sqrt{g^2 + f^2 - c}$

(i) The given straight line touches the circle if

$$r = \frac{|l(-g) + m(-f) - n|}{\sqrt{l^2 + m^2}} \quad \text{(by Corollary 1.3.3)}$$

i.e., $$\sqrt{g^2 + f^2 - c} = \frac{|-(lg + mf - n)|}{\sqrt{l^2 + m^2}}$$

Squaring both sides, we get

$$g^2 + f^2 - c = \frac{(lg + mf - n)^2}{(l^2 + m^2)}$$

(ii) The given line $lx + mx + n = 0$ meets the circle S = 0 in two points if

$$(g^2 + f^2 - c) > \frac{(gl + mf - n)^2}{l^2 + m^2} \quad \text{(by Corollary 1.3.3)}$$

(iii) The given line $lx + my + n = 0$ will not meet the circle S = 0 if

$$(g^2 + f^2 - c) < \frac{(gl + mf - n)^2}{l^2 + m^2} \quad \text{(by Corollary 1.3.3)}$$

2. Problem: *Find the length of the chord intercepted by the circle $x^2 + y^2 + 8x - 4y - 16 = 0$ on the line $3x - y + 4 = 0$.*

Solution: The centre of the given circle C = (–4, 2) and radius $r = \sqrt{16 + 4 + 16} = 6$. Let the perpendicular distance from the centre to the line $3x - y + 4 = 0$ be d. Then

$$d = \frac{|3(-4)-(2)+4|}{\sqrt{3^2+(-1)^2}}$$

$$= \frac{10}{\sqrt{10}} = \sqrt{10} \text{ (see Fig. 1.32)}$$

Length of the chord $= 2\sqrt{r^2 - d^2}$

$$= 2\sqrt{(6)^2 - \sqrt{10}^2}$$

$$= 2\sqrt{26}.$$

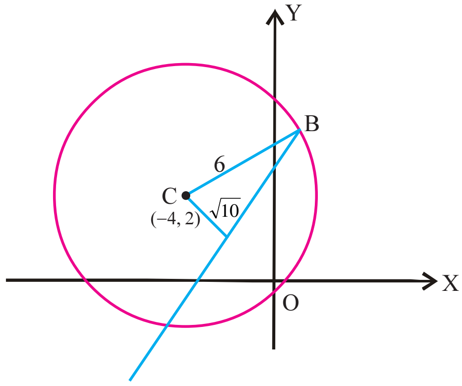


Fig. 1.32

3. Problem : *Find the equation of tangents to* $x^2 + y^2 - 4x + 6y - 12 = 0$ *which are parallel to* $x + 2y - 8 = 0$.

Solution: Here $g=-2$; $f=3$; $r = \sqrt{4+9+12} = 5$ and the slope of the required tangent is $-\frac{1}{2}$. By Note 1.3.4 (eqn. 4) the equations of tangents are

$$y+3 = -\frac{1}{2}(x-2) \pm 5\sqrt{1+\frac{1}{4}}$$

$$2(y+3) = -x+2 \pm 5\sqrt{5}$$

i.e., $x + 2y + (4 \pm 5\sqrt{5}) = 0.$

4. Problem : *Show that the circle* $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ *touches the*

(i) *X-axis if* $g^2 = c$.

(ii) *Y-axis if* $f^2 = c$.

Solution: We know that by Theorem 1.1.8 the intercept made by S = 0 on X-axis is $2\sqrt{g^2 - c}$. If the circle touches the X-axis then $2\sqrt{g^2 - c} = 0 \Rightarrow g^2 = c$. Similarly (ii) can be proved.

1.3.6 Chord joining two points on a circle.

In the next section we derive the equation of the chord joining two points on a circle.

1.3.7 Theorem : *If* $P(x_1, y_1)$ *and* $Q(x_2, y_2)$ *are two points on the circle* $S = 0$ *the equation of the secant* $\overleftrightarrow{PQ}$ *is* $S_1 + S_2 = S_{12}$.

Proof: Observe that the two points on the secant are known. Its equation can be found using two-point form of a straight line. This procedure can be adopted for analytical problems. The intention of this theorem is to find the equation of the chord involving g, f, c, x_1, x_2, y_1 and y_2 which will be used in finding the equation of tangent. The proof of the theorem runs as follows, Since $P(x_1, y_1)$ and $Q(x_2, y_2)$ are distinct, we may suppose that $x_1 \neq x_2$. Then the equation of $\overleftrightarrow{PQ}$ is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) \qquad \text{... (1)}$$

Since $P(x_1, y_1)$ and $Q(x_2, y_2)$ are lying on the circle $S = 0$ we have

$$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0 \text{ and } x_2^2 + y_2^2 + 2gx_2 + 2fy_2 + c = 0.$$

Subtracting and simplifying, we get

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{-(x_1 + x_2 + 2g)}{(y_1 + y_2 + 2f)} \qquad \text{... (2)}$$

Substituting (2) in (1), we obtain

$$y - y_1 = \frac{-(x_1 + x_2 + 2g)}{(y_1 + y_2 + 2f)}(x - x_1) \qquad \text{... (3)}$$

$$(x - x_1)(x_1 + x_2 + 2g) + (y - y_1)(y_1 + y_2 + 2f) = 0$$

$$xx_1 + yy_1 + xx_2 + yy_2 + 2gx + 2fy = x_1x_2 + y_1y_2 + x_1^2 + y_1^2 + 2gx_1 + 2fy_1$$

By adding $g(x_1 + x_2) + f(y_1 + y_2) + 2c$ on both sides to the above equation we obtain

$$S_1 + S_2 = S_{12} + S_{11}$$

$$S_1 + S_2 = S_{12} \qquad (\because S_{11} = 0)$$

The equation of secant $\overleftrightarrow{PQ}$ is $S_1 + S_2 = S_{12}$.

1.3.8 Equation of tangent at a point on the circle

In the next section we derive the equation of tangent at a point on the circle

1.3.9 Theorem : *The equation of the tangent at the point $P(x_1, y_1)$ to the circle*

$S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ is $S_1 = 0$.

Proof : Let $Q(x_2, y_2)$ be a neighbouring point of P and lying on the circle. By Theorem 1.3.7 the equation of $\overleftrightarrow{PQ}$ is $S_1 + S_2 = S_{12}$

i.e.,

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c + xx_2 + yy_2 + g(x + x_2) + f(y + y_2) + c$$

$$= x_1x_2 + y_1y_2 + g(x + x_2) + f(y + y_2) + c$$

As $Q \to P$ (i.e., $x_2 \to x_1, y_2 \to y_1$) this equation becomes

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c + xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c$$

$$= x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c$$

i.e., $S_1 + S_1 = S_{11}$

i.e., $2S_1 = S_{11}$

But $S_{11} = 0$, as $P(x_1, y_1)$ lying on $S = 0$. Hence $S_1 = 0$

i.e., $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$.

1.3.10 Note

The equation of the tangent at the point (x_1, y_1) to the circle $x^2 + y^2 = r^2$ is $xx_1 + yy_1 - r^2 = 0$.

1.3.11 Point of Contact

If a straight line $lx + my + n = 0$ touches the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ at P. (x_1, y_1) then this line is the tangent to the circle $S = 0$ at $P(x_1, y_1)$ and hence by Theorem 1.3.9 its equation is $(x_1 + g)x + (y_1 + f)y + (gx_1 + fy_1 + c) = 0$ and therefore $(x_1 + g)$, $(y_1 + f)$, $gx_1 + fy_1 + c$ are proportional to l, m, n respectively. Using these three proportions it is possible to find the point of contact of the given tangent to the circle $S = 0$.

1.3.12 Solved Problems

1. Problem: *Find the equation of the tangent to $x^2 + y^2 - 6x + 4y - 12 = 0$ at $(-1, 1)$.*

Solution : Here $(x_1, y_1) = (-1, 1)$ and $S \equiv x^2 + y^2 - 6x + 4y - 12 = 0$.

The equation of the tangent is $x(-1) + y(1) - 3(x - 1) + 2(y + 1) - 12 = 0$ (by Theorem 1.3.9)

i.e., $4x - 3y + 7 = 0$.

2. Problem: *Find the equation of the tangent to $x^2 + y^2 - 2x + 4y = 0$ at $(3, -1)$. Also find the equation of tangent parallel to it.*

Solution: Here $(x_1, y_1) = (3, -1)$ and

$$S \equiv x^2 + y^2 - 2x + 4y = 0 \qquad \text{... (1)}$$

$\therefore$ The equation of tangent at $(3, -1)$ is $x(3) + y(-1) - (x + 3) + 2(y - 1) = 0$

i.e., $3x - y - x - 3 + 2y - 2 = 0$

i.e., $$2x + y - 5 = 0 \qquad \text{... (2)}$$

Slope of the tangent is $m = -2$. For the circle (1), $g = -1$; $f = 2$;

radius $r = \sqrt{1 + 4 - 0} = \sqrt{5}$. By Note 1.3.4, the equations of tangents to (1) are

$$y + 2 = -2(x - 1) \pm \sqrt{5}\sqrt{1 + 4}$$

$$(y + 2) = -2(x - 1) \pm 5$$

$$2x + y \pm 5 = 0.$$

One of these equations namely $2x+y-5=0$ is the tangent at $(3,-1)$.

$\therefore$ The tangent parallel to $2x+y-5=0$ is $2x+y+5=0$.

3. Problem: *If $4x-3y+7=0$ is a tangent to the circle represented by $x^2+y^2-6x+4y-12=0$ then find its point of contact.*

Solution: Let (x_1, y_1) be the point of contact. Then by 1.3.11, we have

$$\frac{x_1-3}{4} = \frac{y_1+2}{-3} = \left(\frac{-3x_1+2y_1-12}{7}\right) \quad \text{... (1)}$$

From first and second equalities of (1), we get

$$3x_1+4y_1=1 \quad \text{... (2)}$$

Now by taking first and third equalities of (1), we get

$$19x_1-8y_1=-27 \quad \text{... (3)}$$

Solving (2) and (3) we obtain

$$x_1=-1; \quad y_1=1$$

Hence the point of contact is $(-1, 1)$.

4. Problem: *Find the equations of circles which touch $2x-3y+1=0$ at (1, 1) and having radius $\sqrt{13}$.*

Solution: The centres of the required circles lie on a line (see Fig. 1.33) which is perpendicular to $2x-3y+1=0$ and passing through $(1, 1)$

i.e., on $3x+2y-5=0$.

$\therefore$ The centres are situated on $3x+2y-5=0$ at a distance $\sqrt{13}$ from $(1, 1)$. Thus the centres are given by

$$\left(1+\sqrt{13}\left(\frac{-2}{\sqrt{13}}\right), 1+\sqrt{13}\cdot\frac{3}{\sqrt{13}}\right) \text{ and}$$

$$\left(1-\sqrt{13}\left(\frac{-2}{\sqrt{13}}\right), 1-\sqrt{13}\cdot\frac{3}{\sqrt{13}}\right)$$

(See Intermediate Mathematics - IB Text book)

i.e., $(-1, 4)$ and $(3, -2)$ (see Fig. 1.33)

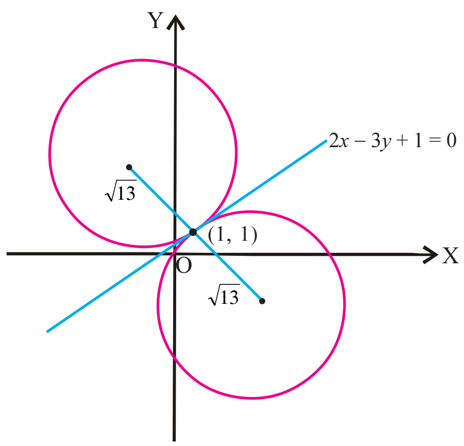


Fig. 1.33

The required circles are

$$(x+1)^2+(y-4)^2=13 \text{ and}$$

$$(x-3)^2+(y+2)^2=13$$

i.e., $x^2+y^2+2x-8y+4=0$ and $x^2+y^2-6x+4y=0$.

5. Problem: *Show that the line $5x + 12y - 4 = 0$ touches the circle $x^2 + y^2 - 6x + 4y + 12 = 0$.*

Solution: Here $g = -3;\ f = 2;\ c = 12$ and radius $= \sqrt{9+4-12} = 1$ ... (1)

The given straight line touches the circle if the perpendicular distance from the centre i.e., $(3, -2)$ to the given straight line equals to the radius of the given circle.

Let d be the perpendicular distance from the centre to the given straight line. Then

$$d = \frac{|5(3)+12(-2)-4|}{\sqrt{(5)^2+(12)^2}}$$

$$= 1 \qquad \text{... (2)}$$

From the facts (1) and (2) we can conclude that the given straight line touches the given circle (by Corollary 1.3.3).

1.3.13 Theorem : *If θ_1 [i.e., $(-g + r\cos\theta_1, -f + r\sin\theta_1)$ where r is the radius of the circle] and θ_2 [i.e., $(-g + r\cos\theta_2, -f + r\sin\theta_2)$] are two points on*

$$S \equiv x^2 + y^2 + 2gx + 2fy + c = 0 \qquad \text{... (1)}$$

then the equation of the chord joining these points is

$$(x+g)\cos\left(\frac{\theta_1+\theta_2}{2}\right) + (y+f)\sin\left(\frac{\theta_1+\theta_2}{2}\right) = r\cos\left(\frac{\theta_1-\theta_2}{2}\right) \qquad \text{... (2)}$$

Proof : Let A and B be the points on the circle (1) corresponding to θ_1 and θ_2 (these are parametric values of θ). Then

$$A = (-g + r\cos\theta_1, -f + r\sin\theta_1)$$

$$B = (-g + r\cos\theta_2, -f + r\sin\theta_2)$$

$\therefore$ The equation of the chord AB is

$$(y + f - r\sin\theta_1) = \frac{r(\sin\theta_2 - \sin\theta_1)}{r(\cos\theta_2 - \cos\theta_1)}(x + g - r\cos\theta_1)$$

Simplifying the above equation, we get

$$(x+g)\cos\left(\frac{\theta_1+\theta_2}{2}\right) + (y+f)\sin\left(\frac{\theta_1+\theta_2}{2}\right) = r\cos\left(\frac{\theta_1-\theta_2}{2}\right).$$

1.3.14 Note

(i) *The equation of the tangent at θ of the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ is given by $(x + g)\cos\theta + (y + f)\sin\theta = r$ where r is the radius of the circle $S = 0$.*

(ii) *For the circle* $x^2+y^2=r^2$*, the equation of the chord joining the points* θ_1 *and* θ_2 *(Particular values of the parameter* θ*) is* $x\cos\dfrac{\theta_1+\theta_2}{2}+y\sin\dfrac{\theta_1+\theta_2}{2}=r\cos\left(\dfrac{\theta_1-\theta_2}{2}\right)$.

(iii) *For the circle* $x^2+y^2=r^2$*, the equation of the tangent at* θ *is given by* $x\cos\theta+y\sin\theta=r$*.*

1.3.15 Solved Problems

1. Problem : *If the parametric values of two points A and B lying on the circle* $x^2+y^2-6x+4y-12=0$ *are* 30^o *and* 60^o *respectively then find the equation of the chord joining A and B.*

Solution: Here $g=-3$; $f=-2$; $r=\sqrt{9+4+12}=5$.

$\therefore$ The equation of the chord joining the points $\theta_1=30^o$, $\theta_2=60^o$ is (by Theorem 1.3.13)

$$(x-3)\cos\frac{60^0+30^0}{2}+(y+2)\sin\left(\frac{60^0+30^0}{2}\right)=5\cos\left(\frac{60^0-30^0}{2}\right)$$

i.e., $$(x-3)\cos 45^0+(y+2)\sin 45^0=5\cos 15^0$$

$$\frac{(x-3)+(y+2)}{\sqrt{2}}=5\times\frac{(\sqrt{3}+1)}{2\sqrt{2}}$$

i.e., $$2x+2y-(7+5\sqrt{3})=0.$$

2. Problem: *Find the equation of the tangent at the point* 30^o *(parametric value of* θ*) of the circle* $x^2+y^2+4x+6y-39=0$*.*

Solution : Here $g=2$; $f=3$; $c=-39$; $r=\sqrt{4+9+39}=\sqrt{52}=2\sqrt{13}$. By Note 1.3.14(i) the required equation is

$$(x+2)\cos 30^0+(y+3)\sin 30^0=2\sqrt{13}$$

i.e., $$(x+2)\frac{\sqrt{3}}{2}+(y+3)\left(\frac{1}{2}\right)=2\sqrt{13}$$

i.e., $$\sqrt{3}x+y+3+2(\sqrt{3}-2\sqrt{13})=0.$$

3. Problem: *Find the area of the triangle formed by the tangent at* $P(x_1, y_1)$ *to the circle*

$$x^2+y^2=a^2 \qquad \ldots(1)$$

with the coordinate axes where $x_1y_1\neq 0$*.*

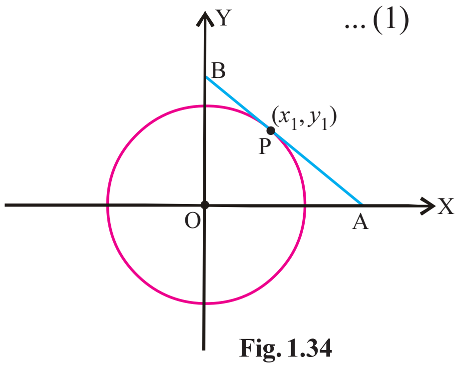


Fig. 1.34

Solution : The equation of the tangent at $P(x_1, y_1)$ to the circle (1) (see Fig. 1.34) is

$$xx_1+yy_1-a^2=0 \qquad \ldots(2)$$

Let this tangent cut the X-axis at A and Y-axis at B. We have to find the area of the triangle OAB.

The x, y intercepts of (2) are $\frac{a^2}{x_1}$ and $\frac{a^2}{y_1}$ respectively.

$\therefore$ Required area of the triangle

$$= \Delta AOB \text{ area}$$

$$= \frac{1}{2}\left|\frac{a^2}{x_1} \cdot \frac{a^2}{y_1}\right|$$

$$= \frac{a^4}{2\,|\,x_1 y_1\,|}.$$

1.3.16 Normal

The normal at any point P of the circle is the line which passes through P and is perpendicular to the tangent at P (see Fig. 1.35)

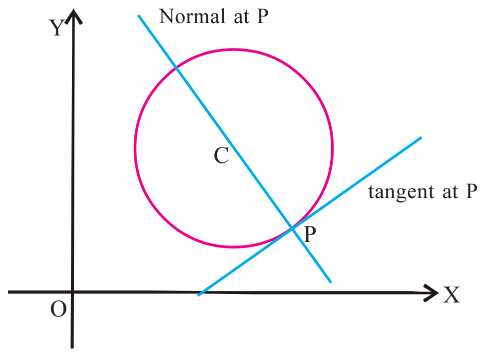


Fig. 1.35

1.3.17 Equation of Normal

We find the equation of normal at a point lying on the circle.

1.3.18 Theorem : *The equation of the normal at $P(x_1, y_1)$ of the circle*

$$S \equiv x^2 + y^2 \; 2gx + 2fy + c = 0 \qquad \ldots (1)$$

is $\quad (x - x_1)\;(y_1 + f) - (y - y_1)\;(x_1 + g) = 0.$

Proof : Let C be the centre of the circle given by (1). Then $C = (-g, -f)$. We know that normal at any point passes through the centre of the circle (see Fig. 1.35).

The slope of $\quad \overline{CP} = \dfrac{y_1 + f}{x_1 + g}$

Hence the equation of the normal at $P(x_1, y_1)$ is

$$(y - y_1) = \frac{(y_1 + f)}{(x_1 + g)}(x - x_1)$$

i.e., $\quad \boxed{(x - x_1)\,(y_1 + f) - (y - y_1)\,(x_1 + g) = 0}$

1.3.19 Note

The equation of the normal to the circle $x^2 + y^2 = r^2$ at $P(x_1, y_1)$ is $xy_1 - yx_1 = 0$.

1.3.20 Solved Problems

1. Problem : *Find the equation of the normal to the circle $x^2 + y^2 - 4x - 6y + 11 = 0$ at (3, 2). Also find the other point where the normal meets the circle.*

Solution: Let A(3, 2), C be the centre of given circle and the normal at A meet the circle at B(a, b). From the hypothesis, we have

$$2g = -4 \quad \text{i.e.,} \quad g = -2$$

$$2f = -6 \quad \text{i.e.,} \quad f = -3;$$

$$x_1 = 3 \quad \text{and} \quad y_1 = 2.$$

By Theorem 1.3.18, the equation of normal at A(3, 2) is

$$(x-3)(2-3) - (y-2)(3-2) = 0$$

i.e., $x + y - 5 = 0.$

The centre of the circle is the mid point of A and B (points of intersection of normal and circle)

$$\frac{a+3}{2} = 2 \Rightarrow a = 1$$

and $$\frac{b+2}{2} = 3 \Rightarrow b = 4.$$

Hence the normal at (3, 2) meets the circle at (1, 4).

2. Problem: *Find the area of the triangle formed by the normal at (3, −4) to the circle $x^2 + y^2 - 22x - 4y + 25 = 0$ with the coordinate axes.*

Solution: Here $2g = -22$; $2f = -4$ and $x_1 = 3$ and $y_1 = -4$.

By Theorem 1.3.18, the equation of normal is

$$(x-3)(-4-2) - (y+4)(3-11) = 0$$

i.e., $$3x - 4y - 25 = 0 \qquad \text{... (1)}$$

The straight line (1) cuts the x-axis at $\left(\frac{25}{3}, 0\right)$ and y-axis $\left(0, -\frac{25}{4}\right)$.

Hence the required area is

$$= \frac{1}{2}\left|\frac{25}{3} \times -\frac{25}{4}\right|$$

$$= \frac{625}{24}.$$

3. Problem: *Show that the line $lx + my + n = 0$ is a normal to the circle $S = 0$ if and only if $gl+mf=n$.*

Solution: The straight line $lx+my+n=0$ is normal to the circle

$$S = x^2 + y^2 + 2gx + 2fy + c = 0$$

$\Leftrightarrow$ if the centre $(-g, -f)$ of the circle lies on $lx+my+n=0$

$\Leftrightarrow$ $l(-g) + m(-f) + n = 0$

$\Leftrightarrow$ $lg + mf = n$.

Exercise 1(c)

I. 1. Find the equation of the tangent at P of the circle S = 0 where P and S are given by

(i) $P = (7, -5)$, $S \equiv x^2 + y^2 - 6x + 4y - 12$

(ii) $P = (-1, 1)$, $S \equiv x^2 + y^2 - 6x + 4y - 12$

(iii) $P = (-6, -9)$, $S \equiv x^2 + y^2 + 4x + 6y - 39$

(iv) $P = (3, 4)$, $S \equiv x^2 + y^2 - 4x - 6y + 11$.

2. Find the equation of the normal at P of the circle S = 0 where P and S are given by

(i) $P = (3, -4)$, $S \equiv x^2 + y^2 + x + y - 24$

(ii) $P = (3, 5)$, $S \equiv x^2 + y^2 - 10x - 2y + 6$

(iii) $P = (1, 3)$, $S \equiv 3(x^2 + y^2) - 19x - 29y + 76$

(iv) $P = (1, 2)$, $S \equiv x^2 + y^2 - 22x - 4y + 25$.

II. 1. Find the length of the chord intercepted by the circle $x^2 + y^2 - x + 3y - 22 = 0$ on the line $y = x - 3$.

2. Find the length of the chord intercepted by the circle $x^2 + y^2 - 8x - 2y - 8 = 0$ on the line $x + y + 1 = 0$.

3. Find the length of the chord formed by $x^2 + y^2 = a^2$ on the line $x\cos\alpha + y\sin\alpha = p$.

4. Find the equation of the circle with centre (2, 3) and touching the line $3x - 4y + 1 = 0$

5. Find the equation of the circle with centre (–3, 4) and touching y-axis.

6. Find the equation of tangents of the circle $x^2 + y^2 - 8x - 2y + 12 = 0$ at the points whose ordinates are 1.

7. Find the equation of tangents of the circle $x^2 + y^2 - 10 = 0$ at the points whose abscissae are 1.

III. 1. If $x^2 + y^2 = c^2$ and $\frac{x}{a} + \frac{y}{b} = 1$ intersect at A and B then find $\overline{AB}$. Hence deduce the condition that the line touches the circle.

2. The line $y = mx + c$ and the circle $x^2 + y^2 = a^2$ intersect at A and B. If $AB = 2\lambda$ then show that $c^2 = (1 + m^2)(a^2 - \lambda^2)$.

3. Find the equation of the circle with centre $(-2, 3)$ cutting a chord length 2 units on $3x + 4y + 4 = 0$.

4. Find the equation of tangent and normal at $(3, 2)$ of the circle $x^2 + y^2 - x - 3y - 4 = 0$.

5. Find the equation of tangent and normal at $(1, 1)$ to the circle $2x^2 + 2y^2 - 2x - 5y + 3 = 0$.

6. Prove that the tangent at $(3, -2)$ of the circle $x^2 + y^2 = 13$ touches the circle $x^2 + y^2 + 2x - 10y - 26 = 0$ and find its point of contact.

7. Show that the tangent at $(-1, 2)$ of the circle $x^2 + y^2 - 4x - 8y + 7 = 0$ touches the circle $x^2 + y^2 + 4x + 6y = 0$ and also find its point of contact.

8. Find the equations of the tangents to the circle $x^2 + y^2 - 4x + 6y - 12 = 0$ which are parallel to $x + y - 8 = 0$.

9. Find the equations of the tangents to the circle $x^2 + y^2 + 2x - 2y - 3 = 0$ which are perpendicular to $3x - y + 4 = 0$.

10. Find the equation of the tangents to the circle $x^2 + y^2 - 4x - 6y + 3 = 0$ which makes an angle 45^0 with X-axis.

11. Find the equation of the circle passing through $(-1, 0)$ and touching $x + y - 7 = 0$ at $(3, 4)$.

12. Find the equations of circles passing through $(1, -1)$, touching the lines $4x + 3y + 5 = 0$ and $3x - 4y - 10 = 0$.

13. Show that $x + y + 1 = 0$ touches the circle $x^2 + y^2 - 3x + 7y + 14 = 0$ and find its point of contact.

1.4 Chord of contact and polar

1.4.1 Theorem : *If* $P(x_1, y_1)$ *is an exterior point of the circle.*

$$S \equiv x^2 + y^2 + 2gx + 2fy + c = 0 \quad \text{...(1)}$$

then there exist two tangents from P *to the circle* $S = 0$ (see Fig. 1.36).

Proof: Let C be the centre and r be the radius of the circle (1). Then $C = (-g, -f)$ and $r = \sqrt{g^2 + f^2 - c}$. Let m be the slope of a tangent passing through (x_1, y_1). By Note 1.3.4 (iii) the equation of the tangent with slope m is

$$(y+f) = m(x+g) \pm r\sqrt{1+m^2}$$

If it passes through (x_1, y_1), we have

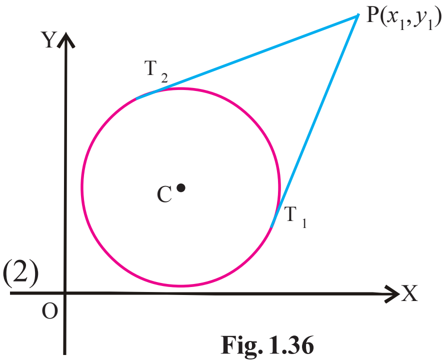


Fig. 1.36

$$(y_1+f) = m(x_1+g) \pm r\sqrt{1+m^2}$$

or $\quad [(y_1+f) - m(x_1+g)]^2 = r^2(1+m^2)$

or $\quad m^2[(x_1+g)^2 - r^2] - 2m(x_1+g)(y_1+f) + (y_1+f)^2 - r^2 = 0 \quad \ldots (2)$

The discriminant of (2) is

$$4(x_1+g)^2(y_1+f)^2 - 4[(x_1+g)^2 - r^2]\,[(y_1+f)^2 - r^2]$$

or $\quad 4r^2\,[x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c] \qquad (\because r^2 = g^2 + f^2 - c)$

or $\quad 4r^2\, S_{11}$

Since $P(x_1, y_1)$ is an exterior point to the circle $S = 0$, we have $S_{11} > 0$.

$\therefore$ The discriminant of (2) is positive and hence we have two real and distinct values of m say m_1 and m_2.

For these two values we have two tangents from $P(x_1, y_1)$ to the circle (1).

1.4.2 Note

(i) *If the discriminant of (2) is zero then the roots of equation (2) coincide and hence the tangents described above coincide. This situation arises when the point is on the circle.*

(ii) *When* $P(x_1, y_1)$ *is a point in the interior of the circle* $S = 0$ *then* $S_{11} < 0$ *and hence the discriminant of (2) is negative so that equation (2) has no real roots and hence there are no tangents passing through P to the circle.*

(iii) *If* θ *is the angle between the tangents through a point* $P(x_1, y_1)$ *to the circle* $S = 0$ *then* $\tan\left(\frac{\theta}{2}\right) = \frac{r}{\sqrt{S_{11}}}$ *where r is the radius of the circle.*

For if $\overleftrightarrow{PT}$ and $\overleftrightarrow{PT'}$ are two tangents to the circle $S = 0$ through P (which is an exterior point with respect to the circle $S = 0$) then the triangles ΔPTC, $\Delta PT'C$ are identical (see Fig. 1.37)

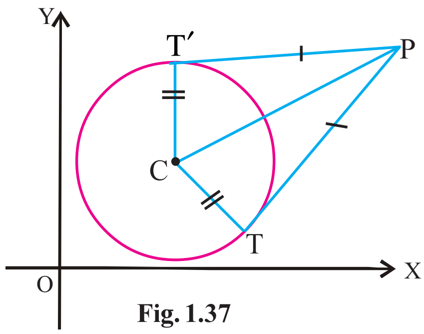


Fig. 1.37

$$\therefore \; T\hat{P}C = T'\hat{P}C = \theta/2$$

$$\therefore \; \tan\left(\frac{\theta}{2}\right) = \frac{TC}{PT} = \frac{r}{\sqrt{S_{11}}}$$

(iv) The area of the triangle ΔPTC as shown in the Fig. 1.37 is $\frac{1}{2} \cdot r\sqrt{S_{11}}$.

1.4.3 Solved Problems

1. Problem : *Find the condition that the tangents drawn from the exterior point (g,f) to $S \equiv x^2+y^2+2gx+2fy+c=0$ are perpendicular to each other.*

Solution : If the angle between the tangents drawn from $P(x_1,y_1)$ to $S=0$ is θ then

$$\tan\left(\frac{\theta}{2}\right) = \frac{r}{\sqrt{S_{11}}} \qquad \text{(by Note 1.4.2 (iii))}$$

where r is the radius of the circle $S=0$.

Here $(x_1,y_1) = (g,f)$

$\therefore\ S_{11} = g^2+f^2+2g^2+2f^2+c$

i.e., $S_{11} = 3g^2+3f^2+c$

$$\theta = 90^0 \Leftrightarrow \tan\left(\frac{90}{2}\right) = \frac{\sqrt{g^2+f^2-c}}{\sqrt{3g^2+3f^2+c}}$$

$$\Leftrightarrow 3g^2+3f^2+c = g^2+f^2-c$$

$$\Leftrightarrow 2g^2+2f^2+2c=0$$

$$\Leftrightarrow g^2+f^2+c=0.$$

Thus the tangents drawn from (g,f) to the circle $S=0$ are perpendicular if and only if $g^2+f^2+c=0$. In this case note that $c<0$.

2. Problem : *If θ_1, θ_2 are the angles of inclination of tangents through a point P to the circle $x^2+y^2=a^2$ then find the locus of P when $\cot\theta_1+\cot\theta_2=k$.*

Solution: The equation of the tangent to $x^2+y^2=a^2$ having the slope m is

$$y = mx \pm a\sqrt{1+m^2} \qquad \text{... (1)}$$

Let $P(x_1,y_1)$ be a point on the locus. If the tangents (1) passes through P then

$$y_1 = mx_1 \pm a\sqrt{1+m^2}$$

or $$y_1 - mx_1 = \pm a\sqrt{1+m^2}$$

or $$(y_1-mx_1)^2 = a^2(1+m^2)$$

or $$m^2(x_1^2-a^2) - 2mx_1y_1 + y_1^2 - a^2 = 0.$$

If m_1, m_2 are the roots of the above equation then

$$m_1+m_2 = \tan\theta_1+\tan\theta_2 = \frac{2x_1y_1}{x_1^2-a^2} \qquad \text{... (2)}$$

and $$m_1\,m_2 = \tan\theta_1.\tan\theta_2 = \frac{y_1^2-a^2}{x_1^2-a^2} \qquad \text{... (3)}$$

Given that $\cot\theta_1 + \cot\theta_2 = k$

$$\therefore \quad \frac{\tan\theta_2 + \tan\theta_1}{\tan\theta_1 \tan\theta_2} = k$$

$$\frac{2x_1y_1}{y_1^2 - a^2} = k \qquad \text{(from (2) and (3))}$$

or $$k(y_1^2 - a^2) = 2x_1y_1 \qquad \text{... (4)}$$

Also, conversely if $P(x_1, y_1)$ satisfies the equation (4) then it can be shown that $\cot\theta_1 + \cot\theta_2 = k$, thus the locus of P is $k(y^2 - a^2) = 2xy$.

1.4.4 Chord of Contact

If the tangents drawn through $P(x_1, y_1)$ to a circle $S = 0$ touch the circle at points A and B then the secant $\overleftrightarrow{AB}$ is called the chord of contact of P with respect to $S = 0$ (see Fig. 1.38).

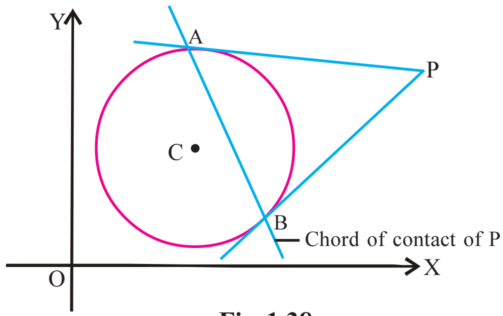


Fig. 1.38

1.4.5 Theorem : *If* $P(x_1, y_1)$ *is an exterior point to the circle* $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ *then the equation of the chord of contact of* P *with respect to* $S = 0$ *is* $S_1 = 0$.

Proof: Let the tangents drawn from $P(x_1, y_1)$ to the circle $S = 0$ touch at $A(x_2, y_2)$ and $B(x_3, y_3)$.

The equation of tangent at $A(x_2, y_2)$ is $xx_2 + yy_2 + g(x + x_2) + f(y + y_2) + c = 0$.

Similarly the equation of tangent at $B(x_3, y_3)$ is $xx_3 + yy_3 + g(x + x_3) + f(y + y_3) + c = 0$.

These two tangents are passing through $P(x_1, y_1)$

$$\therefore \quad x_1x_2 + y_1y_2 + g(x_1 + x_2) + f(y_1 + y_2) + c = 0 \qquad \text{... (1)}$$

and $$x_1x_3 + y_1y_3 + g(x_1 + x_3) + f(y_1 + y_3) + c = 0 \qquad \text{... (2)}$$

Thus the two points $A(x_2, y_2)$ and $B(x_3, y_3)$ satisfy the following linear equation in x and y

$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0. \qquad \text{... (3)}$$

Note that equation (3) can be written as

$$x(x_1 + g) + y(y_1 + f) + (gx_1 + fy_1 + c) = 0. \qquad \text{... (4)}$$

Clearly equation (4) represents a straight line. Equations (2) and (3) show that the points $A(x_2, y_2)$ and $B(x_3, y_3)$ are satisfying equation (4) (hence equation (3)).

$\therefore$ The equation of the chord of contact $\overleftrightarrow{AB}$ is given by (3) i.e., $S_1 = 0$.

1.4.6 Note

(i) *If the point* $P(x_1, y_1)$ *is on the circle* $S = 0$ *then the tangent itself can be defined as the chord of contact.*

(ii) *If the point* $P(x_1, y_1)$ *is an interior point of the circle* $S = 0$ *then the chord of contact does not exist.*

1.4.7 Solved Problems

1. **Problem :** Find the chord of contact of (2, 5) with respect to the circle $x^2 + y^2 - 5x + 4y - 2 = 0$.

Solution : Here $(x_1, y_1) = (2, 5)$. By Theorem 1.4.5 the required chord of contact is

$$xx_1 + yy_1 - \frac{5}{2}(x + x_1) + 2(y + y_1) - 2 = 0.$$

Substituting x_1 and y_1 values, we get

$$x(2) + y(5) - \frac{5}{2}(x + 2) + 2(y + 5) - 2 = 0$$

i.e., $x - 14y - 6 = 0$.

2. **Problem :** *If the chord of contact of a point* P *with respect to the circle*

$$x^2 + y^2 = a^2 \quad \dots (1)$$

cut the circle at A and B such that $A\hat{O}B = 90^0$ then show that P lies on the circle

$$x^2 + y^2 = 2a^2.$$

Solution : Let $P(x_1, y_1)$ be a point and let the chord of contact of it cut the circle in A and B such that $A\hat{O}B = 90^0$. The equation of the chord of contact of $P(x_1, y_1)$ with respect to (1) is

$$xx_1 + yy_1 - a^2 = 0 \quad \dots (2)$$

The equation to the pair of lines $\overleftrightarrow{OA}$ and $\overleftrightarrow{OB}$ is given by

$$x^2 + y^2 - a^2\left(\frac{xx_1 + yy_1}{a^2}\right)^2 = 0$$

or $$a^2(x^2 + y^2) - (xx_1 + yy_1)^2 = 0$$

or $$x^2(a^2 - x_1^2) - 2x_1y_1xy + y^2(a^2 - y_1^2) = 0 \quad \dots (3)$$

Since $A\hat{O}B = 90^0$, we have the coefficient of x^2 in (3) + coefficient of y^2 in (3) = 0

$$\therefore\ a^2 - x_1^2 + a^2 - y_1^2 = 0$$

i.e., $$x_1^2 + y_1^2 = 2a^2$$

Hence the point $P(x_1, y_1)$ lies on the circle $x^2 + y^2 = 2a^2$.

1.4.8 Pole and polar

Let $S = 0$ *be a circle and* P *be any point in the plane other than the centre of* $S = 0$*. If any line drawn through the point* P *meets the circle in two points* Q *and* R*, then the locus of the points of intersection of tangents drawn at* Q *and* R *is a line, called polar of* P *and* P *is called pole of the polar* (*see Fig.* 1.39).

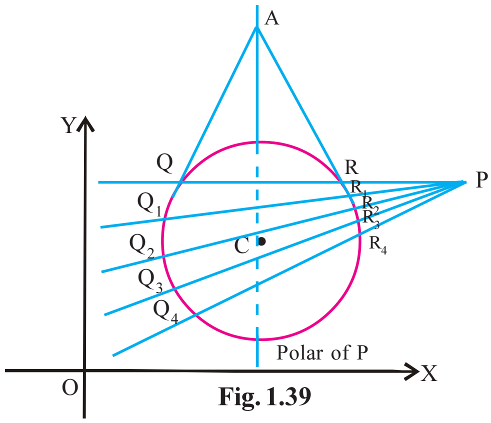


Fig. 1.39

1.4.9 Theorem : *The equation of the polar of* $P(x_1, y_1)$ *with respect to* $S = 0$ *is* $S_1 = 0$.

Proof: Let QR be any chord drawn through $P(x_1, y_1)$ and let the tangents at Q and R meet at the point $A(\alpha, \beta)$. Then $\overleftrightarrow{QR}$ is a chord of contact of $A(\alpha, \beta)$.

$\therefore$ The equation of $\overleftrightarrow{QR}$ is

$$x\alpha + y\beta + g(x+\alpha) + f(y+\beta) + c = 0.$$

It is passes through $P(x_1, y_1)$, therefore

$$x_1\alpha + y_1\beta + g(x_1+\alpha) + f(y_1+\beta) + c = 0$$

$\therefore$ $A(\alpha, \beta)$ satisfies $S_1 = 0$.

$\therefore$ The equation of polar of $P(x_1, y_1)$ is $S_1 = 0$.

1.4.10 Note

(i) *Suppose a point* $A(\alpha, \beta)$ *which lies outside or on the circle satisfies the equation*

$$S_1 \equiv xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0.$$

Then $x_1\alpha + y_1\beta + g(x_1 + \alpha) + f(y_1 + \beta) + c = 0$ *so that the point* $P(x_1, y_1)$ *lies on the straight line* $\alpha x + \beta y + g(\alpha + x) + f(\beta + y) + c = 0$ *which is the chord of contact of* $A(\alpha, \beta)$ *of the circle* $S = 0$. *i.e.,* $A(\alpha, \beta)$ *is a point on the polar of the point* $P(x_1, y_1)$.

Thus every point $A(\alpha, \beta)$ *on the line* $S_1 = 0$ *which is outside or on the circle is the intersection of the tangents at the points of intersection of a line viz.*

$$\alpha x + \beta y + g(x + \alpha) + f(\beta + y) + c = 0$$

through $P(x_1, y_1)$ *and the circle.*

(ii) *If* $P(x_1, y_1)$ *is an exterior point of the circle then the chord of contact of* P *will be the polar of* P (see Fig. 1.40).

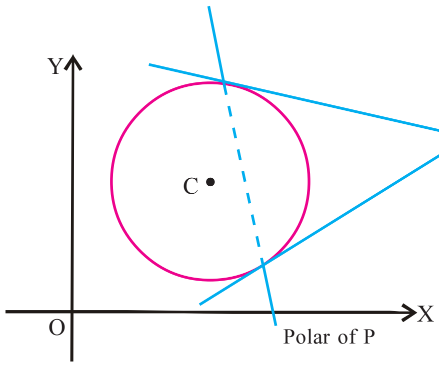


Fig. 1.40

(iii) *If* $P(x_1, y_1)$ *lies on the circle then the polar of* P *coincides with the tangent at* $P(x_1, y_1)$ (see Fig.1.41).

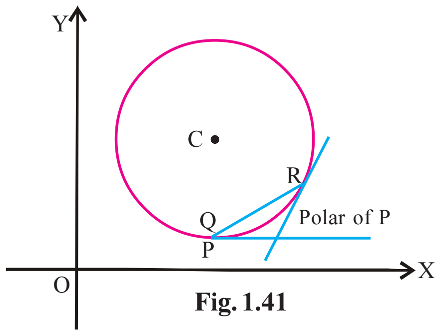


Fig. 1.41

(iv) *If* $P(x_1, y_1)$ *is inside the circle then the polar of* P *does not intersect the circle*. (see Fig. 1.42).

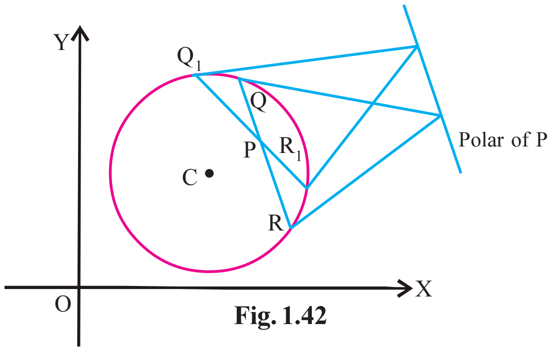


Fig. 1.42

(v) If C is the centre of the circle then the polar of P has slope $-\dfrac{(x_1+g)}{(y_1+f)}$ and hence it is perpendicular to CP (whose slope is $\dfrac{(y_1+f)}{(x_1+g)}$).

(vi) *If* P *is the centre of a circle* $S=0$ *then the polar of* P *does not exist i.e., the polar of* $P(-g, -f)$ *of the circle* $S = x^2+y^2+2gx+2fy+c=0$ *does not exist.*

(vii)

$P(x_1, y_1)$	Tangent at P	Chord of contact at P	Polar of P
Interior of the circle	Does not exist	Does not exist	$S_1=0$ P is different from centre
on the circle	$S_1=0$	$S_1=0$	$S_1=0$
Exterior of the circle	Does not exist	$S_1=0$	$S_1=0$

1.4.11 Theorem : *The pole of* $lx+my+n=0$ $(n \neq 0)$ *with respect to the circle* $x^2+y^2=a^2$ *is*

$$\left(-\frac{a^2 l}{n}, -\frac{a^2 m}{n}\right).$$

Proof: Let $P(x_1, y_1)$ be the pole of

$$lx+my+n=0 \qquad \dots (1)$$

with respect to the circle

$$x^2 + y^2 = a^2 \qquad \text{... (2)}$$

By Theorem 1.4.9, the polar of P with respect to the circle (2) is

$$xx_1 + yy_1 - a^2 = 0 \qquad \text{... (3)}$$

The equations (1) and (3) represent the same straight line

$$\therefore \quad \frac{x_1}{l} = \frac{y_1}{m} = -\frac{a^2}{n}$$

Hence $\quad x_1 = \dfrac{-a^2 l}{n}, \quad y_1 = \dfrac{-a^2 m}{n}.$

Therefore the pole of $lx + my + n = 0$ with respect to the circle (2) is $\left(\dfrac{-a^2 l}{n}, \ \dfrac{-a^2 m}{n}\right)$.

1.4.12 Note

The pole of $lx + my + n = 0$ with respect to $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ is

$$\left(-g + \frac{lr^2}{lg + mf - n}, \ -f + \frac{mr^2}{lg + mf - n}\right)$$

where r is the radius of the circle if $lg + mf - n \neq 0$.

1.4.13 Solved Problems

1. Problem : *Find the equation of the polar of* (2, 3) *with respect to the circle* $x^2 + y^2 + 6x + 8y - 96 = 0$.

Solution : Here $(x_1, y_1) = (2, 3)$ and $S \equiv x^2 + y^2 + 6x + 8y - 96 = 0$. Hence by Theorem 1.4.9, the equation of polar is $x(2) + y(3) + 3(x + 2) + 4(y + 3) - 96 = 0$ i.e., $5x + 7y - 78 = 0$.

2. Problem : *Find the pole of* $x + y + 2 = 0$ *with respect to the circle* $x^2 + y^2 - 4x + 6y - 12 = 0$.

Solution: Here $lx + my + n = 0$ is $x + y + 2 = 0$ and $S = 0$ is $S \equiv x^2 + y^2 - 4x + 6y - 12 = 0$. Therefore, $l = 1$; $m = 1$; $n = 2$; $g = -2$; $f = 3$; and radius of the circle $r = \sqrt{4 + 9 + 12} = 5$. By Note 1.4.12, the pole of $lx + my + n = 0$ with respect to $S = 0$ is

$$\left(-g + \frac{lr^2}{lg + mf - n}, \ -f + \frac{mr^2}{lg + mf - n}\right).$$

$\therefore$ The pole of $x + y + 2 = 0$ with respect to the circle $x^2 + y^2 - 4x + 6y - 12 = 0$ is

$$\left(2 + \frac{(1)\,(5)^2}{(1)\,(-2) + (1)(3) - 2}, \ -3 + \frac{(1)\,(5)^2}{(1)\,(-2) + (1)\,(3) - 2}\right)$$

i.e., $\quad (2 - 25, \ -3 - 25)$

i.e., $\quad (-23, \ -28)$

$\therefore$ The pole of $x + y + 2 = 0$ with respect to the given circle is $(-23, -28)$.

3. Problem : *Show that the poles of the tangents to the circle*

$$x^2 + y^2 = a^2 \quad \text{... (1)}$$

with respect to the circle

$$(x + a)^2 + y^2 = 2a^2 \quad \text{... (2)}$$

lie on $y^2 + 4ax = 0.$

Solution : Let $P(x_1, y_1)$ be the pole of the tangent to the circle (1) with respect to the circle (2). Then the polar of P with respect to the circle given by (2) is a tangent to the circle given by (1). Now, the polar of P with respect to (2) is

$$xx_1 + yy_1 + a(x + x_1) - a^2 = 0$$

i.e., $$x(x_1 + a) + yy_1 + (ax_1 - a^2) = 0. \quad \text{... (3)}$$

This line is a tangent to circle (1)

$$\therefore \quad a = \frac{\left|0 + 0 + ax_1 - a^2\right|}{\sqrt{(x_1 + a)^2 + y_1^2}}$$

i.e., $$y_1^2 + 4ax_1 = 0.$$

$\therefore$ The poles of the tangents to circle (1) with respect to (2) lie on the curve

$$y^2 + 4ax = 0.$$

1.4.14 Theorem : *The polar of $P(x_1, y_1)$ with respect to the circle $S = 0$ passes through $Q(x_2, y_2)$ $\Leftrightarrow$ the polar of Q passes through P.*

Proof : Suppose that the polar of $P(x_1, y_1)$ with respect to the circle

$$S \equiv x^2 + y^2 + 2gx + 2fy + c = 0 \quad \text{... (1)}$$

passes through $Q(x_2, y_2)$.

We shall prove that the polar of $Q(x_2, y_2)$ passes through P.

The polar of P with respect to (1) is $S_1 = 0$. If it passes through $Q(x_2, y_2)$ then

$$S_{12} = 0 \quad \text{... (2)}$$

Now the polar of $Q(x_2, y_2)$ with respect to the circle (1) is

$$S_2 = 0. \quad \text{... (3)}$$

It passes through P if $S_{12} = 0$. In view of (2) the condition $S_{12} = 0$ is satisfied. Hence the polar of P passes through Q. Then the polar of Q with respect to the circle S = 0 passes through P. Similarly the converse part can be proved.

1.4.15 Conjugate points

Two points P and Q are said to be conjugate points with respect to the circle S = 0 if Q lies on the polar of P (observe that if Q lies on the polar of P then P lies on the polar of Q).

1.4.16 Note

The condition that the two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ are conjugate points with respect to the circle $S = 0$ is $S_{12} = 0$.

1.4.17 Conjugate lines

If P *and* Q *are conjugate points with respect to the circle* $S = 0$ *then the polars of* P *and* Q *are called conjugate lines with respect to the circle* $S = 0$.

1.4.18 Theorem : *Let* $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ ... (1)

be a circle with radius r and $l_1 x + m_1 y + n_1 = 0,$ ... (2)

$l_2 x + m_2 y + n_2 = 0$ ... (3)

be two straight lines. Then the following statements are equivalent.

(i) $l_1 x + m_1 y + n_1 = 0,\ l_2 x + m_2 y + n_2 = 0$ *are conjugate lines with respect to the circle (1)*

(ii) $r^2 (l_1 l_2 + m_1 m_2) = (l_1 g + m_1 f - n_1)(l_2 g + m_2 f - n_2)$

Proof : (i) $\Rightarrow$ (ii)

Suppose that the lines given by (2) and (3) are conjugate. Then the pole of (2) i.e.,

$$\left(-g + \frac{l_1 r^2}{l_1 g + m_1 f - n_1},\ -f + \frac{m_1 r^2}{l_1 g + m_1 f - n_1}\right)$$

lies on (3). Here r is the radius of the circle (1). Since (2) and (3) are conjugate, this point lies on (3).

$$\therefore\ l_2 \left\{-g + \frac{l_1 r^2}{(l_1 g + m_1 f - n_1)}\right\} + m_2 \left\{-f + \frac{m_1 r^2}{(l_1 g + m_1 f - n_1)}\right\} + n_2 = 0$$

$$\text{i.e. } (-l_2 g - m_2 f + n_2) + \frac{(l_1 l_2 + m_1 m_2)\, r^2}{(l_1 g + m_1 f - n_1)} = 0$$

i.e., $r^2 (l_1 l_2 + m_1 m_2) = (l_1 g + m_1 f - n_1)(l_2 g + m_2 f - n_2)$.

Thus (i) $\Rightarrow$ (ii) is proved. Now we prove (ii) $\Rightarrow$ (i)

Suppose $r^2 (l_1 l_2 + m_1 m_2) = (l_1 g + m_1 f - n_1)(l_2 g + m_2 f - n_2)$

$$\Rightarrow \frac{r^2\ (l_1 l_2 + m_1 m_2)}{l_1 g + m_1 f - n_1} = l_2 g + m_2 f - n_2$$

$$\Rightarrow\ l_2 \left\{-g + \frac{l_1\ r^2}{(l_1 g + m_1 f - n_1)}\right\} + m_2 \left\{-f + \frac{m_1\ r^2}{(l_1 g + m_1 f - n_1)}\right\} + n_2 = 0$$

$\Rightarrow$ The pole of (2) lies on (3)

$\Rightarrow$ The lines given by (2) and (3) are conjugate.

1.4.19 Note

Two lines $l_1x+m_1y+n_1=0$ and $l_2x+m_2y+n_2=0$ are conjugate with respect to the circle $x^2+y^2=a^2$ if and only if $a^2(l_1l_2+m_1m_2)=n_1n_2$.

1.4.20 Solved Problems

1. Problem : *Show that* $(4,-2)$ *and* $(3,-6)$ *are conjugate with respect to the circle* $x^2+y^2-24=0$.

Solution : Here $(x_1,y_1)=(4,-2)$ and $(x_2,y_2)=(3,-6)$ and $S\equiv x^2+y^2-24=0$. Two points (x_1,y_1) and (x_2,y_2) are conjugate with respect to the circle $S=0$ if $S_{12}=0$. In this case

$$x_1x_2+y_1y_2-24=0.$$

For the given points $\quad S_{12}=(4)(3)+(-2)(-6)-24=0.$

$\therefore$ The given points are conjugate with respect to the given circle.

2. Problem : *If* $(4,k)$ *and* $(2,3)$ *are conjugate points with respect to the circle* $x^2+y^2=17$ *then find* k.

Solution : Here $(x_1,y_1)=(4,k),(x_2,y_2)=(2,3)$ and $S\equiv x^2+y^2-17=0$. Since the given points are conjugate, we have $S_{12}=0$.

i.e., $\quad x_1x_2+y_1y_2-17=0$

i.e., $\quad (4)(2)+(k)(3)-17=0$

$\Rightarrow \quad k=3.$

3. Problem : *Show that the lines* $2x+3y+11=0$ *and* $2x-2y-1=0$ *are conjugate with respect to the circle* $x^2+y^2+4x+6y-12=0$.

Solution: Here $l_1=2, m_1=3, n_1=11; l_2=2, m_2=-2, n_2=-1$ and $g=2, f=3, c=12$. Further the radius of the circle $r=\sqrt{4+9-12}=1$. By Theorem 1.4.18, we have $l_1x+m_1y+n_1=0$, $l_2x+m_2y+n_2=0$ are conjugate with respect to $S=0$ if

$$r^2(l_1l_2+m_1m_2)=(l_1g+m_1f-n_1)(l_2g+m_2f-n_2) \qquad \text{... (1)}$$

L.H.S. of $(1)=(1)^2\ [(2)(2)+(3)(-2)]=-2$

R.H.S. of $(1)=(4+9-11)\ (4-6+1)=-2$

$\therefore$ Condition (1) is satisfied by the given lines with respect to the given circle. Hence they are conjugate lines.

4. Problem : *Show that the area of the triangle formed by the two tangents through* $P(x_1, y_1)$ *to the circle.* $S\equiv x^2+y^2+2gx+2fy+c=0$ *and the chord of contact of* P *with respect to* $S=0$ *is* $\dfrac{r(S_{11})^{3/2}}{S_{11}+r^2}$ *where* r *is the radius of the circle.*

Solution : Let PA and PB be two tangents through P to the circle S = 0 and θ be the angle between these two tangents. We know that

$$\tan\left(\frac{\theta}{2}\right) = \frac{r}{\sqrt{S_{11}}} \quad \text{(By Note 1.4.2(iii))}$$

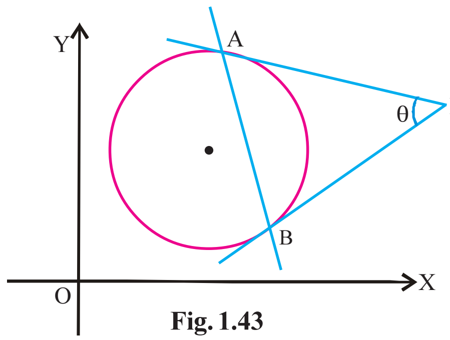


Fig. 1.43

Required area (see Fig. 1.43)

$$= \Delta APB \text{ area}$$

$$= \frac{1}{2} PA.PB. \sin\theta$$

$$= \frac{1}{2}\sqrt{S_{11}}\sqrt{S_{11}} . \frac{2\tan(\frac{\theta}{2})}{1+\tan^2(\frac{\theta}{2})}$$

$$= \frac{S_{11}\left(\frac{r}{\sqrt{S_{11}}}\right)}{1+\frac{r^2}{S_{11}}}$$

$$= \frac{r(S_{11})^{3/2}}{S_{11}+r^2}.$$

1.4.21 Definition

Let C *be the centre and r be the radius of the circle* S = 0. *Two points* P *and* Q *are said to be inverse points with respect to the circle* S = 0 *if* C, P, Q *are collinear such that* P, Q *are on the same side of* C *and* CP. CQ = r^2.

1.4.22 Theorem : *Let* C *be the centre and r be the radius of the circle* $S \equiv x^2 + y^2 - r^2 = 0$. *Two points P and Q are inverse points if and only if Q is the point of intersection of the polar of P with respect to S = 0 and the line joining P and C.*

Proof: Suppose that $P(x_1, y_1)$ and $Q(x_2, y_2)$ are the inverse points. Then

(i) CP. CQ = r^2

(ii) C, P, Q are collinear

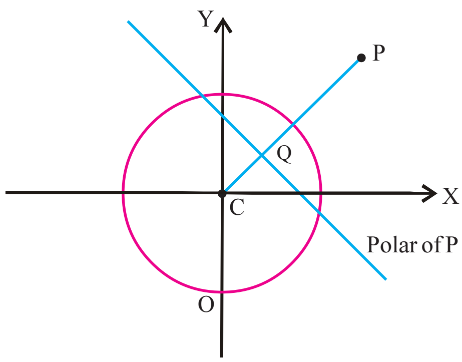


Fig. 1.44

From (i), we get

$$(x_1^2 + y_1^2)(x_2^2 + y_2^2) = r^4 \quad \ldots (1)$$

From (ii), we get ΔCPQ area = 0

$$\text{i.e.,} \quad x_1y_2 - x_2y_1 = 0 \quad \ldots (2)$$

Equation (1) is equivalent to

$$x_1^2x_2^2 + y_1^2y_2^2 + x_1^2y_2^2 + x_2^2y_1^2 = r^4$$

$$x_1^2x_2^2 + y_1^2y_2^2 + (x_1y_2 - x_2y_1)^2 + 2x_1x_2y_1y_2 = r^4$$

$$(x_1x_2 + y_1y_2)^2 + 0 = r^4 \qquad \text{(by (2))}$$

$$x_1x_2 + y_1y_2 = \pm r^2$$

Since P, Q lie on the same side of C, we get $x_1x_2 > 0, y_1y_2 > 0$.

$\therefore \quad x_1x_2 + y_1y_2 = r^2$.

Thus P, Q are conjugate points.

$\therefore$ Q lies on the polar of P.

Thus Q is the intersection of CP and the polar of P.

Conversely suppose Q is the intersection of the polar of P and CP. We shall prove that CP.CQ $= r^2$

Equation of polar of P is $xx_1 + yy_1 - r^2 = 0$

$$\text{CQ} = \frac{|0+0-r^2|}{\sqrt{x_1^2+y_1^2}} = \frac{r^2}{\sqrt{x_1^2+y_1^2}}$$

Thus $\text{CP}\,.\,\text{CQ} = \sqrt{x_1^2+y_1^2}\,.\,\dfrac{r^2}{\sqrt{x_1^2+y_1^2}} = r^2$.

Hence P and Q are inverse points.

1.4.23 Note

The inverse of the point P with respect to the circle S = 0 is the foot of the perpendicular from the centre of the circle S = 0 to the polar of P.

1.4.24 Example

Let us find the inverse point of (–2, 3) with respect to the circle $x^2 + y^2 - 4x - 6y + 9 = 0$.

Let P = (–2, 3) and C be the centre of the circle. Then C = (2, 3). The polar of P is

$$x(-2) + y(3) - 2(x-2) - 3(y+3) + 9 = 0$$

i.e., $x = 1$.

The equation of the line $\overleftrightarrow{CP}$ is

$$y - 3 = \frac{3-3}{2+1}(x+2)$$

i.e., $y = 3$.

From (1) and (2), we get the common point of $x = 1$ and $y = 3$ as (1, 3).

$\therefore$ The inverse point of (–2, 3) is (1, 3).

1.4.25 Equation of chord with given middle point

Now, we derive the equation of chord when the middle point of it is known.

1.4.26 Theorem : *If* $P(x_1, y_1)$ *is the mid - point of a chord* AB (*other than the diameter*) *of the circle* $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ *then the equation of secant* $\overleftrightarrow{AB}$ is $S_1 = S_{11}$.

Proof : Let C be the centre of the circle $S = 0$. Then $C = (-g, -f) \neq (x_1, y_1)$. We know that $\overline{AB}$ is perpendicular to $\overline{CP}$ (see Fig. 1.45). We may suppose that $y_1 \neq -f$.

Slope of AB

$$= -\frac{1}{\text{Slope of CP}}$$

$$= -\frac{(x_1 + g)}{(y_1 + f)}$$

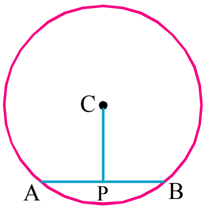


Fig. 1.45

Thus the equation of $\overleftrightarrow{AB}$ is given by

$$y - y_1 = -\frac{(x_1 + g)}{(y_1 + f)}(x - x_1)$$

i.e., $(y - y_1)(y_1 + f) + (x - x_1)(x_1 + g) = 0$

i.e., $xx_1 + yy_1 + gx + fy$

$$= x_1^2 + y_1^2 + gx_1 + fy_1$$

Adding $gx_1 + fy_1 + c$ on both sides to the above equation, we obtain

$$S_1 = S_{11}.$$

Note that if $y_1 = -f$ then the equation of secant is $x = x_1$.

1.4.27 Solved Problems

1. Problem : *Find the mid point of the chord intercepted by*

$$x^2 + y^2 - 2x - 10y + 1 = 0 \quad \dots (1)$$

on the line $x - 2y + 7 = 0$. ... (2)

Solution : Let $P(x_1, y_1)$ be the mid-point of the chord intercepted by the circle (1) on the line given by (2).

The equation of secant along the chord is $S_1 = S_{11}$

i.e., $xx_1 + yy_1 - (x + x_1) - 5(y + y_1) + 1 = x_1^2 + y_1^2 - 2x_1 - 10y_1 + 1$

i.e., $x(x_1 - 1) + y(y_1 - 5) - (x_1^2 + y_1^2 - x_1 - 5y_1) = 0$ ... (3)

Equations (2) and (3) represent the same chord

$$\therefore \frac{x_1 - 1}{1} = \frac{y_1 - 5}{-2} = \frac{-(x_1^2 + y_1^2 - x_1 - 5y_1)}{7} = \text{K(say)}$$

$x_1 = K + 1$ ... (4)

$$y_1 = -2K + 5 \qquad \text{... (5)}$$

$$x_1^2 + y_1^2 - x_1^2 - 5y_1 = -7K \qquad \text{... (6)}$$

Substituting (4) and (5) in (6) we get

$$(K+1)^2 + (-2K+5)^2 - (K+1)^2 - 5(-2K+5) = -7K$$

i.e., $5K^2 - 2K = 0$

$\therefore \quad K = 0 \text{ or } K = \frac{2}{5}$.

For $K = 0$, the point $(x_1, y_1) = (1, 5)$ which is not a point on the chord $x - 2y + 7 = 0$. Hence $K = 0$ rejected.

For $K = \frac{2}{5}$, the point $(x_1, y_1) = \left(\frac{7}{5}, \frac{21}{5}\right)$

$\therefore$ Mid point of the chord is $\left(\frac{7}{5}, \frac{21}{5}\right)$.

Other Method

Let C be the centre of the circle. Then C(1, 5). Let $P(x_1, y_1)$ be mid point of the chord intersected by (2) on the circle (1). Then (x_1, y_1) is the foot of the perpendicular of C to the chord given by (2).

We have (by a result proved in Intermediate Mathematics - IB Text Book) that

$$\frac{x_1 - 1}{1} = \frac{y_1 - 5}{-2} = \frac{-(1-10+7)}{(1+4)}$$

i.e.,
$$\frac{x_1 - 1}{1} = \frac{y_1 - 5}{-2} = \frac{2}{5}$$

$$\therefore \quad x_1 = \frac{7}{5}, \quad y_1 = \frac{21}{5}$$

Thus $\left(\frac{7}{5}, \frac{21}{5}\right)$ is the mid point of the given chord.

2. Problem : *Find the locus of mid-points of the chords of contact of $x^2 + y^2 = a^2$ from the points lying on the line $lx + my + n = 0$.*

Solution : Let $P(x_1, y_1)$ be a point on the locus. Then the point P is the mid-point of a chord of the circle

$$x^2 + y^2 = a^2 \qquad \text{... (1)}$$

and this chord is chord of contact of a point lying on

$$lx + my + n = 0 \qquad \text{... (2)}$$

i.e., the pole of this chord is on the line given by (2). The equation of the chord of the circle (1) having $P(x_1, y_1)$ as its midpoint is

$$xx_1 + yy_1 = x_1^2 + y_1^2$$

i.e., $$xx_1 + yy_1 - (x_1^2 + y_1^2) = 0 \quad \text{... (3)}$$

The pole of (3) with respect to the circle (1) is

$$\left(\frac{-a^2 x_1}{-(x_1^2 + y_1^2)}, \frac{-a^2 y_1}{-(x_1^2 + y_1^2)}\right) \quad \text{(by Theorem 1.4.11)}$$

i.e., $$\left(\frac{a^2 x_1}{x_1^2 + y_1^2}, \frac{a^2 y_1}{x_1^2 + y_1^2}\right)$$

This point lies on the line given by (2)

$$\therefore \; l\frac{a^2 x_1}{x_1^2 + y_1^2} + m\frac{a^2 y_1}{x_1^2 + y_1^2} + n = 0$$

$$\therefore \; a^2(lx_1 + my_1) + n(x_1^2 + y_1^2) = 0$$

Hence the locus of P is $a^2(lx + my) + n(x^2 + y^2) = 0$.

Exercise 1(d)

I. 1. Find the condition that the tangents drawn from (0, 0) to $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ be perpendicular to each other.

2. Find the chord of contact of (0, 5) with respect to the circle $x^2 + y^2 - 5x + 4y - 2 = 0$.
3. Find the chord of contact of (1, 1) to the circle $x^2 + y^2 = 9$.
4. Find the polar of (1, 2) with respect to $x^2 + y^2 = 7$.
5. Find the polar of (3, –1) with respect to $2x^2 + 2y^2 = 11$.
6. Find the polar of (1, –2) with respect to $x^2 + y^2 - 10x - 10y + 25 = 0$.
7. Find the pole of $ax + by + c = 0$ $(c \neq 0)$ with respect to $x^2 + y^2 = r^2$.
8. Find the pole of $3x + 4y - 45 = 0$ with respect to $x^2 + y^2 - 6x - 8y + 5 = 0$
9. Find the pole of $x - 2y + 22 = 0$ with respect to $x^2 + y^2 - 5x + 8y + 6 = 0$.
10. Show that the points (–6, 1) and (2, 3) are conjugate points with respect to the circle $x^2 + y^2 - 2x + 2y + 1 = 0$.
11. Show that the points (4, 2) and (3, –5) are conjugate points with respect to the circle $x^2 + y^2 - 3x - 5y + 1 = 0$.

12. Find the value of k if $kx+3y-1=0$, $2x+y+5=0$ are conjugate lines with respect to the circle $x^2+y^2-2x-4y-4=0$.
13. Find the value of k if $x+y-5=0$ and $2x+ky-8=0$ are conjugate with respect to the circle $x^2+y^2-2x-2y-1=0$.
14. Find the value of k if the points $(1,3)$ and $(2,k)$ are conjugate with respect to the circle $x^2+y^2=35$.
15. Find the value of k if the points $(4,2)$ and $(k,-3)$ are conjugate points with respect to the circle $x^2+y^2-5x+8y+6=0$

II. 1. Find the acute angle between the tangents drawn from $(3,2)$ to the circle $x^2+y^2-6x+4y-2=0$.

2. Find the acute angle between the pair of tangents drawn from $(1,3)$ to the circle
$$x^2+y^2-2x+4y-11=0.$$
3. Find the acute angle between the pair of tangents drawn from $(0,0)$ to the circle $x^2+y^2-14x+2y+25=0$.
4. Find the locus of P if the tangents drawn from P to $x^2+y^2=a^2$ include an angle α.
5. Find the locus of P if the tangents drawn from P to $x^2+y^2=a^2$ are perpendicular to each other.
6. Find the slope of the polar of $(1,3)$ with respect to the circle $x^2+y^2-4x-4y-4=0$. Also find the distance from the centre to it.
7. If $ax+by+c=0$ is the polar of $(1,1)$ with respect to the circle $x^2+y^2+4x+2y+1=0$ and H.C.F. of a, b, c is equal to one then find $a^2+b^2+c^2$.

III. 1. Find the coordinates of the point of intersection of tangents at the points where $x+4y-14=0$ meets the circle $x^2+y^2-2x+3y-5=0$.

2. If the polar of the points on the circle $x^2+y^2=a^2$ with respect to the circle $x^2+y^2=b^2$ touches the circle $x^2+y^2=c^2$ then prove that a, b, c are in Geometrical progression.
3. Tangents are drawn to the circle $x^2+y^2=16$ from the point $P(3,5)$. Find the area of the triangle formed by these tangents and the chord of contact of P.
4. Find the locus of the point whose polars with respect to the circles $x^2+y^2-4x-4y-8=0$ and $x^2+y^2-2x+6y-2=0$ are mutually perpendicular.
5. Find the locus of the foot of the perpendicular drawn from the origin to any chord of the circle $S \equiv x^2+y^2+2gx+2fy+c=0$ which subtends a right angle at the origin.

1.5 Relative Positions of two circles

The number of common tangents that can be drawn to two given circles depend on their relative positions. We shall describe the various possible relative positions of two circles. First, let us recall that any two intersecting common tangents of two circles and the line joining the centres of the circles are concurrent, equivalently the point of intersection of two common tangents (if exists) of two circles and the centres of these two circles are collinear. In this section we learn the different possible relative position of two circles and the number of common tangents exists in each case.

1.5.1 Definition

A straight line L *is said to be a common tangent to the circles* $S=0$ *and* $S'=0$ *if it is tangent to both* $S=0$ *and* $S'=0$ (see Fig. 1.46).

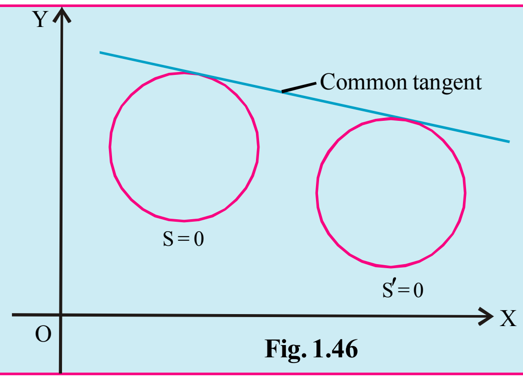


Fig. 1.46

1.5.2 Definition

Two circles are said to be touching each other if they have only one common point (see Fig. 1.47(a), 1.47(b))

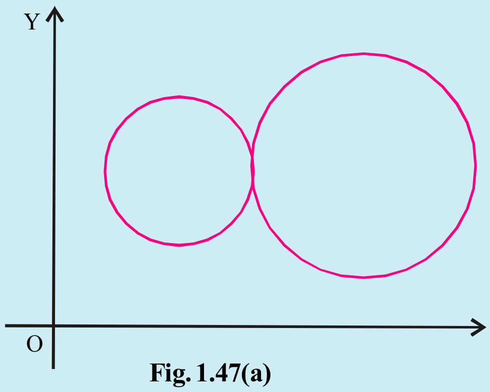


Fig. 1.47(a)

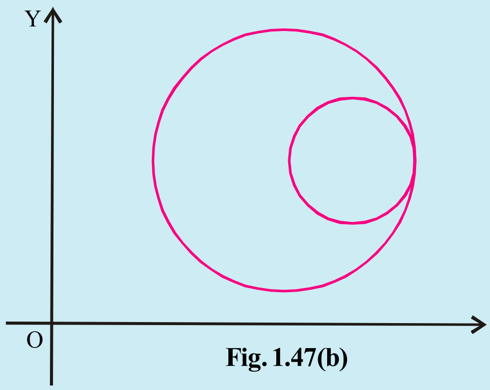


Fig. 1.47(b)

1.5.3 Relative positions of two circles

Let C_1, C_2 be the centres and r_1, r_2 be the radii of two circles $S=0$ and $S'=0$ respectively. Further let $\overline{C_1C_2}$ represents the line segment from C_1 to C_2. The following cases arise with regard to the relative position of two circles.

(i) $C_1C_2 > r_1 + r_2$

In this case the two circles will be apart i.e., one will be away from the other (see Fig. 1.48).

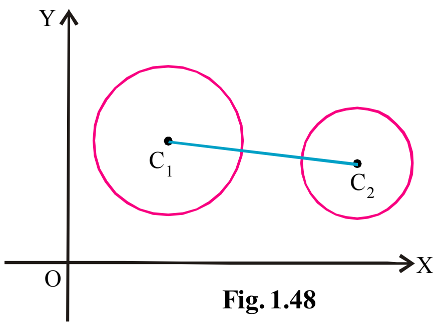


Fig. 1.48

(ii) $C_1C_2 = r_1 + r_2$

In this case the two circles touch each other externally (see Fig. 1.49).

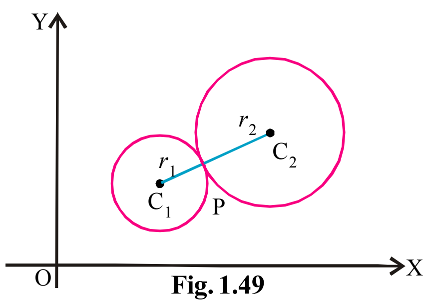


Fig. 1.49

(iii) $|r_1 - r_2| < C_1C_2 < r_1 + r_2$

In this case the two circles intersect in two distinct points (see Fig.1.50).

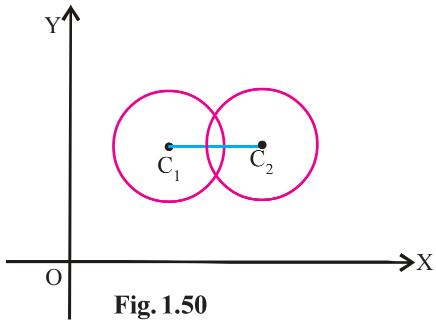


Fig. 1.50

(iv) $C_1C_2 = |r_1 - r_2|$

The two circles touch each other internally (see Fig.1.51) in this case.

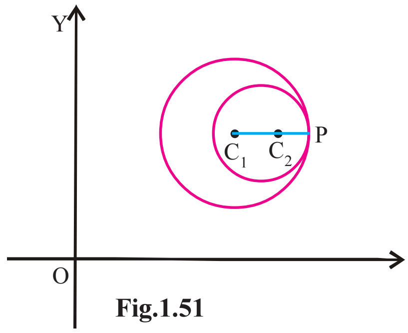


Fig.1.51

(v) $C_1C_2 < |r_1 - r_2|$

In this case the two circles do not intersect / touch and one circle will be completely inside the other (see Fig. 1.52).

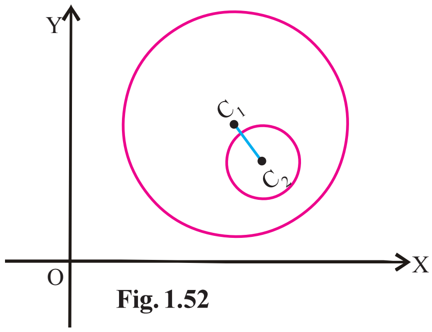


Fig. 1.52

1.5.4 Note

If $C_1C_2 = 0$ then the centres of the two circles coincide and they are concentric circles (see Fig. 1.53).

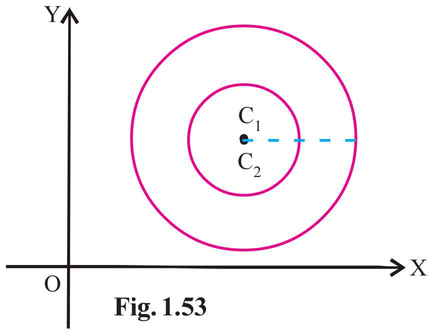


Fig. 1.53

Before the discussion on the number of common tangents in the above cases, we give a proof of a useful result. In the next two sections the figures are drawn without drawing the axes for convenience.

1.5.5 A useful result

Let (i) C_1 and C_2 be the centres of two circles (ii) r_1 and r_2 are radii of these circles (iii) one pair of common tangents meet $\overline{C_1C_2}$ in P and other pair meet in Q (see Fig. 1.54). Let these common tangents meet the circles at $T_1, T_2, T_3, T_4, T_5, T_6, T_7$ and T_8 as shown in Fig. 1.54. Then ΔPC_1T_1 and ΔPC_2T_2 are similar triangles.

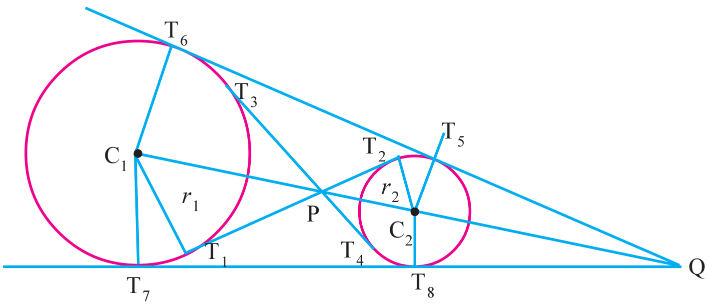


Fig. 1.54

$$\therefore \quad \frac{C_1T_1}{C_2T_2} = \frac{C_1P}{C_2P}$$

$$\frac{r_1}{r_2} = \frac{C_1P}{C_2P}$$

Similarly ΔQC_1T_7, ΔQC_2T_8 are similar,

$$\therefore \quad \frac{C_1T_7}{C_2T_8} = \frac{C_1Q}{C_2Q}.$$

$$\frac{r_1}{r_2} = \frac{C_1Q}{C_2Q}.$$

$\therefore$ The points P and Q divide $C_1\,C_2$ in the ratio of the radii (i.e., $r_1 : r_2$).

1.5.6 Common tangents, Centre of similitude

Now we discuss the number of common tangents that exist for the cases specified in the section 1.5.3

Case (i) : Each of the given pair of circles lies in the exterior of the other i.e., $C_1C_2 > r_1 + r_2$.

Subcase (i) : $r_1 \neq r_2$ (r_1, r_2 are radii of the circles)

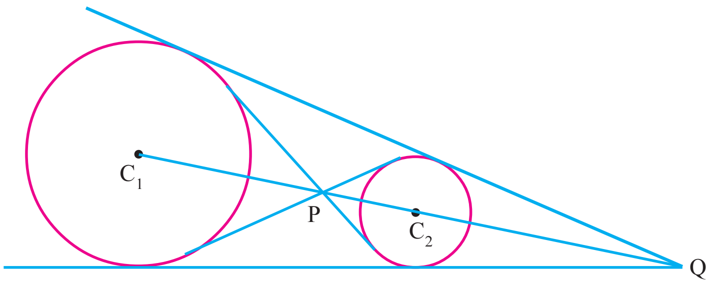


Fig. 1.55

In this case, there is a possibility of having two pairs of common tangents. The pair of common tangents intersecting at a point on the line segment $\overline{C_1C_2}$ is called transverse pair of common tangents and the pair of common tangents intersecting at a point not in $\overline{C_1C_2}$ (see Fig. 1.55) is called as direct pair of common tangents. The points P, Q are collinear with the centres C_1 and C_2 of given circles. The point of intersection of transverse pair of common tangents P is called the internal centre of similitude and the point of intersection of direct pair of common tangents Q is called external centre of similitude. Note that P divides C_1C_2 in the ratio $r_1 : r_2$ internally and Q divides $\overline{C_1C_2}$ in the same ratio externally. Also note that $C_1C_2 > r_1 + r_2$. In this case the number of distinct common tangents is 4.

subcase (ii) : $\overline{C_1C_2} > r_1 + r_2$ and $r_1 = r_2$. In this case the direct common tangents are parallel and the external centre of similitude doesn't exist. (see Fig. 1.56).

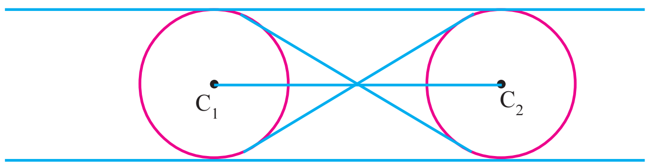


Fig. 1.56

To find the equations of parallel common tangents, suppose the tangent equation as $y = mx + c$. The slope $m =$ slope of C_1C_2. From this fact the value of m is known.

$$r_1 = \frac{|\, m(-g_1) - f_1 + c \,|}{\sqrt{1+m^2}} \quad \text{(radius is equal to perpendicular distance)}$$

Using the above equation we can find c. In this case the number of common tangents is 4.

Case (ii) : $\overline{C_1C_2} = r_1 + r_2$

Given circles touch each other externally (see Fig. 1.57).

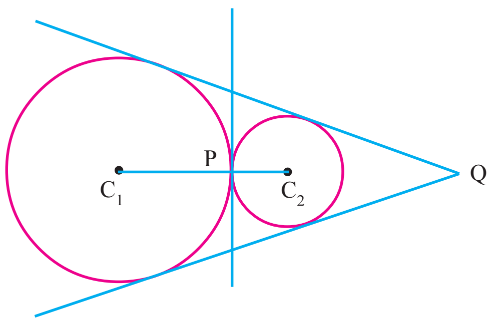


Fig. 1.57

In this case the internal centre of similitude P is the point of contact of two given circles. At P there is only one common tangent. Through Q, there will be two common tangents. In this case the number of common tangents is 3.

Case (iii) : $|r_1 - r_2| < \overline{C_1C_2} < r_1 + r_2$

(i.e., Given circles intersecting each other)

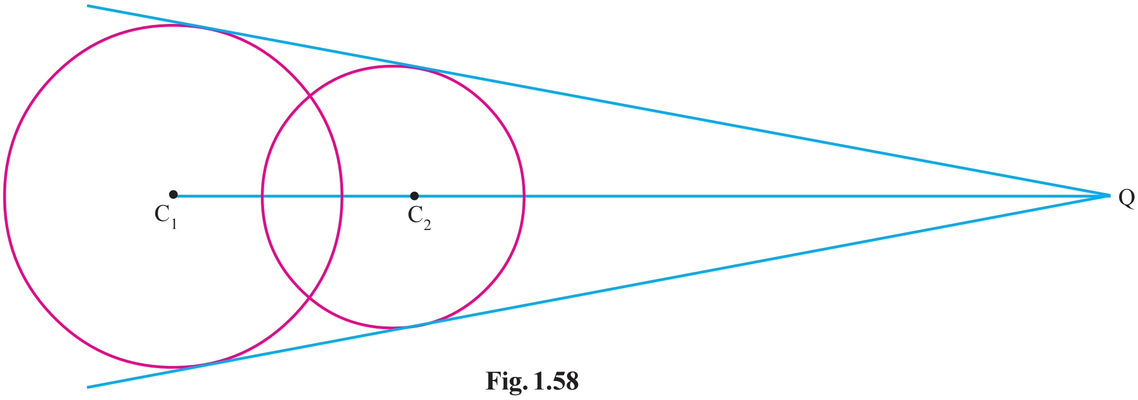


Fig. 1.58

In this case the internal centre of similitude does not exist. Only two common tangents through Q can be drawn (see Fig.1.58)

Case (iv) : $C_1C_2 = |r_1 - r_2|$

(i.e., Given circles touch each other internally)

In this case internal centre of similitude does not exist and the external centre of similitude Q is the point of contact of the two circles. Only one common tangent exists at Q. Thus the number of common tangents in the present case is one (see Fig.1.59)

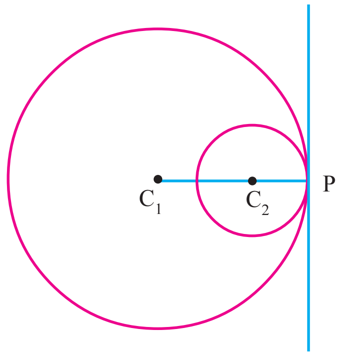


Fig.1.59

case (v): $\overline{C_1C_2} < |r_1 - r_2|$

(i.e., one circle lies entirely in the interior of the other circle)

In this case the number of common tangents is zero (see Fig. 1.60).

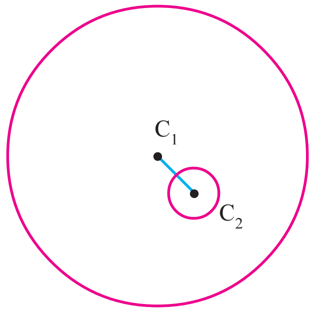


Fig. 1.60

1.5.7 Solved Problems

1. Problem : *Show that four common tangents can be drawn for the circles given*

by $x^2 + y^2 - 14x + 6y + 33 = 0$ ... (1)

and $x^2 + y^2 + 30x - 2y + 1 = 0$ ... (2)

and find the internal and external centres of similitude.

Solution : Centres of the given circles are $C_1 = (7, -3)$ and $C_2 = (-15, 1)$ and radii of given circles are $r_1 = 5, r_2 = 15$.

Now $C_1C_2 = \sqrt{(7+15)^2 + (-3-1)^2} = 10\sqrt{5}$.

$r_1 + r_2 = 20$

$\therefore \quad C_1C_2 > r_1 + r_2 \qquad (\because C_1C_2 = \sqrt{500},\ r_1 + r_2 = \sqrt{400})$

$\therefore$ Four common tangents exist for the given circles (by 1.5.6 sub case(i))

Now $r_1 : r_2 = 5 : 15 = 1 : 3$.

The internal centre of similitude

$$= \left(\frac{(3)\ (7) + (1)(-15)}{3+1}, \frac{(3)\ (-3) + (1)\ (1)}{3+1}\right)$$

$$= \left(\frac{3}{2}, -2\right)$$

The external centre of similitude

$$= \left(\frac{(3)\ (7) - (1)\ (-15)}{3-1}, \frac{3(-3) - (1)\ (1)}{3-1}\right)$$

$= (18, -5)$.

2. Problem : *Prove that the circles*

$$x^2 + y^2 - 8x - 6y + 21 = 0$$

and $$x^2 + y^2 - 2y - 15 = 0$$

have exactly two common tangents. Also find the point of intersection of those tangents

Solution : Let C_1, C_2 be the centres and r_1, r_2 be the radii of circles given by (1) and (2) respectively. Then $C_1 = (4, 3)$, $C_2 = (0, 1)$; $r_1 = 2$ and $r_2 = 4$.

$$\therefore \ \overline{C_1C_2} = \sqrt{20} = 2\sqrt{5}$$

$|r_1 - r_2| = |2 - 4| = 2$ and $r_1 + r_2 = 6$.

$|r_1 - r_2| < C_1\ C_2 < r_1 + r_2 \qquad (\because \sqrt{4} < \sqrt{20} < \sqrt{36})$

$\therefore$ Given circles intersect each other and have exactly two common tangents.

Now $r_1 : r_2 = 2 : 4 = 1 : 2$.

The external centre of similitude is

$$\left(\frac{8-0}{2-1}, \frac{6-1}{2-1}\right) = (8,5).$$

Thus the point of intersection of common tangents is (8, 5).

3. Problem : *Show that the circles* $x^2 + y^2 - 4x - 6y - 12 = 0$ ... (1)

and $x^2 + y^2 + 6x + 18y + 26 = 0$ ... (2)

touch each other. Also find the point of contact and common tangent at this point of contact.

Solution : Let C_1, C_2 be the centres of the circles (1) and (2) and r_1, r_2 be the radii of these circles. Then $C_1 = (2, 3)$, $C_2 = (-3, -9)$; $r_1 = 5$, $r_2 = 8$.

Now $\overline{C_1C_2} = \sqrt{(2+3)^2+(3+9)^2} = 13$

$r_1 + r_2 = 5 + 8 = 13$

$\therefore \overline{C_1C_2} = r_1 + r_2$.

$\therefore$ The given circles touch each other externally.

The point of contact $P(x_1, y_1)$ divides $\overline{C_1C_2}$ in the ratio $r_1 : r_2 = 5 : 8$.

$$\therefore P(x_1, y_1) = \left(\frac{16-15}{8+5}, \frac{24-45}{8+5}\right)$$

$$= \left(\frac{1}{13}, \frac{-21}{13}\right)$$

The common tangent at this point of contact is

$$x\left(\frac{1}{13}\right) + y\left(-\frac{21}{13}\right) - 2\left(x+\frac{1}{13}\right) - 3\left(y-\frac{21}{13}\right) - 12 = 0$$

i.e., $5x + 12y + 19 = 0$.

4. Problem : *Show that the circles* $x^2 + y^2 - 4x - 6y - 12 = 0$. ... (1)

and $5(x^2 + y^2) - 8x - 14y - 32 = 0$ ... (2)

touch each other and find their point of contact.

Solution : Here the centres of (1) and (2) are $C_1 = (2, 3)$, $C_2 = \left(\frac{4}{5}, \frac{7}{5}\right)$. The radii of (1) and (2) are $r_1 = 5$, $r_2 = 3$ and $\overline{C_1C_2} = 2$.

Note that $\overline{C_1C_2} = |r_1 - r_2|$

Hence the circles (1) and (2) are touch each other internally. The point of contact P divides $\overline{C_1C_2}$ in the ratio 5 : 3 externally.

$$\therefore P = \left(\frac{(3)(2) - 5\left(\frac{4}{5}\right)}{3-5}, \frac{(3)(3) - 5\left(\frac{7}{5}\right)}{3-5}\right)$$

$$= (-1, -1).$$

Thus the point of contact of the given circles is $(-1, -1)$.

Now we shall derive the combined equation of the pair of tangents drawn from an external point to a circle.

1.5.8 Theorem : *The combined equation of the pair of tangents drawn from an external point* $P(x_1, y_1)$ *to the circle* $S = 0$ *is* $SS_{11} = S_1^2$.

Proof

Suppose that the tangents drawn from P to the circle S = 0 touch the circle at A and B (see Fig. 1.61).

The equation of AB is $S_1 = 0$.

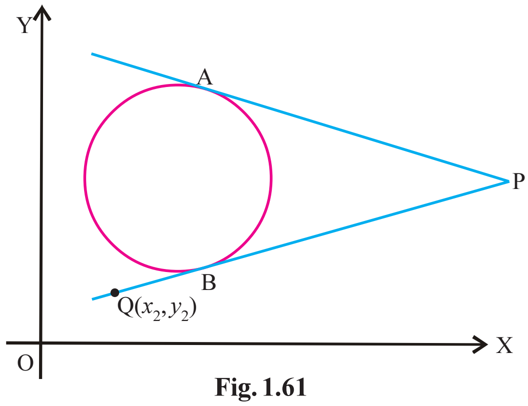


Fig. 1.61

i.e. $xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$ ... (1)

Let Q(x_2, y_2) be any point on these tangents. Now the locus of Q will be the equation of pair of tangents drawn from P.

The segment $\overline{PQ}$ is divided by the line AB (whose equation is $S_1 = 0$) in the ratio $-S_{11} : S_{12}$

$\therefore$ PB : QB = $-S_{11} : S_{12}$ or $S_{11} : S_{12}$

according as $S_{11} S_{12} < 0$ or $S_{11} S_{12} > 0$

$$\Rightarrow \frac{PB}{QB} = \left|\frac{S_{11}}{S_{12}}\right| \quad \text{... (2)}$$

But PB = $\sqrt{S_{11}}$ and QB = $\sqrt{S_{22}}$ (lengths of tangents from P and Q)

$$\therefore \frac{PB}{QB} = \frac{\sqrt{S_{11}}}{\sqrt{S_{22}}} \quad \text{... (3)}$$

From (2) and (3), we get

$$\frac{S_{11}^2}{S_{12}^2} = \frac{S_{11}}{S_{22}}$$

$$S_{11} S_{22} = S_{12}^2$$

Hence the locus of Q(x_2, y_2) is

$$S_{11}\, S = S_1^2 .$$

1.5.9 Solved Problems

1. Problem : *Find the equation of the pair of tangents from (10, 4) to the circle $x^2 + y^2 = 25$.*

Solution : Here $(x_1, y_1) = (10, 4)$. By Theorem 1.5.8, the equation of the pair of tangents is given by $(100 + 16 - 25)\ (x^2 + y^2 - 25) = (10x + 4y - 25)^2$

i.e., $9x^2 + 80xy - 75y^2 - 500x - 200y + 2900 = 0$.

2. Problem : *Find the equations to all possible common tangents of the circles*

$$x^2 + y^2 - 2x - 6y + 6 = 0 \quad \text{... (1)}$$

and

$$x^2 + y^2 = 1 \quad \text{... (2)}$$

Solution : Let C_1, C_2 be the centres and r_1, r_2 be the radii of the circles given by (1) and (2). Then $C_1 = (1, 3)$; $C_2 = (0, 0)$; $r_1 = 2$; $r_2 = 1$. Here $\overline{C_1C_2} = \sqrt{10}$, $r_1 + r_2 = 3$, $|C_1C_2| > r_1 + r_2$ and $r_1 \neq r_2$.

Here there exist four common tangents. The centres of similitudes are $\left(\frac{1}{3}, 1\right)$ and $(-1, -3)$. The required common tangents are given by

$$(x^2 + y^2 - 1)\left(\frac{1}{9} + 1 - 1\right) = \left(\frac{x}{3} + y - 1\right)^2 \quad \text{... (3)}$$

and

$$(x^2 + y^2 - 1)(1 + 9 - 1) = (-x - 3y - 1)^2 \quad \text{... (4)}$$

Equation (3) is equivalent to

$$4y^2 + 3xy - 3x - 9y + 5 = 0$$

i.e.,

$$(y - 1)(4y + 3x - 5) = 0 \quad \text{... (5)}$$

Now equation (4) can be expressed as

$$(x + 1)\ (4x - 3y - 5) = 0 \quad \text{... (6)}$$

From equations (5) and (6), we get the equations of common tangents as $y - 1 = 0$, $3x + 4y - 5 = 0$, $x + 1 = 0$ and $4x - 3y - 5 = 0$.

Exercise 1(e)

I. 1. Discuss the relative position of the following pair of circles

(i) $x^2 + y^2 - 4x - 6y - 12 = 0$, $x^2 + y^2 + 6x + 18y + 26 = 0$

(ii) $x^2 + y^2 + 6x + 6y + 14 = 0$, $x^2 + y^2 - 2x - 4y - 4 = 0$

(iii) $(x - 2)^2 + (y + 1)^2 = 9$, $(x + 1)^2 + (y - 3)^2 = 4$

(iv) $x^2 + y^2 - 2x + 4y - 4 = 0$ $x^2 + y^2 + 4x - 6y - 3 = 0$

2. Find the number of possible common tangents that exist for the following pairs of circles.

(i) $x^2 + y^2 + 6x + 6y + 14 = 0$, $x^2 + y^2 - 2x - 4y - 4 = 0$

(ii) $x^2 + y^2 - 4x - 2y + 1 = 0$, $x^2 + y^2 - 6x - 4y + 4 = 0$

(iii) $x^2 + y^2 - 4x + 2y - 4 = 0$ $x^2 + y^2 + 2x - 6y + 6 = 0$

(iv) $x^2 + y^2 = 4$, $x^2 + y^2 - 6x - 8y + 16 = 0$

(v) $x^2 + y^2 + 4x - 6y - 3 = 0$
$x^2 + y^2 + 4x - 2y + 4 = 0$.

3. Find the internal centre of similitude for the circles
$x^2 + y^2 + 6x - 2y + 1 = 0$ and $x^2 + y^2 - 2x - 6y + 9 = 0$

4. Find the external centre of similitude for the circles
$x^2 + y^2 - 2x - 6y + 9 = 0$ and $x^2 + y^2 = 4$.

II. 1. (i) Show that the circles $x^2 + y^2 - 6x - 2y + 1 = 0$, $x^2 + y^2 + 2x - 8y + 13 = 0$ touch each other. Find the point of contact and the equation of common tangent at their point of contact.

(ii) Show that $x^2 + y^2 - 6x - 9y + 13 = 0$, $x^2 + y^2 - 2x - 16y = 0$ touch each other. Find the point of contact and the equation of common tangent at their point of contact.

2. Find the equation of the circle which touches the circle $x^2 + y^2 - 2x - 4y - 20 = 0$ externally at $(5, 5)$ with radius 5.

3. Find the direct common tangents of the circles $x^2 + y^2 + 22x - 4y - 100 = 0$ and $x^2 + y^2 - 22x + 4y + 100 = 0$.

4. Find the transverse common tangents of the circles $x^2 + y^2 - 4x - 10y + 28 = 0$ and $x^2 + y^2 + 4x - 6y + 4 = 0$.

5. Find the pair of tangents from $(4, 10)$ to the circle $x^2 + y^2 = 25$.

6. Find the pair of tangents drawn from $(0, 0)$ to $x^2 + y^2 + 10x + 10y + 40 = 0$.

III. 1. Find the equation of the circle which touches $x^2 + y^2 - 4x + 6y - 12 = 0$ at $(-1, 1)$ internally with a radius of 2.

2. Find all common tangents of the following pairs of circles.
(i) $x^2 + y^2 = 9$ and $x^2 + y^2 - 16x + 2y + 49 = 0$
(ii) $x^2 + y^2 + 4x + 2y - 4 = 0$ and $x^2 + y^2 - 4x - 2y + 4 = 0$.

3. Find the pair of tangents drawn from $(3, 2)$ to the circle $x^2 + y^2 - 6x + 4y - 2 = 0$.

4. Find the pair of tangents drawn from $(1, 3)$ to the circle $x^2 + y^2 - 2x + 4y - 11 = 0$ and also find the angle between them.

5. Find the pair of tangents from the origin to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ and hence deduce a condition for these tangents to be perpendicular.

6. From a point on the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ two tangents are drawn to the circle $x^2 + y^2 + 2gx + 2fy + c\sin^2\alpha + (g^2 + f^2)\cos^2\alpha = 0$ $(0 < \alpha < \pi/2)$. Prove that the angle between them is 2α.

Key Concepts

- The locus of a point in a plane such that its distance from a fixed point in the plane is always the same is called a circle.
- The equation of a circle with centre $(h_1 k)$ and radius r is $(x-h)^2+(y-k)^2=r^2$
- The equation of a circle in standard form is $x^2+y^2=r^2$
- The equation of a circle in general form is $x^2+y^2+2gx+2fy+c=0$ and its centre is $(-g,-f)$, radius is $\sqrt{g^2+f^2-c}$.
- The intercept made by $x^2+y^2+2gx+2fy+c=0$

  (i) on X-axis is $2\sqrt{g^2-c}$ if $g^2 \geq c$

  (ii) on Y-axis is $2\sqrt{f^2-c}$ of $f^2 \geq c$.
- If the extremities of a diameter of a circle are (x_1,y_1) and (x_2,y_2) then its equation is $(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0$
- The equation of a circle passing through three non- collinear points $(x_1,y_1),(x_2,y_2)$ and (x_3,y_3) is

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}(x^2+y^2)+\begin{vmatrix} c_1 & y_1 & 1 \\ c_2 & y_2 & 1 \\ c_3 & y_3 & 1 \end{vmatrix}x+\begin{vmatrix} x_1 & c_1 & 1 \\ x_2 & c_2 & 1 \\ x_3 & c_3 & 1 \end{vmatrix}y+\begin{vmatrix} x_1 & y_1 & c_1 \\ x_2 & y_2 & c_2 \\ x_3 & y_3 & c_3 \end{vmatrix}=0.$$

  where $c_i=-(x_i^2+y_i^2)$
- The centre of the circle passing through three non- collinear points $(x_1,y_1),(x_2,y_2)$ and (x_3,y_3) is

$$\left(\frac{\begin{vmatrix} c_1 & y_1 & 1 \\ c_2 & y_2 & 1 \\ c_3 & y_3 & 1 \end{vmatrix}}{(-2)\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}}, \frac{\begin{vmatrix} x_1 & c_1 & 1 \\ x_2 & c_2 & 1 \\ x_3 & c_3 & 1 \end{vmatrix}}{(-2)\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}}\right)$$

- The parametric equations of a circle with centre (h, k) and radius $(r \geq 0)$ are given by

  $x=h+r\cos\theta$

  $y=k+r\sin\theta \qquad 0 \leq \theta < 2\pi$

- A point $P(x_1, y_1)$ is an interior point or on the circumference or an exterior point of a circle $S = 0 \Leftrightarrow S_{11} \lesseqgtr 0$.
- The power of $P(x_1, y_1)$ with respect to the circle $S = 0$ is S_{11}.
- A point $P(x_1, y_1)$ is an interior point or on the circumference or exterior point of the circle $S = 0 \Leftrightarrow$ the power of P with respect to $S = 0$ is negative, zero and positive.
- If a straight line through a point $P(x_1, y_1)$ meets the circle $S = 0$ at A and B then the power of P is equal to PA . PB.
- The length of the tangent from $P(x_1, y_1)$ to $S = 0$ is $\sqrt{S_{11}}$.
- The straight line $L = 0$ intersects, touches or does not meet the circle $S = 0$ according as $l < r, l = r$ or $l > r$ where l is the perpendicular distance from the centre of the circle to the line $L = 0$ and r is the radius.
- For every real value of m the straight line $y = mx \pm r\sqrt{1+m^2}$ is a tangent to the circle $x^2 + y^2 = r^2$.
- If r is the radius of the circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ then for every real value of m the straight line
$$y + f = m(x + g) \pm r\sqrt{1+m^2}$$
will be a tangent to the circle.
- If $P(x_1, y_1)$ and $Q(x_2, y_2)$ are two points on the circle $S = 0$ then the secant's ($\overleftrightarrow{PQ}$) equation is
$$S_1 + S_2 = S_{12}.$$
- The equation of tangent at (x_1, y_1) of the circle $S = 0$ is $S_1 = 0$.
- If θ_1, θ_2 are two points on $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ then the equation of the chord joining the points θ_1, θ_2 is
$$(x+g)\cos\left(\frac{\theta_1+\theta_2}{2}\right) + (y+f)\sin\left(\frac{\theta_1+\theta_2}{2}\right) = r\cos\left(\frac{\theta_1-\theta_2}{2}\right)$$
- The equation of the tangent at θ of the circle $S = 0$ is $(x+g)\cos\theta + (y+f)\sin\theta = r$.
- The equation of normal at (x_1, y_1) of the circle $S = 0$ is $(x - x_1)(y_1 + f) - (y - y_1)(x_1 + g) = 0$.
- The chord of contact of $P(x_1, y_1)$ (exterior point) with respect to $S = 0$ is $S_1 = 0$.
- The equation of the polar of a point $P(x_1, y_1)$ with respect to $S = 0$ is $S_1 = 0$.

❖

$P(x_1,y_1)$	Tangent at P	Chord of contact at P	Polar of P
Interior of the circle	Does not exist	Does not exist (not defined)	$S_1 = 0$ (P is different from the centre of the circle)
on the circle	$S_1 = 0$	$S_1 = 0$	$S_1 = 0$
Exterior of the circle	Does not exist	$S_1 = 0$	$S_1 = 0$

❖ The pole of $lx + my + n = 0$ with respect to $S = 0$ is

$$\left(-g + \frac{lr^2}{lg + mf - n}, -f + \frac{mr^2}{lg + mf - n}\right)$$

where r is the radius of the circle.

❖ The polar of $P(x_1, y_1)$ with respect to $S = 0$ passes through $Q(x_2, y_2) \Leftrightarrow$ the polar of Q with respect to $S = 0$ passes through P.

❖ The points (x_1, y_1) and (x_2, y_2) are conjugate points with respect to $S = 0$ if $S_{12} = 0$.

❖ Two lines $l_1x + m_1y + n_1 = 0$, $l_2x + m_2y + n_2 = 0$ are conjugate with respect to $x^2 + y^2 = a^2 \Leftrightarrow a^2(l_1l_2 + m_1m_2) = n_1n_2$.

❖ Two points P, Q are said to be inverse points with respect to $S = 0$ if $CP. CQ = r^2$ where C is the centre and r is radius of the circle $S = 0$.

❖ If (x_1, y_1) is the mid-point of a chord of the circle $S = 0$ then its chord equation is $S_1 = S_{11}$.

❖ The pair of common tangents to the circles $S = 0$, $S' = 0$ touching at a point on the line segment $\overline{C_1C_2}$ (C_1, C_2 are centres of the circles) is called transverse pair of common tangents.

❖ The pair of common tangents to the circles $S = 0$, $S' = 0$ intersecting at a point not in $\overline{C_1C_2}$ is called as direct pair of common tangents.

❖ The point of intersection of transverse (direct) common tangents is called internal (external) Centre of similitude.

❖

Situation	No.of common tangents
1. $\overline{C_1C_2} > r_1 + r_2$	4
2. $r_1 + r_2 = \overline{C_1C_2}$	3
3. $\lvert r_1 - r_2\rvert < \overline{C_1C_2} < r_1 + r_2$	2
4. $C_1C_2 = \lvert r_1 - r_2\rvert$	1
5. $C_1C_2 < \lvert r_1 - r_2\rvert$	0

❖ The combined equation of the pair of tangents drawn from an external point $P(x_1, y_1)$ to the circle $S = 0$ is $SS_{11} = S_1^2$.

Historical Note

It is not easy to trace the origin of the studies on circle. Babylonians, ancient Egyptians, Greeks, Chinese and Indians contributed to the studies on circle to begin with.

Probably the first writings about the circle and the circular shapes are in *Rigveda*. For construction of *Yagna Vedikas* - sacrifical altars, many geometrical shapes were in use. These are referred to in sulbasutras. Ever since the shape of a circle was identified there were attempts to find the circumferences and areas of the circles.

Answers

Exercise 1(a)

I. 1. (i) $x^2+y^2-4x+6y-3=0$ (ii) $x^2+y^2+2x-4y-20=0$

(iii) $x^2+y^2-2ax+2by-2ab=0$ (iv) $x^2+y^2+2ax+2by+2b^2=0$

(v) $x^2+y^2-2x\cos\alpha-2y\sin\alpha=0$ (vi) $x^2+y^2+14x+6y+42=0$

(vii) $4x^2+4y^2+4x+72y+225=0$ (viii) $36x^2+36y^2-180x+96y-1007=0$

(ix) $4x^2+4y^2-8x-56y+175=0$ (x) $x^2+y^2-81=0$

2. $x^2+y^2+8x+6y=0$

3. $x^2+y^2-4x-6y-3=0$

4. $x^2+y^2-13=0$

5. $x^2+y^2+6x-8y-11=0$

6. $a=2$, radius $=\sqrt{21}/4$

7. $a=3$; $b=0$; radius $=\sqrt{65}/6$, center $=\left(\frac{5}{6},-\frac{1}{3}\right)$.

8. $g=-2$; $f=-3$; radius $=5$

9. $g=4$; $f=3$; radius $=5$

10. $c=-23$

11. (i) centre $=(2,4)$; radius $=\sqrt{61}$ (ii) centre $=\left(\frac{5}{6},1\right)$; radius $=\sqrt{13}/6$

(iii) centre $=(-1,2)$; radius $=\frac{4}{\sqrt{3}}$ (iv) centre $=(-3,-4)$; radius $=11$

(v) centre $=\left(1,\ -\dfrac{3}{2}\right)$; radius $=\dfrac{\sqrt{19}}{2}$ (vi) centre $\left(\dfrac{3}{4},\ -\dfrac{1}{2}\right)$; radius $=\dfrac{\sqrt{21}}{4}$

(vii) centre $=\left(\dfrac{c}{\sqrt{1+m^2}},\ \dfrac{mc}{\sqrt{1+m^2}}\right)$; radius $=c$

(viii) centre $=(-a, b)$; radius $=a$.

12. (i) $x^2+y^2-5x-8y+16=0$ (ii) $x^2+y^2+x+y-24=0$

(iii) $x^2+y^2-9x-8y+20=0$ (iv) $x^2+y^2-5x-7y+14=0$

(v) $x^2+y^2-10x-2y+6=0$ (vi) $x^2+y^2-3x+1=0$

(vii) $x^2+y^2-8x-5y=0$ (viii) $x^2+y^2-5x-8y+13=0$.

13. (i) $x=2\cos\theta, y=2\sin\theta,\quad 0\le\theta<2\pi$

(ii) $x=\dfrac{3}{2}\cos\theta;\quad y=\dfrac{3}{2}\sin\theta,\ 0\le\theta<2\pi$

(iii) $x=\sqrt{\dfrac{7}{2}}\cos\theta,\ y=\sqrt{\dfrac{7}{2}}\sin\theta,\quad 0\le\theta<2\pi$

(iv) $x=3+8\cos\theta,\ y=4+8\sin\theta,\quad 0\le\theta<2\pi$

(v) $x=2+5\cos\theta,\ y=3+5\sin\theta,\quad 0\le\theta<2\pi$

(vi) $x=3+5\cos\theta,\ y=-2+5\sin\theta,\quad 0\le\theta<2\pi$

II. 1. $x^2+y^2+2ax+2py-(b^2+q^2)=0$ 2. (i) $(-1, -3)$ (ii) $(-5, -12)$

3. $x^2+y^2+2x-2y-23=0$ 4. $x^2+y^2-6x-8y+15=0$

5. $x^2+y^2-6x-4y-156=0$ 6. $3(x^2+y^2)-14x-67=0$

III. 1. (i) $x^2+y^2-4x-6y+11=0$ (ii) $x^2+y^2-22x-4y+25=0$

(iii) $x^2+y^2+x-12y+5=0$ (iv) $3(x^2+y^2)-29x-19y+56=0$

(v) $x^2+y^2-2x-2y=0$

2. (i) $x^2+y^2 \pm 4x \pm 3y = 0$ (ii) $x^2+y^2 \pm 6x \pm 4y = 0$

4. $c = \frac{14}{3}$

5. (i) $x^2+y^2-17x-19y+50=0$ (ii) $x^2+y^2+12x+12y+7=0$

(iii) $49(x^2+y^2)+280x-259y+245=0$ (iv) $x^2+y^2-24x+16y-52=0.$

Exercise 1(b)

I. 1. (i) interior (ii) exterior (iii) exterior (iv) exterior

2. (i) 44 (ii) 0 (iii) 10 (iv) −24

3. (i) 2 (ii) 5 (iii) $\sqrt{34}$

II. 1. $k=-5$ (2) $k=-2$

III. 1. $5(x^2+y^2)-60x-126y-212=0$ 2. $4x+6y+9=0$

Exercise 1(c)

I. 1. (i) $4x-3y-43=0$ (ii) $4x-3y+7=0$

(iii) $2x+3y+39=0$ (iv) $x+y-7=0$

2. (i) $x+y+1=0$ (ii) $2x+y-11=0$

(iii) $11x-13y+28=0$ (iv) $y=2$

II. 1. $4\sqrt{6}$ 2. $2\sqrt{7}$

3. $2\sqrt{a^2-p^2}$ 4. $x^2+y^2-4x-6y+12=0$

5. $x^2+y^2+6x-8y+16=0$ 6. $x=4+\sqrt{5},\ x=4-\sqrt{5}$

7. $x+3y-10=0$

$x-3y-10=0$

III. 1. $\overline{AB} = 2\sqrt{c^2-\frac{a^2b^2}{(a^2+b^2)}}$ condition is $\frac{1}{a^2}+\frac{1}{b^2}=\frac{1}{c^2}$

3. $x^2+y^2+4x-6y+8=0$ 4. $5x+y-17=0,\ x-5y+7=0$

5. $2x - y - 1 = 0$; $x + 2y - 3 = 0$

6. $(5, 1)$ 7. $(1, -1)$ 8. $x + y + 1 \pm 5\sqrt{2} = 0$

9. $x + 3y - 2 \pm 5\sqrt{2} = 0$ 10. $x - y + 1 \pm 2\sqrt{5} = 0$ 11. $x^2 + y^2 - 2x - 4y - 3 = 0$

12. $x^2 + y^2 - 2x - 4y - 4 = 0$ or $25(x^2 + y^2) - 26x + 68y + 44 = 0$

13. $(2, -3)$

Exercise 1(d)

I. 1. $g^2 + f^2 = 2c$ 2. $5x - 14y - 16 = 0$ 3. $x + y - 9 = 0$

4. $x + 2y - 7 = 0$ 5. $6x - 2y = 11$ 6. $4x + 7y - 30 = 0$

7. $\left(\frac{-ar^2}{c}, \frac{-br^2}{c}\right)$ 8. $(6, 8)$ 9. $(2, -3)$

12. 2 13. 2 14. 11

15. $\frac{28}{3}$.

II. 1. $\text{Cos}^{-1}(\frac{7}{8})$ 2. $\text{Cos}^{-1}(\frac{7}{25})$ 3. $\frac{\pi}{2}$

4. $x^2 + y^2 = a^2 \operatorname{cosec}^2(\alpha/2)$ 5. $x^2 + y^2 - 2a^2 = 0$

6. Slope $= 1$, distance $= 6\sqrt{2}$ 7. 29

III. 1. $\left(\frac{109}{76}, \frac{9}{38}\right)$ 3. $\frac{108\sqrt{2}}{17}$

4. $x^2 + y^2 - 3x + y - 4 = 0$ 5. $2(x^2 + y^2) + 2gx + 2fy + c = 0$

Exercise 1(e)

I 1. (i) touch each other (ii) each lies on the exterior of the other

(iii) touch each other (iv) Cut each other in two points

2. (i) 4 (ii) 2 (iii) 3

(iv) 3 (v) 0

3. $(0, \frac{5}{2})$ 4. $(2, 6)$

II. 1. (i) $\left(\frac{3}{5}, \frac{14}{5}\right)$, $4x - 3y + 6 = 0$ (ii) $(5, 1)$, $4x - 7y - 13 = 0$

2. $x^2 + y^2 - 18x - 16y + 120 = 0$

3. $3x + 4y - 50 = 0$, $7x - 24y - 250 = 0$

4. $x - 1 = 0$, $3x + 4y - 21 = 0$

5. $75x^2 - 9y^2 - 80xy + 200x + 500y - 2900 = 0$

6. $3x^2 - 10xy + 3y^2 = 0$.

III. 1. $5x^2 + 5y^2 - 2x + 6y - 18 = 0$

2. (i) $4x - 3y - 15 = 0$, $12x + 5y - 39 = 0$
$y - 3 = 0$, $16x + 63y + 195 = 0$

(ii) $y - 2 = 0$, $4x - 3y - 10 = 0$
$x - 1 = 0$, $3x + 4y - 5 = 0$

3. $x^2 - 15y^2 - 6x + 60y - 51 = 0$

4. $9x^2 - 16y^2 - 18x + 96y - 135 = 0$, $\text{Cos}^{-1}\left(\frac{7}{25}\right)$

5. $(gx + fy)^2 = c(x^2 + y^2)$; $g^2 + f^2 = 2c$.

Chapter 2

System of Circles

"All man's miseries derive from not being able to sit quietly in a room alone"

- Blaise Pascal

Ptolemy

(ca. 83 - 161)

Ptolemy, *was a Greek -Egyptian mathematician, geographer, astronomer, and astrologer who flourished in Alexandria, Roman Egypt. The first notable value of* π *after that of Archimedes, was given by Ptolemy of Alexandria, as 377/120 i.e., 3.1416. His famous book was Almagest.*

Introduction

In this chapter, we shall discuss the angle between two intersecting circles and obtain a condition for their orthogonality. Also, we shall learn about the radical axis of two circles, its properties, common chord, common tangent of two circles and the radical centre.

2.1 Angle between two intersecting circles

We have learnt that two circles will intersect each other if the distance between their centres lies between the absolute value of the difference of their radii and the sum of their radii. For such circles we define the angle between them.

2.1.1 Definition

The angle between two intersecting circles is defined as the angle between the tangents at the point of intersection of the two circles (see Fig. 2.1)

$T_1 \hat{P} T_2$ *is the angle between the circles at P.*

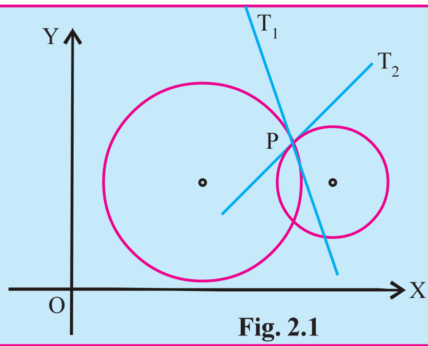


Fig. 2.1

2.1.2 Note

If two circles S = 0, S' = 0 intersect at P and Q then the angle between the two circles at the points P and Q are equal.

2.1.3 Theorem : *If (i) C_1, C_2 are the centres of two given intersecting circles (ii) $d = C_1C_2$ (iii) r_1, r_2 are radii of these circles (iv) θ is the angle between these circles, then*

$$\cos\theta = \frac{d^2 - r_1^2 - r_2^2}{2r_1r_2}.$$

Proof : Let P be a point of intersection of two given circles. Let the tangents drawn to two circles at P intersect the line joining the centres at T_1 and T_2 (see Fig. 2.2). Then $\angle T_1PT_2 = \theta$.

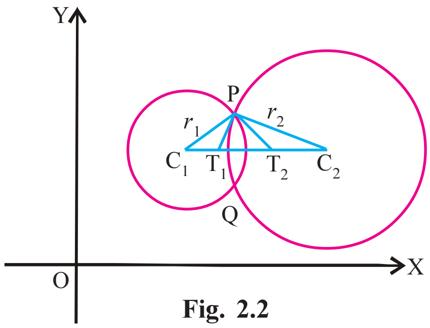


Fig. 2.2

Consider $\angle C_1PC_2 = \angle C_1PT_2 + \angle T_2PC_2$

$= 90^0 + 90^0 - \theta$

$= 180^0 - \theta$

From ΔC_1PC_2, we have

$$C_1C_2^2 = C_1P^2 + C_2P^2 - 2(C_1P)(C_2P)\cos\angle C_1PC_2$$

i.e., $$d^2 = r_1^2 + r_2^2 - 2r_1r_2\cos(180^0 - \theta)$$

$$\therefore \quad \cos\theta = \frac{d^2 - r_1^2 - r_2^2}{2r_1r_2}.$$

Note that cos θ is independent of the point of intersection (coordinates of the point of intersection are not involved). Therefore, the angle at Q is also equal to θ.

2.1.4 Theorem : *If θ is the angle between the intersecting circles*

$$x^2 + y^2 + 2gx + 2fy + c = 0 \quad \text{... (1)}$$

and $$x^2 + y^2 + 2g'x + 2f'y + c' = 0 \quad \text{... (2)}$$

then $\cos\theta = \dfrac{c + c' - 2gg' - 2ff'}{2\sqrt{g^2 + f^2 - c}\,\sqrt{g'^2 + f'^2 - c'}}$.

Proof : Let C_1, C_2 be the centres and r_1, r_2 be the radii of the two given circles (1) and (2). Then $C_1 = (-g, -f)$; $C_2 = (-g', -f')$; $r_1 = \sqrt{g^2 + f^2 - c}$; $r_2 = \sqrt{g'^2 + f'^2 - c'}$.

By Theorem 2.1.3, we have

$$\cos\theta = \frac{(g' - g)^2 + (f' - f)^2 - (g^2 + f^2 - c) - (g'^2 + f'^2 - c')}{2\sqrt{g^2 + f^2 - c}\,\sqrt{g'^2 + f'^2 - c'}}$$

i.e.,

$$\cos\theta = \frac{c + c' - 2gg' - 2ff'}{2\sqrt{g^2 + f^2 - c}\,\sqrt{g'^2 + f'^2 - c'}}$$

2.1.5 Solved Problems

1. Problem: *Find the angle between the circles*

$$x^2 + y^2 + 4x - 14y + 28 = 0 \quad \text{... (1)}$$

$$x^2 + y^2 + 4x - 5 = 0 \quad \text{... (2)}$$

Solution : Here $g = 2$; $f = -7$; $c = 28$; $g' = 2$; $f' = 0$; $c' = -5$.

Let θ be the angle between the circles (1) and (2). Then by Theorem 2.1.4. we have

$$\cos\theta = \frac{28 - 5 - 2(2)\,(2) - 2(-7)\,(0)}{2\sqrt{4 + 49 - 28}\,\sqrt{4 + 0 + 5}}$$

$$= 1/2$$

$$\therefore\ \theta = 60^\circ.$$

Hence the angle between the two given circles (1) and (2) is 60°.

2. Problem: *If the angle between the circles*

$$x^2 + y^2 - 12x - 6y + 41 = 0 \quad \text{... (1)}$$

and $$x^2 + y^2 + kx + 6y - 59 = 0 \quad \text{... (2)}$$

is 45° find k.

Solution: Here $g = -6$; $f = -3$; $c = 41$; $g' = \dfrac{k}{2}$; $f' = 3$; $c' = -59$.

Given that $\theta = 45^\circ$.

∴ By Theorem 2.1.4, we have

$$\cos 45^0 = \frac{41-59-2(-6)\left(\frac{k}{2}\right)-2(-3)(3)}{2\sqrt{36+9-41}\sqrt{\frac{k^2}{4}+9+59}}$$

i.e., $$\frac{1}{\sqrt{2}} = \frac{3k}{2\sqrt{\frac{k^2}{4}+68}}$$

$$\Rightarrow \quad k = \pm 4.$$

2.1.6 Definition

Two intersecting circles are said to be orthogonal if the angle between them is a right angle (i.e., 90^0)

2.1.7 Condition for orthogonality

Let the two intersecting circles be given by

$$x^2+y^2+2gx+2fy+c=0 \qquad \text{... (1)}$$

and $$x^2+y^2+2g'x+2f'y+c'=0 \qquad \text{... (2)}$$

These two circles are orthogonal

$$\Leftrightarrow \quad \frac{c+c'-2gg'-2ff'}{2\sqrt{g^2+f^2-c}\sqrt{g'^2+f'^2-c'}}=0 \qquad \text{(By Theorem 2.1.4)}$$

$$\Leftrightarrow \quad 2(gg'+ff')=c+c'.$$

Thus, the condition for orthogonality of the two intersecting circles (1) and (2) is

$$\boxed{2(gg'+ff') = c+c'}$$

2.1.8 Note

(i) *Two intersecting circles are orthogonal if and only if the square of the distance between their centres is equal to the sum of the squares of their radii. In this case, a tangent of one circle at the point of intersection will be normal to the other circle and hence it passes through the centre of the other circle.*

(ii) *If two circles are orthogonal, then $d^2 = r_1^2 + r_2^2$ where d is the distance between the centres of the circles and r_1, r_2 are their radii.*

2.1.9 Theorem

(i) *if* $S = 0$ $S' = 0$ *are two circles intersecting at two distinct points, then* $S - S' = 0$ *(or* $S' - S = 0$*) represents a common chord of these circles.*

(ii) *if* $S = 0$ $S' = 0$ *are two circles touching each other, then* $S - S' = 0$ *(or* $S' - S = 0$*) is a common tangent.*

Proof : Let

$$S \equiv x^2 + y^2 + 2gx + 2fy + c = 0 \qquad ...(1)$$

and

$$S' \equiv x^2 + y^2 + 2g'x + 2f'y + c' = 0 \qquad ...(2)$$

(i) Let $P(x_1, y_1)$, $Q(x_2, y_2)$ be the points of intersection of (1) and (2).

Consider $S - S' = 0$

$$2(g - g')x + 2(f - f')y + (c - c') = 0 \qquad ... (3)$$

Clearly the points P, Q lie on (3), since $S_{11} = 0$, $S_{22} = 0$, $S'_{11} = 0$, $S'_{22} = 0$.

Further, the equation (3) is linear in x and y and hence it represents a line. Therefore $S - S' = 0$ is the equation of common chord of circles (1) and (2).

(ii) Let (1) and (2) touch each other at $P(x_1, y_1)$

Consider $S - S' = 0$

i.e., $$2(g - g')x + 2(f - f')y + (c - c') = 0 \qquad ...(3)$$

$P(x_1, y_1)$ is a point on (3) and it represents a line and the slope of (3) is $= -\dfrac{(g - g')}{(f - f')}$

The slope of the line joining the centres of the circles $= \dfrac{-f' + f}{-g' + g}$.

Thus the line given by (3) is perpendicular to the line of centres and it passes through the point of contact of the two circles. Hence it is a common tangent.

2.1.10 Theorem

(i) *If*

$$S \equiv x^2 + y^2 + 2gx + 2fy + c = 0 \qquad ... (1)$$

and

$$L \equiv lx + my + n = 0 \qquad ... (2)$$

are the equations of a circle and a straight line respectively intersecting each other, then $S + kL = 0$ *represents a circle passing through the points of intersection of* $S = 0$ *and* $L = 0$ *for all real values of k.*

(ii) *If*

$$S \equiv x^2 + y^2 + 2gx + 2fy + c = 0 \qquad \text{... (1)}$$

and

$$S' \equiv x^2 + y^2 + 2g'x + 2g'y + c' = 0 \qquad \text{... (3)}$$

are the equations of two intersecting circles, λ *and* μ *are any real numbers such that* $\lambda + \mu \neq 0$, *then* $\lambda S + \mu S' = 0$ *represents a circle passing through the points of intersection of (1) and (3).*

Proof

(i) Let $P(x_1, y_1)$ be one of the points of intersection of (1) and (2).

Clearly for any real number k

$$S + kL \equiv (x^2 + y^2 + 2gx + 2fy + c) + k(lx + my + n) = 0$$

passes through $P(x_1, y_1)$. Hence $S + kL = 0$ represents a circle for any real number k (problem 10 of 1.1.7)

(ii) Let

$$L \equiv S - S' = 0 \qquad \text{... (4)}$$

By Theorem 2.1.9 (i) and (ii), L is the common chord or tangent.

Consider

$$\lambda S + \mu S' = 0 \qquad \text{... (5)}$$

where λ, μ are any real numbers such that $\lambda + \mu \neq 0$

Clearly it passes through the points of intersection of (1) and (3). Further equation (5) is equivalent to

$$S + kL = 0 \qquad \text{... (6)}$$

where $k = \dfrac{-\mu}{(\lambda + \mu)}$.

Now $S + kL = 0$ represents a circle, hence $\lambda S + \mu S' = 0$ represents a circle. Hence the theorem.

2.1.11 Note

(i) *The equation* $\lambda S + \mu S' = 0$ *can also be written as* $S + kS' = 0$. *For, since* $\lambda + \mu \neq 0$, *we can assume that* $\lambda \neq 0$ *and hence we can express* $\lambda S + \mu S' = 0$ *by* $S + kS' = 0$ *where* $k = \dfrac{\mu}{\lambda} \neq -1$.

(ii) *If* $k = -1$ *then* $S + kS' = S - S' = 0$ *represents a line passing through the points of intersection of the circles* $S = 0$ *and* $S' = 0$. *In this case it is the common chord.*

(iii) *If the circles* $S = 0$ *and* $S' = 0$ *touch each other i.e., the points of intersection coincide, then* $S - S' = 0$ *is a common tangent to the circles.*

2.1.12 Solved Problems

1. Problem : *Find the equation of the circle which passes through (1, 1) and cuts orthogonally each of the circles*

$$x^2 + y^2 - 8x - 2y + 16 = 0 \qquad \text{... (1)}$$

and

$$x^2 + y^2 - 4x - 4y - 1 = 0. \qquad \text{... (2)}$$

Solution : Let the equation of the required circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0 \qquad \text{... (3)}$$

Then the circle (3) is orthogonal to (1) and (2).

$\therefore$ By applying the condition of orthogonality given in 2.1.7, we get

$$2g(-4) + 2f(-1) = c + 16 \qquad \text{... (4)}$$

and

$$2g(-2) + 2f(-2) = c - 1 \qquad \text{... (5)}$$

Given that the circle (3) is passing through (1, 1)

$$\therefore \quad 1^2 + 1^2 + 2g(1) + 2f(1) + c = 0$$

$$2g + 2f + c + 2 = 0 \qquad \text{... (6)}$$

Solving (4), (5) and (6) for g, f and c, we get

$$g = -\frac{7}{3}, \quad f = \frac{23}{6}, \quad c = -5$$

Thus the equation of the required circle is

$$3(x^2 + y^2) - 14x + 23y - 15 = 0.$$

2. Problem : *Find the equation of the circle which is orthogonal to each of the following three circles*

$$x^2 + y^2 + 2x + 17y + 4 = 0 \qquad \text{... (1)}$$

$$x^2 + y^2 + 7x + 6y + 11 = 0 \qquad \text{... (2)}$$

and

$$x^2 + y^2 - x + 22y + 3 = 0 \qquad \text{... (3)}$$

Solution : Let the equation of the required circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0 \qquad \text{... (4)}$$

Since this circle is orthogonal to (1), (2) and (3), by applying the condition of orthogonality given in 2.1.7, we have

$$2(g)(1) + 2(f)\left(\frac{17}{2}\right) = c + 4 \qquad \text{... (5)}$$

$$2(g)\left(\frac{7}{2}\right) + 2(f)(3) = c + 11 \qquad \text{... (6)}$$

and

$$2(g)\left(-\frac{1}{2}\right) + 2(f)(11) = c + 3 \qquad \text{... (7)}$$

Solving (5), (6) and (7) for $g,\ f,\ c$ we get $g = -3,\ f = -2$ and $c = -44$.

Thus the equation of the required circle is $x^2 + y^2 - 6x - 4y - 44 = 0$.

3. Problem : *If the straight line represented by*

$$x \cos \alpha + y \sin \alpha = p \qquad \text{... (1)}$$

intersects the circle

$$x^2 + y^2 = a^2 \qquad \text{... (2)}$$

at the points A and B, then show that the equation of the circle with $\overline{AB}$ *as diameter is*

$(x^2 + y^2 - a^2) - 2p(x \cos \alpha + y \sin \alpha - p) = 0.$

Solution : The equation of the circle passing through the points A and B is (by Theorem 2.1.10(i))

$$(x^2 + y^2 - a^2) + \lambda\,(x \cos \alpha + y \sin \alpha - p) = 0 \qquad \text{... (3)}$$

The centre of this circle is

$$\left(-\frac{\lambda \cos \alpha}{2}, -\frac{\lambda \sin \alpha}{2}\right).$$

If the circle given by (3) has $\overline{AB}$ as diameter then the centre of it must lie on (1).

$$\therefore \quad -\frac{\lambda\cos\alpha}{2}(\cos\alpha) - \frac{\lambda\sin\alpha}{2}(\sin\alpha) = p.$$

i.e., $$-\frac{\lambda}{2}(\cos^2\alpha + \sin^2\alpha) = p$$

i.e., $$\lambda = -2p.$$

Hence the equation of the required circle is

$$(x^2 + y^2 - a^2) - 2p\,(x\cos\alpha + y\sin\alpha - p) = 0.$$

4. Problem : *Find the equation of the circle passing through the points of intersection of the circles*

$$x^2 + y^2 - 8x - 6y + 21 = 0 \quad \text{... (1)}$$

$$x^2 + y^2 - 2x - 15 = 0 \quad \text{... (2)}$$

and (1, 2).

Solution : The equation of circle passing through the points of intersection of (1) and (2) is

$$(x^2 + y^2 - 8x - 6y + 21) + \lambda(x^2 + y^2 - 2x - 15) = 0 \quad \text{... (3)}$$

If it passes through (1, 2), we obtain

$$(1 + 4 - 8 - 12 + 21) + \lambda(1 + 4 - 2 - 15) = 0$$

i.e., $$6 + \lambda(-12) = 0$$

i.e., $$\lambda = 1/2$$

Hence the equation of the required circle is

$$(x^2 + y^2 - 8x - 6y + 21) + \frac{1}{2}(x^2 + y^2 - 2x - 15) = 0$$

i.e., $$3(x^2 + y^2) - 18x - 12y + 27 = 0.$$

Exercise 2(a)

I. 1. Find k if the following pairs of circles are orthogonal

(i) $x^2 + y^2 + 2by - k = 0,\quad x^2 + y^2 + 2ax + 8 = 0.$

(ii) $x^2 + y^2 - 6x - 8y + 12 = 0,\quad x^2 + y^2 - 4x + 6y + k = 0$

(iii) $x^2 + y^2 - 5x - 14y - 34 = 0,\quad x^2 + y^2 + 2x + 4y + k = 0$

(iv) $x^2 + y^2 + 4x + 8 = 0,\quad x^2 + y^2 - 16y + k = 0$

2. Find the angle between the circles given by the equations

   (i) $x^2 + y^2 - 12x - 6y + 41 = 0, \quad x^2 + y^2 + 4x + 6y - 59 = 0.$

   (ii) $x^2 + y^2 + 6x - 10y - 135 = 0, \quad x^2 + y^2 - 4x + 14y - 116 = 0.$

3. Show that the angle between the circles $x^2 + y^2 = a^2$, $x^2 + y^2 = ax + ay$ is $\dfrac{3\pi}{4}$.

4. Show that the circles given by the following equations intersect each other orthogonally

   (i) $x^2 + y^2 - 2x - 2y - 7 = 0, \; 3x^2 + 3y^2 - 8x + 29y = 0.$

   (ii) $x^2 + y^2 + 4x - 2y - 11 = 0, \quad x^2 + y^2 - 4x - 8y + 11 = 0$

   (iii) $x^2 + y^2 - 2x + 4y + 4 = 0, \quad x^2 + y^2 + 3x + 4y + 1 = 0$

   (iv) $x^2 + y^2 - 2lx + g = 0, \; x^2 + y^2 + 2my - g = 0$

II. 1. Find the equation of the circle which passes through the origin and intersects the circles below, orthogonally

   (i) $x^2 + y^2 - 4x + 6y + 10 = 0, \; x^2 + y^2 + 12y + 6 = 0.$

   (ii) $x^2 + y^2 - 4x - 6y - 3 = 0, \quad x^2 + y^2 - 8y + 12 = 0$

2. Find the equation of the circle which passes through the point (0, −3) and intersects the circles given by the equations $x^2 + y^2 - 6x + 3y + 5 = 0$ and $x^2 + y^2 - x - 7y = 0$ orthogonally.

3. Find the equation of the circle passing through the origin, having its centre on the line $x + y = 4$ and intersecting the circle $x^2 + y^2 - 4x + 2y + 4 = 0$ orthogonally.

4. Find the equation of the circle which passes through the points (2, 0), (0, 2) and orthogonal to the circle $2x^2 + 2y^2 + 5x - 6y + 4 = 0$.

5. Find the equation of the circle which cuts orthogonally the circle $x^2 + y^2 - 4x + 2y - 7 = 0$ and having the centre at (2, 3).

III. 1. Find the equation of the circle which intersects the circle $x^2 + y^2 - 6x + 4y - 3 = 0$ orthogonally and passes through the point (3, 0) and touches Y-axis.

2. Find the equation of the circle which cuts the circles $x^2 + y^2 - 4x - 6y + 11 = 0$ and $x^2 + y^2 - 10x - 4y + 21 = 0$ orthogonally and has the diameter along the straight line $2x + 3y = 7$.

3. If P, Q are conjugate points with respect to a circle $S \equiv x^2 + y^2 + 2gx + 2fy + c = 0$ then prove that the circle PQ as diameter cuts the circle S = 0 orthogonally.

4. If the equations of two circles whose radii are a, a' are $S = 0$ and $S' = 0$, then show that the circles $\frac{S}{a} + \frac{S'}{a'} = 0$ and $\frac{S}{a} - \frac{S'}{a'} = 0$ intersect orthogonally.

5. Find the equation of the circle which intersects each of the following circles orthogonally.

   (i) $x^2 + y^2 + 2x + 4y + 1 = 0$, $x^2 + y^2 - 2x + 6y - 3 = 0$, $2(x^2 + y^2) + 6x + 8y - 3 = 0$

   (ii) $x^2 + y^2 + 4x + 2y + 1 = 0$, $2(x^2 + y^2) + 8x + 6y - 3 = 0$, $x^2 + y^2 + 6x - 2y - 3 = 0$

6. If the straight line $2x + 3y = 1$ intersects the circle $x^2 + y^2 = 4$ at the points A and B, then find the equation of the circle having AB as diameter.

7. If $x + y = 3$ is the equation of the chord AB of the circle $x^2 + y^2 - 2x + 4y - 8 = 0$, find the equation of the circle having $\overline{AB}$ as diameter.

8. Find the equation of the circle passing through the intersection of the circles $x^2 + y^2 = 2ax$ and $x^2 + y^2 = 2by$ and having its centre on the line $\frac{x}{a} - \frac{y}{b} = 2$.

2.2 Radical axis of two circles

In this section we shall define the radical axis of two circles and study its properties. Also we discuss about the common chord, common tangent of two circles and the radical centre of three circles.

2.2.1 Definition

The radical axis of two circles is defined to be the locus of a point which moves so that its powers with respect to the two circles are equal.

2.2.2 Theorem : *If*

$$S \equiv x^2 + y^2 + 2gx + 2fy + c = 0 \quad \text{... (1)}$$

and

$$S' \equiv x^2 + y^2 + 2g'x + 2f'y + c' = 0 \quad \text{... (2)}$$

are two non-concentric circles, then the radical axis of (1) and (2) is a straight line represented by $S - S' = 0$.

i.e., $$2(g - g')x + 2(f - f')y + (c - c') = 0 \quad \text{... (3)}$$

Proof : Let $P(x_1, y_1)$ be a point on the radical axis. Then by the definition of radical axis, we have that the powers of $P(x_1, y_1)$ with respect to (1) and (2) are equal

$$x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = x_1^2 + y_1^2 + 2g'x_1 + 2f'y_1 + c'$$

($\because$ The power of $P(x_1, y_1)$ with respect to the circle $S = 0$ is S_{11})

i.e., $2(g-g')x_1 + 2(f-f')y_1 + (c-c') = 0$

Hence the equation of the locus of $P(x_1, y_1)$ is

$$2(g-g')x + 2(f-f')y + (c-c') = 0 \qquad ...(3)$$

Note that this equation represents a straight line, since the circles are non-concentric and therefore $g \neq g'$ or $f \neq f'$. Equation (3) can be written as $S - S' = 0$.

2.2.3 Note

(i) *For the concentric circles with distinct radii, the radical axis does not exist, since there is no point whose powers with respect to two distinct concentric circles are equal. However if their radii are equal then the locus is the whole plane.*

(ii) *While using the formula $S - S' = 0$ to find the equation of the radical axis, first reduce the equations of the circles to general form (if they are not in general form).*

(iii) *Whenever we consider the radical axis of two circles, it means that two circles are non-concentric.*

2.2.4 Examples

1. Example : Let us find the equation of the radical axis of the circles

$$S \equiv x^2 + y^2 - 5x + 6y + 12 = 0 \qquad ... (1)$$

and $$S' \equiv x^2 + y^2 + 6x - 4y - 14 = 0 \qquad ... (2)$$

The given equations of circles are in general form. Therefore their radical axis is $(S - S' = 0)$

i.e., $11x - 10y - 26 = 0.$

2. Example : Let us find the equation of the radical axis of the circles

$$2x^2 + 2y^2 + 3x + 6y - 5 = 0 \qquad ... (1)$$

and $$3x^2 + 3y^2 - 7x + 8y - 11 = 0 \qquad ... (2)$$

Here, the given equations of the circles are not in the general form. Reducing them into general form, we get

$$x^2 + y^2 + \frac{3}{2}x + 3y - \frac{5}{2} = 0,$$

and $$x^2 + y^2 - \frac{7}{3}x + \frac{8}{3}y - \frac{11}{3} = 0$$

Now the equation of radical axis of given circles is

$$\left(\frac{3}{2} + \frac{7}{3}\right)x + \left(3 - \frac{8}{3}\right)y + \left(-\frac{5}{2} + \frac{11}{3}\right) = 0$$

i.e., $23x + 2y + 7 = 0.$

2.2.5 Theorem : *The radical axis of any two circles is perpendicular to the line joining their centres.*

Proof : Let the equations of two non-concentric circles be

$$S \equiv x^2 + y^2 + 2gx + 2fy + c = 0 \qquad \text{... (1)}$$

and $$S' \equiv x^2 + y^2 + 2g'x + 2f'y + c' = 0 \qquad \text{... (2)}$$

Then $(-g, -f) \neq (-g', -f')$. The equation of the radical axis is

$$2(g - g')x + 2(f - f')y + c - c' = 0 \qquad \text{... (3)}$$

$\therefore$ The slope of the radical axis

$$= -\frac{(g-g')}{(f-f')}$$

The slope of the line joining the centres is

$$= \frac{-f' + f}{-g' + g} = \frac{f - f'}{g - g'}$$

Since (the slope of radical axis) $\times$ (slope of the line joining centres)

$$= -\frac{(g-g')}{(f-f')} \times \frac{(f-f')}{(g-g')} = -1,$$

the radical axis is perpendicular to the line joining the centres.

2.2.6 Theorem : *If the centres of any three circles are non-collinear, then the radical axes of each pair of the circles chosen from these three circles are concurrent.*

Proof : Let the equations of three circles (whose centres are not collinear) be

$$S \equiv x^2 + y^2 + 2gx + 2fy + c = 0 \qquad \text{... (1)}$$

$$S' \equiv x^2 + y^2 + 2g'x + 2f'y + c' = 0 \qquad \text{... (2)}$$

and $$S'' \equiv x^2 + y^2 + 2g''x + 2f''y + c'' = 0 \qquad \text{... (3)}$$

(see Fig. 2.3, the figure is drawn for the case of all centres lying in the first quadrant)

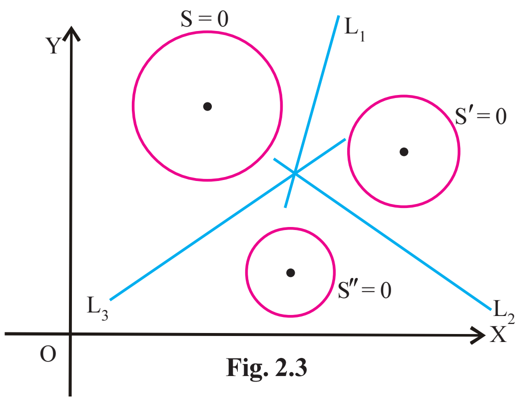


Fig. 2.3

The radical axis L_1(say) of (1) and (2) is

$$L_1 \equiv 2(g-g')x + 2(f-f')y + (c-c') = 0 \quad \text{... (4)}$$

Similarly the radical axis L_2 (say) of (2) and (3) is

$$L_2 \equiv 2(g'-g'')x + 2(f'-f'')y + (c'-c'') = 0 \quad \text{... (5)}$$

and the radical axis L_3(say) of (3) and (1) is

$$L_3 \equiv 2(g''-g)x + 2(f''-f)y + (c''-c) = 0 \quad \text{... (6)}$$

Now $L_1 + L_2 + L_3 = 0$ gives that L_1, L_2 and L_3 are concurrent.

2.2.7 Definition

The point of concurrence of the radical axes of each pair of the three circles whose centres are not collinear is called the radical centre.

2.2.8 Note

The lengths of tangents from the radical centre to these three circles are equal.

2.2.9 Example

Let us find the radical centre of the circles

$$x^2 + y^2 - 2x + 6y = 0 \quad \text{... (1)}$$

$$x^2 + y^2 - 4x - 2y + 6 = 0 \quad \text{... (2)}$$

and $$x^2 + y^2 - 12x + 2y + 3 = 0 \quad \text{... (3)}$$

The radical axis of (1) and (2); (2) and (3); (3) and (1) are respectively

$$x + 4y - 3 = 0 \quad \text{... (4)}$$

$$8x - 4y + 3 = 0 \quad \text{... (5)}$$

$$10x + 4y - 3 = 0 \quad \text{... (6)}$$

Solving (4) and (5) for the point of intersection we get $\left(0, \frac{3}{4}\right)$ which is the required radical centre. Observe that the coordinates of this point satisfies (6) also.

2.2.10 Theorem : *If the circle*

$$S \equiv x^2 + y^2 + 2gx + 2fy + c = 0 \quad \text{... (1)}$$

cuts each of the two circles

$$S' \equiv x^2 + y^2 + 2g'x + 2f'y + c' = 0 \quad \text{... (2)}$$

and $$S'' \equiv x^2 + y^2 + 2g''x + 2f''y + c'' = 0 \quad \text{... (3)}$$

orthogonally then the centre of $S = 0$ lies on the radical axis of $S' = 0$ and $S'' = 0$.

Proof : The radical axis of (2) and (3) is

$$2(g'-g'')x+2(f'-f'')y+(c'-c'') = 0 \qquad \text{... (4)}$$

We shall prove that $(-g, -f)$ (which is the centre of $S = 0$) lies on (4).

Since the circles (1) and (2); (1) and (3) are orthogonal, we have

$$2gg'+2ff' = c+c' \qquad \text{... (5)}$$

$$2gg''+2ff'' = c+c'' \qquad \text{... (6)}$$

Subtracting (6) from (5), we get

$$2(g'-g'')g+2(f'-f'')f = c'-c''$$

i.e., $$2(g'-g'')(-g)+2(f'-f'')(-f)+(c'-c'') = 0$$

Therefore $(-g, -f)$ lies on (4). Hence the centre of (1) lies on the radical axis of (2) and (3).

i.e., if any circle cuts two other circles orthogonally then the centre of the circle lies on the radical axis of other two circles.

2.2.11 Theorem : *The radical axis of two circles is*

(i) *the 'common chord' when the two circles intersect at two distinct points.*

(ii) *the 'common tangent' at the point of contact when the two circles touch each other.*

Proof : Let the equations of two circles be

$$S \equiv x^2+y^2+2gx+2fy+c = 0 \qquad \text{... (1)}$$

$$S' \equiv x^2+y^2+2g'x+2f'y+c' = 0 \qquad \text{... (2)}$$

and the radical axis of these circles be L, then

$$L \equiv S-S' = 2(g-g')x+2(f-f')y+c-c' = 0 \qquad \text{... (3)}$$

(i) Let the circles given by (1) and (2) intersect at two distinct points P and Q (see Fig. 2.4)

By Theorem 2.1.9, $S-S'=0$ is the common chord. Hence (i) is proved.

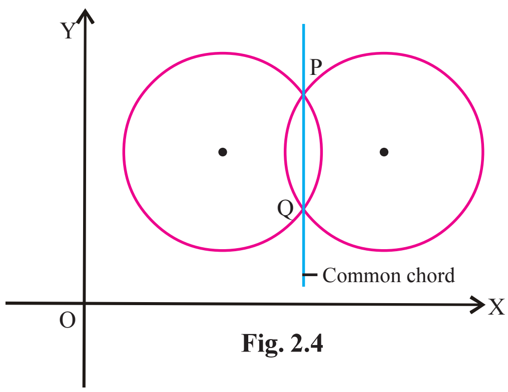


Fig. 2.4

(ii) Let the two circles given by (1) and (2) touch each other at P (see Fig. 2.5, 2.6)

By Theorem 2.1.9 (ii) the common tangent is $S - S' = 0$ i.e., $L = 0$. Hence (ii) is proved.

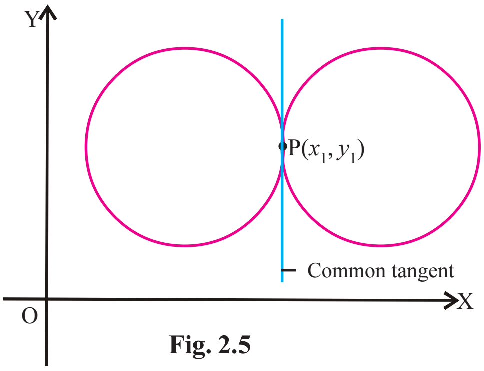


Fig. 2.5

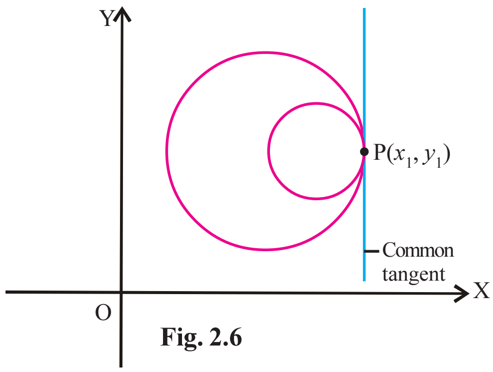


Fig. 2.6

2.2.12 Theorem : *The radical axis of any two circles (whose common tangent is not perpendicular to the line join of their centres) bisects the line joining the points of contact of common tangent to the circles.*

Proof : Let

$$S \equiv x^2 + y^2 + 2gx + 2fy + c = 0 \qquad \text{... (1)}$$

and

$$S' = x^2 + y^2 + 2g'x + 2f'y + c' = 0 \qquad \text{... (2)}$$

be two circles and T_1, T_2 be the points of contact of common tangent to the circles $S = 0$ and $S' = 0$ (see Fig. 2.7)

We know that radical axis of two circles is perpendicular to the line joining the centres of the circles (by Theorem 2.2.5) and the common tangent is not perpendicular to the line joining the centres (hypothesis). Therefore common tangent and radical axis intersect at a point.

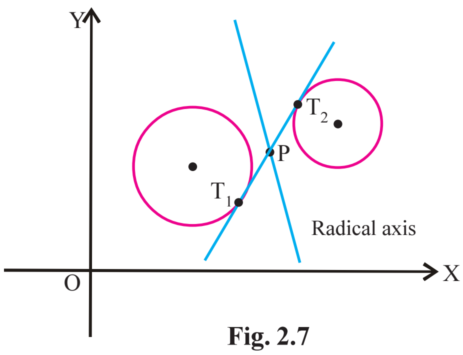


Fig. 2.7

Let $T_1 T_2$ (common tangent) intersect the radical axis of (1) and (2) at $P(x_1, y_1)$.

The powers of P with respect to the circles $S = 0$ and $S' = 0$ are equal. Therefore

$$PT_1 \, . \, PT_1 = PT_2 \, . \, PT_2 \qquad \text{(by Theorem 1.2.11)}$$

$$PT_1^2 = PT_2^2.$$

i.e.,
$$\overline{PT_1} = \overline{PT_2}$$

i.e., P is the mid point of T_1 and T_2. Thus the radical axis of the two circles bisects each of their common tangents (see Fig. 2.7)

2.2.13 Solved Problems

1. Problem : *Find the equation and length of the common chord of the two circles.*

$$S \equiv x^2 + y^2 + 3x + 5y + 4 = 0 \quad \text{... (1)}$$

and

$$S' \equiv x^2 + y^2 + 5x + 3y + 4 = 0 \quad \text{... (2)}$$

Solution: The common chord of two intersecting circles is their radical axis (by Theorem 2.2.11(i)).

$\therefore$ The equation of common chord is $S - S' = 0$.

i.e., $x - y = 0$ ... (3)

The centre of the circle (1) is

C_1 (say) $= \left(-\frac{3}{2}, -\frac{5}{2}\right)$ and radius $r_1 = \frac{3}{\sqrt{2}}$

(see Fig. 2.8)

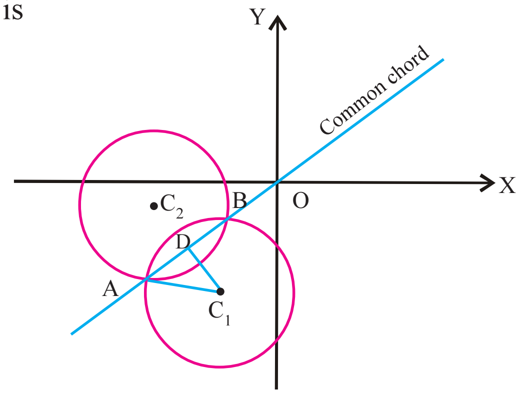


Fig. 2.8

C_1D = length of the perpendicular from C_1 to AB

$$= \frac{\left|-\frac{3}{2} - \left(-\frac{5}{2}\right)\right|}{\sqrt{(1)^2 + (-1)^2}}$$

$$= \frac{1}{\sqrt{2}}$$

Length of the common chord AB

$$= 2 \times AD$$

$$= 2\sqrt{AC_1^2 - C_1D^2}$$

$$= 2\sqrt{\frac{9}{2} - \frac{1}{2}}$$

$$= 4.$$

2. Problem : *Show that the circles*

$$S \equiv x^2 + y^2 - 2x - 4y - 20 = 0 \quad \text{... (1)}$$

and

$$S' \equiv x^2 + y^2 + 6x + 2y - 90 = 0 \quad \text{... (2)}$$

touch each other internally. Find their point of contact and the equation of common tangent.

Solution : Let C_1, C_2 be the centres and r_1, r_2 be the radii of the given circles (1) and (2). Then $C_1 = (1, 2)$; $C_2 = (-3, -1)$; $r_1 = 5$; $r_2 = 10$.

C_1C_2 = distance between the centres = 5

$|r_1 - r_2| = |5 - 10| = 5 = C_1C_2.$

$\therefore$ The given two circles touch internally. In this case, the common tangent is nothing but the radical axis (by Theorem 2.2.11(ii)). Therefore its equation is $S - S' = 0.$

i.e., $4x + 3y - 35 = 0$

Now we find the point of contact. The point of contact divides $\overline{C_1C_2}$ in the ratio 5 : 10 i.e., 1 : 2 (externally)

$\therefore$ Point of contact

$$= \left(\frac{(1)(-3) - 2(1)}{1-2}, \frac{(1)(-1) - 2(2)}{1-2} \right)$$

$= (5, 5).$

3. Problem : *Find the equation of the circle whose diameter is the common chord of the circles*

$$S \equiv x^2 + y^2 + 2x + 3y + 1 = 0 \quad \text{... (1)}$$

and

$$S' \equiv x^2 + y^2 + 4x + 3y + 2 = 0 \quad \text{... (2)}$$

Solution : Here the common chord is the radical axis of (1) and (2). The equation of the radical axis is $S - S' = 0.$

i.e., $2x + 1 = 0$ ...(3)

The equation of any circle passing through the points of intersection of (1) and (3) is $(S + \lambda L = 0)$

$$(x^2 + y^2 + 2x + 3y + 1) + \lambda(2x + 1) = 0$$

$$x^2 + y^2 + 2(\lambda + 1)x + 3y + (1 + \lambda) = 0 \quad \text{... (4)}$$

The centre of this circle is $\left(-(\lambda + 1), \frac{3}{2} \right).$

For the circle (4), $2x + 1 = 0$ is one chord. This chord will be a diameter of the circle (4) if the centre of (4) lies on (3).

$\therefore$ $2\{-(\lambda + 1)\} + 1 = 0$

$\Rightarrow$ $\lambda = -\frac{1}{2}.$

Thus the equation of the circle whose diameter is the common chord of (1) and (2) is (Put $\lambda = -\frac{1}{2}$ in equation (4))

$$2(x^2 + y^2) + 2x + 6y + 1 = 0.$$

2.2.14 Theorem : Let $S'=0,\ S''=0,\ S'''=0$ *be three circles whose centres are non-collinear and no two circles of these are intersecting then the circle having*

(i) *radical centre of these circles as the centre and*

(ii) *length of the tangent from the radical centre to any one of these three circles as radius cuts the given three circles orthogonally.*

Proof : Let C be the radical centre of three given circles. As no two circles of given three circles are intersecting the point C is exterior to these circles. Choose C as the origin. Let the equations of the three circles be

$$S' \equiv x^2+y^2+2g'x+2f'y+c'=0 \qquad \text{... (1)}$$

$$S'' \equiv x^2+y^2+2g''x+2f''y+c''=0 \qquad \text{... (2)}$$

$$S''' \equiv x^2+y^2+2g'''x+2f'''y+c'''=0 \qquad \text{... (3)}$$

Since, the origin is an external point to these circles,

we have c', c'', c''' are positive.

The lengths of the tangents from C to circle (1), (2) and (3) are

$\sqrt{c'}$ $\sqrt{c}$, $\sqrt{c}$ respectively

Since, these lengths are equal, we have $\sqrt{c'} = \sqrt{c''} = \sqrt{c'''} = r$ say ...(4)

Now, the equation of the circle with centre C and radius r is

$$x^2+y^2-r^2=0 \qquad \text{...(5)}$$

Then $2[g'.0+f'.0]=0=c'-r^2$ (using (4))

Hence, the circle (5) cuts the circle (1) orthogonally.

Similarly, the circle (5) cuts the circles (2) and (3) orthogonally.

Thus the circle having radical centre of three circles as the centre of the circle and having the length of tangent from the radical centre to one of these circles as radius cuts the given three circles orthogonally.

2.2.15 Example

Let us find the equation of a circle which cuts each of the following circles orthogonally

$$S' \equiv x^2 + y^2 + 3x + 2y + 1 = 0 \qquad \text{... (1)}$$

$$S'' \equiv x^2 + y^2 - x + 6y + 5 = 0 \qquad \text{... (2)}$$

and

$$S''' \equiv x^2 + y^2 + 5x - 8y + 15 = 0 \qquad \text{... (3)}$$

The centre of the required circle is radical centre of (1), (2) and (3) and the radius is the length of the tangent from this point to any one of the given three circles. First we shall find the radical centre. For, the radical axis of (1) and (2) is

$$x - y = 1 \qquad \text{... (4)}$$

and the radical axis of (2) and (3) is

$$3x - 7y = -5. \qquad \text{... (5)}$$

The point of intersection (3, 2) of (4) and (5) is the radical centre of the circles (1), (2) and (3).

The length of tangent from (3, 2) to the circle (1)

$$= \sqrt{3^2 + 2^2 + 3(3) + 2(2) + 1} = 3\sqrt{3}.$$

Thus the required circle is

$$(x-3)^2 + (y-2)^2 = (3\sqrt{3})^2$$

i.e.,

$$x^2 + y^2 - 6x - 4y - 14 = 0.$$

Exercise 2(b)

I. 1. Find the equation of the radical axis of the following circles.

(i) $x^2 + y^2 - 3x - 4y + 5 = 0$, $3(x^2 + y^2) - 7x + 8y - 11 = 0$

(ii) $x^2 + y^2 + 2x + 4y + 1 = 0$, $x^2 + y^2 + 4x + y = 0$

(iii) $x^2 + y^2 + 4x + 6y - 7 = 0$, $4(x^2 + y^2) + 8x + 12y - 9 = 0$

(iv) $x^2 + y^2 - 2x - 4y - 1 = 0$, $x^2 + y^2 - 4x - 6y + 5 = 0$

2. Find the equation of the common chord of the following pair of circles.

(i) $x^2 + y^2 - 4x - 4y + 3 = 0, \quad x^2 + y^2 - 5x - 6y + 4 = 0$

(ii) $x^2 + y^2 + 2x + 3y + 1 = 0, \quad x^2 + y^2 + 4x + 3y + 2 = 0$

(iii) $(x - a)^2 + (y - b)^2 = c^2, \quad (x - b)^2 + (y - a)^2 = c^2 \ (a \neq b)$

II. 1. Find the equation of the common tangent of the following circles at their point of contact

(i) $x^2 + y^2 + 10x - 2y + 22 = 0, \quad x^2 + y^2 + 2x - 8y + 8 = 0$

(ii) $x^2 + y^2 - 8y - 4 = 0, \quad x^2 + y^2 - 2x - 4y = 0$

2. Show that the circles $x^2 + y^2 - 8x - 2y + 8 = 0$ and $x^2 + y^2 - 2x + 6y + 6 = 0$ touch each other and find the point of contact.

3. If the two circles $x^2 + y^2 + 2gx + 2fy = 0$ and $x^2 + y^2 + 2g'x + 2f'y = 0$ touch each other then show that $f'g = fg'$.

4. Find the radical centre of the following circles.

(i) $x^2 + y^2 - 4x - 6y + 5 = 0, \quad x^2 + y^2 - 2x - 4y - 1 = 0, \quad x^2 + y^2 - 6x - 2y = 0$

(ii) $x^2 + y^2 + 4x - 7 = 0, \quad 2x^2 + 2y^2 + 3x + 5y - 9 = 0, \quad x^2 + y^2 + y = 0$

III. 1. Show that the common chord of the circles $x^2 + y^2 - 6x - 4y + 9 = 0$ and $x^2 + y^2 - 8x - 6y + 23 = 0$ is the diameter of the second circle and also find its length.

2. Find the equation and length of the common chord of the following circles.

(i) $x^2 + y^2 + 2x + 2y + 1 = 0, \quad x^2 + y^2 + 4x + 3y + 2 = 0$

(ii) $x^2 + y^2 - 5x - 6y + 4 = 0, \quad x^2 + y^2 - 2x - 2 = 0$

3. Prove that the radical axis of the circles $x^2 + y^2 + 2gx + 2fy + c = 0$ and $x^2 + y^2 + 2g'x + 2f'y + c' = 0$ is the diameter of the latter circle (or the former bisects the circumference of the latter) if $2g'(g - g') + 2f'(f - f') = c - c'$.

4. Show that the circles $x^2 + y^2 + 2ax + c = 0$ and $x^2 + y^2 + 2by + c = 0$ touch each other if $\dfrac{1}{a^2} + \dfrac{1}{b^2} = \dfrac{1}{c}$.

5. Show that the circles $x^2 + y^2 - 2x = 0$ and $x^2 + y^2 + 6x - 6y + 2 = 0$ touch each other. Find the coordinates of the point of contact. Is the point of contact external or internal?

6. Find the equation of the circle which cuts the following circles orthogonally

(i) $x^2 + y^2 + 4x - 7 = 0, \quad 2x^2 + 2y^2 + 3x + 5y - 9 = 0, \quad x^2 + y^2 + y = 0$

(ii) $x^2 + y^2 + 2x + 4y + 1 = 0, \quad 2x^2 + 2y^2 + 6x + 8y - 3 = 0, \quad x^2 + y^2 - 2x + 6y - 3 = 0$

(iii) $x^2 + y^2 + 2x + 17y + 4 = 0, \quad x^2 + y^2 + 7x + 6y + 11 = 0, \quad x^2 + y^2 - x + 22y + 3 = 0$

(iv) $x^2 + y^2 + 4x + 2y + 1 = 0, \quad 2(x^2 + y^2) + 8x + 6y - 3 = 0, \quad x^2 + y^2 + 6x - 2y - 3 = 0$

Key Concepts

- We denote $x^2+y^2+2gx+2fy+c$ by S and $x^2+y^2+2g'x+2f'y+c'$ by S′
- If C_1, C_2 are the centres and r_1, r_2 are radii of two intersecting circles S = 0 and S′ = 0, $C_1C_2 = d$ and θ is the angle between them, then $\cos\theta = \dfrac{d^2-r_1^2-r_2^2}{2r_1r_2}$
- If θ is the angle between the two intersecting circles S = 0 and S′ = 0 then
$$\cos\theta = \frac{c+c'-2gg'-2ff'}{2\sqrt{g^2+f^2-c}\sqrt{g'^2+f'^2-c'}}$$
- Two circles S = 0 and S′ = 0 are orthogonal iff $2(gg'+ff') = c+c'$.
- If S = 0, S′ = 0 are any two intersecting circles and λ, μ are any two real numbers such that $\lambda+\mu \neq 0$ then $\lambda S+\mu S' = 0$ represents a circle passing through the intersection of the circles S = 0, S′ = 0.
- If S = 0, S′ = 0 are any two intersecting circles and k is any real number where $k \neq -1$, then S + kS′ = 0 represents a circle passing through the points of intersection of them.
- If S = 0 and a straight line L = 0 intersect then for any real number k, S + kL = 0 represents a circle passing through their intersection.
- The equation of the common chord of two intersecting circles S = 0, S′ = 0 is S – S′ = 0.
- The equation of common tangent at the point of contact when the circles S = 0, S′ = 0 touch each other is S – S′ = 0.
- The radical axis of two circles is defined to be the locus of a point which moves so that its powers with respect to the two circles are equal.
- The radical axis of S = 0 and S′ = 0 is S – S′ = 0.
- If the centres of any three circles are non-collinear, then the radical axes of each pair of circles chosen from these three circles are concurrent.
- The radical axis of two circles S = 0 and S′ = 0 is
  (i) the common chord when the two circles intersect at two distinct points.
  (ii) the common tangent at the point of contact when the circles touch each other.
- The radical axis of any two circles bisects the line segment joining the points of contact of common tangent of these two circles.

Historical Note

Ptolemy believed in the geocentric theory of revolving universe and stated that the other heavenly bodies revolved in epicycles and small circles.

The study of circles goes back beyond history. The invention of the wheel was a fundamental discovery of the properties of a circle.

The first theorems relating to circles are attributed to *Thales* (624 - 547 B.C.). Book III of Euclid's **elements** deal with properties of circles and related properties.

In India Sulbasutras (First Millennium B.C.) contain a discussion of circles.

Answers

Exercise 2(a)

I. 1. (i) 8 (ii) −24 (iii) 1 (iv) −8

2. (i) $\frac{\pi}{4}$ (ii) $\frac{2\pi}{3}$

II. 1. (i) $2(x^2+y^2)-7x+2y=0$ (ii) $x^2+y^2+6x-3y=0$

2. $3(x^2+y^2)+2x+4y-15=0$

3. $x^2+y^2-4x-4y=0$

4. $7(x^2+y^2)-8x-8y-12=0$

5. $x^2+y^2-4x-6y+9=0$

III. 1. $x^2+y^2-6x-6y+9=0$

2. $x^2+y^2-4x-2y+3=0$

5. (i) $x^2+y^2-5x-14y-34=0$ (ii) $x^2+y^2-14x-5y-34=0$

6. $13(x^2+y^2)-4x-6y-50=0$

7. $x^2+y^2-6x+4=0$

8. $x^2+y^2-3ax+by=0$

Exercise 2(b)

I. 1. (i) $x+10y-13=0$ (ii) $2x-3y-1=0$

(iii) $8x+12y-19=0$ (iv) $x+y-3=0$

2. (i) $x + 2y - 1 = 0$ (ii) $2x + 1 = 0$

(iii) $x - y = 0$

II. 1. (i) $4x + 3y + 7 = 0$ (ii) $x - 2y - 2 = 0$

2. $\left(\frac{11}{5}, -\frac{7}{5}\right)$

4. (i) (7/6, 11/6) (ii) (2, 1)

III. 1. $2\sqrt{2}$

2. (i) $2x + y + 1 = 0, \frac{2}{\sqrt{5}}$ (ii) $x + 2y - 2 = 0,\ 2\sqrt{\frac{14}{5}}$.

5. $\left(\frac{1}{5}, \frac{3}{5}\right)$, internal.

6. (i) $x^2 + y^2 - 4x - 2y - 1 = 0$ (ii) $x^2 + y^2 - 5x - 14y - 34 = 0$

(iii) $x^2 + y^2 - 6x - 4y - 44 = 0$ (iv) $x^2 + y^2 - 14x - 5y - 34 = 0$

Chapter 3

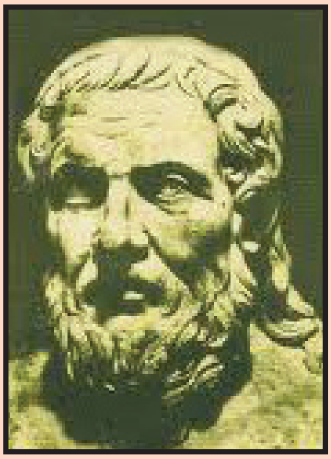
Parabola

"The universe cannot be read until we have learnt the language and become familiar with the characters in which it is written. It is written in mathematical language and the letters are triangles, circles and other geometrical figures"

- Galilei Galilieo

Introduction

In the preceding chapters, we have studied various forms of the equations of circles and systems of circles. In this chapter we shall study about parabola. The name "parabola" (the shape described when you throw a ball in air) was given by Apollonius (Ca. 262 B.C. - Ca. 190 B.C.).



Apollonius
(ca. 262 - 190 B.C.)

Apollonius *was born in ca. 262 B.C., some 25 years after Archimedes. He flourished in the reigns of Ptolemy Euergetes and Ptolemy Philopator (247-205 B.C). His treatise on conics earned him fame as "The Great Geometer", an achievement that has assured his fame for ever.*

3.1 Conic Sections

In fact circle, parabola, ellipse, hyperbola, a pair of a straight lines, a straight line and a point are called as conic sections because each is a section of a double napped right circular cone with a plane. These curves have a very wide range of applications in planetary motion, design of telescopes and antennas, reflectors in flash lights etc.

More generally, suppose the cutting plane makes an angle 'β' with the axis of the cone and suppose the generating angle of the cone is α. Then the section is

(i) a circle if $\beta = \frac{\pi}{2}$, (Fig. 3.1(a))

(ii) an ellipse if $\alpha < \beta < \frac{\pi}{2}$, (Fig. 3.1(b))

(iii) a parabola if $\alpha = \beta$, (Fig. 3.1(c))

(iv) a hyperbola if $0 \leq \beta < \alpha$, (Fig. 3.1(d))

(v) **Degenerated conic sections**

We get the degenerated sections when the plane passes through the vertex of the cone.

(a) a point when $\alpha < \beta \leq \frac{\pi}{2}$, (Fig. 3.1(e))

(b) a straight line when $\beta = \alpha$, (Fig. 3.1(f)) a generator of the cone.

(c) a pair of intersecting straight lines when $0 \leq \beta < \alpha$, (Fig. 3.1(g), 3.1(h))
It is the degenerated case of a hyperbola.

Note : A pair of parallel straight lines, however, is not a conic section as there is no plane which cuts the cone along two parallel lines.

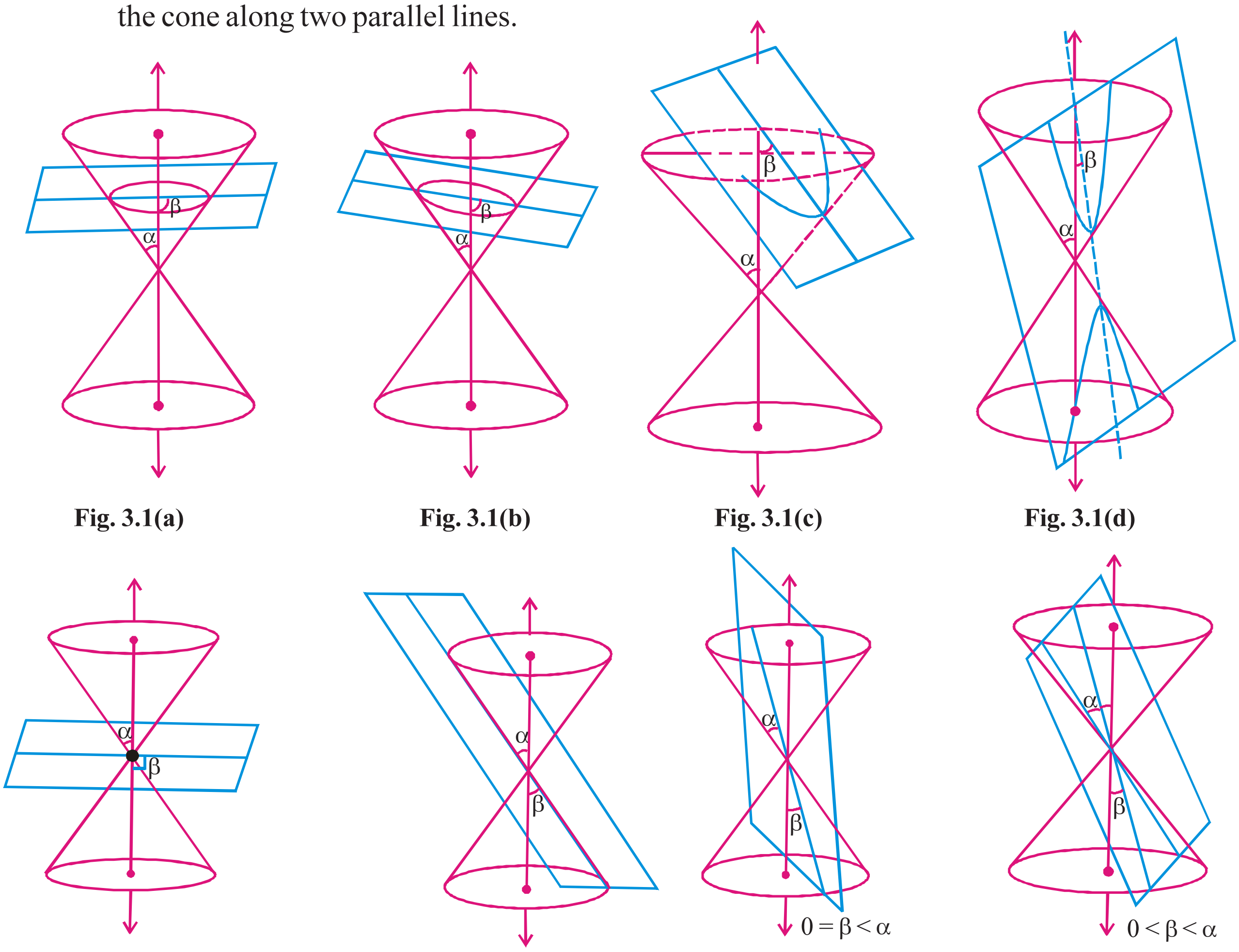


Fig. 3.1(a) **Fig. 3.1(b)** **Fig. 3.1(c)** **Fig. 3.1(d)**

Fig. 3.1(e) **Fig. 3.1(f)** **Fig. 3.1(g)** **Fig. 3.1(h)**

Fig. 3.1

A conic section, can also be defined as the locus of a point moving on a plane such that its distances from a fixed point and a fixed straight line are in constant ratio.

It can be proved that these two approaches to define a conic section (as plane section of a cone and as locus) are equivalent. But it is beyond the scope of this book. Further, in view of the analytic approach of the second definition, we shall adopt the same throughout this book.

3.1.1 Conic

The locus of a point moving on a plane such that its distances from a fixed point and a fixed straight line in the plane are in a constant ratio 'e', is called a conic.

The fixed point is called the **focus** and is usually denoted by S.

The fixed straight line is called the **directrix**.

The constant ratio 'e' is called the **eccentricity**.

The straight line of the plane passing through the focus and perpendicular to the directrix is called the **axis**.

Therefore the locus of a point P moving on a plane such that $\frac{SP}{PM} = e$ (constant) where PM is the perpendicular distance from P to the directrix, is called a conic

If $e = 1$, the conic is called a **parabola**.

If $0 < e < 1$, the conic is called an **ellipse**.

If $e > 1$, the conic is called a **hyperbola**.

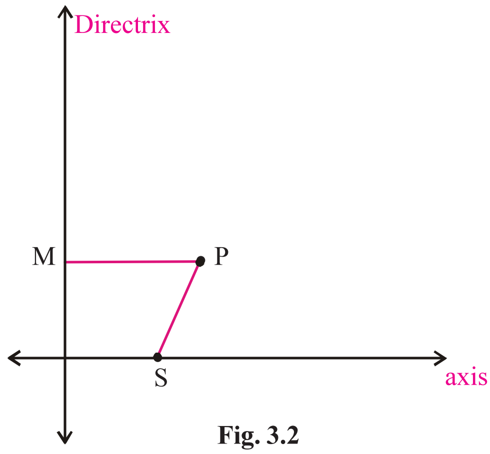


Fig. 3.2

3.1.2 Equation of a parabola

In this section we derive the equation of a parabola in the general form.

Let S(α, β) be the focus and the directrix be $lx + my + n = 0$. Thus, by definition of the parabola, the equation of the parabola is

$$\sqrt{(x-\alpha)^2 + (y-\beta)^2} = \frac{|lx+my+n|}{\sqrt{l^2+m^2}}$$

(or)

$$(x-\alpha)^2 + (y-\beta)^2 = \frac{(lx+my+n)^2}{l^2+m^2},$$

a general equation of second degree in x and y.

The equation of the axis of the above parabola is $m(x-\alpha) - l(y-\beta) = 0$.

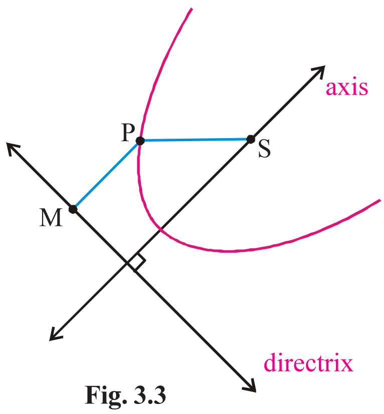


Fig. 3.3

3.1.3 Equation of a parabola in standard form

To study the nature of the curve, we prefer its equation in the simplest possible form. We proceed as follows to derive such an equation.

Let S be the focus, l be the directrix as shown in Fig. 3.4. Let Z be the projection of 'S' on l and 'A' be the midpoint of $\overline{SZ}$. A lies on the parabola because SA = AZ. A is called the *vertex* of the parabola. Let $\overline{YAY'}$ be the straight line through A and parallel to the directrix. Now take $\overleftrightarrow{ZX}$ as the X-axis and $\overleftrightarrow{YY'}$ as the Y-axis.

Then A is the origin (0, 0). Let S = $(a, 0)$, $(a > 0)$. Then Z = $(-a, 0)$ and the equation of the directrix is $x + a = 0$.

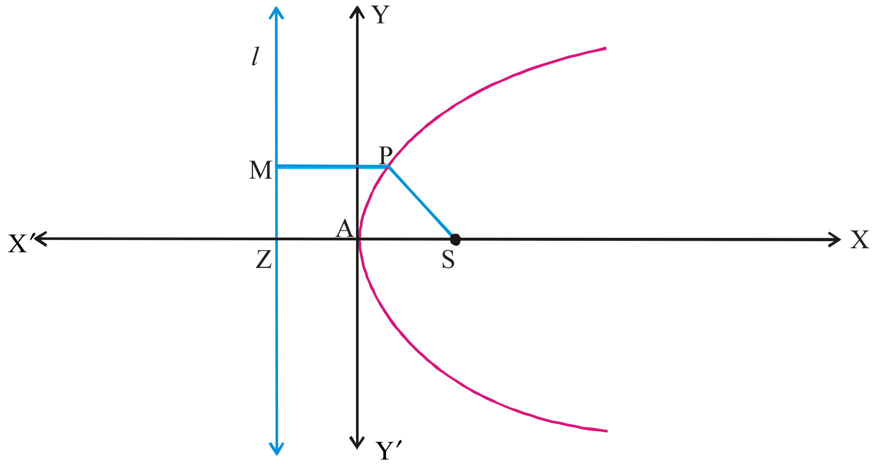


Fig. 3.4

If P(x, y) is a point on the parabola and PM is the perpendicular distance from P to the directrix l, then $\dfrac{SP}{PM} = e = 1$.

$$\therefore \quad (SP)^2 = (PM)^2$$

$$\Rightarrow \quad (x-a)^2 + y^2 = (x+a)^2$$

$$\therefore \quad y^2 = 4ax.$$

Conversely if P(x, y) is any point such that $y^2 = 4ax$ then

$$SP = \sqrt{(x-a)^2 + y^2} = \sqrt{x^2 + a^2 - 2ax + 4ax} = \sqrt{(x+a)^2} = |x + a| = PM.$$

Hence P(x, y) is on the locus. In other words, a necessary and sufficient condition for the point P(x, y) to be on the parabola is that $y^2 = 4ax$.

Thus the equation of the parabola is $y^2 = 4ax$.

3.1.4 Remark

(i) If the focus is situated on the left side of the directrix, the equation of the parabola with the vertex as origin and the axis as X-axis is $y^2 = -4ax$ [Since in this case the focus S is $(-a, 0)$].

(ii) The vertex being the origin, if the axis of the parabola is taken as Y-axis, equation of parabola is $x^2 = 4ay$ or $x^2 = -4ay$ according as the focus is above (or) below the X-axis.

(iii) If S lies on l, then the locus is a straight line passing through S and perpendicular to l. We take this case as the degenerated parabola.

3.1.5 Nature of the curve

In this section we shall study the nature of the parabola or trace the curve represented by the equation $y^2 = 4ax$, $(a > 0)$.

(i) If $y = 0$, then $4ax = 0$ and $x = 0$.

$\therefore$ The curve passes through the origin (0, 0).

(ii) If $x = 0$ then $y^2 = 0$, which gives $y = 0, 0$ (twice). Hence Y-axis is a tangent to the parabola at the origin.

(iii) Let $P(x, y)$ be any point on the parabola. Since $a > 0$ and $y^2 = 4ax$, we have $x \geq 0$ and $y = \pm\sqrt{4ax}$.

$\therefore$ For any positive real value of x, we obtain two values of y of equal magnitude but of opposite in signs. This shows that the curve is symmetric about X-axis and lies in first and fourth quadrants.

The curve does not exist on the left side of the Y-axis (i.e., second and the third quadrants) since $x \geq 0$ for any point (x, y) on the parabola.

(iv) As x increases infinitely, the two values of y also increase infinitely in magnitude. So the two branches of the parabola lying on opposite sides of the X-axis extended to infinity towards the positive direction of the X-axis. Hence it is an open curve.

3.1.6 Note

(i) As noted earlier, S is called the focus and the line l is called the directrix of the parabola. For the parabola $y^2 = 4ax$ $(a > 0)$, the focus is $S(a, 0)$, directrix is $x + a = 0$ and axis is $y = 0$. The point $A(0, 0)$ is called vertex of the parabola.

(ii) If the vertex is at (h, k) and the axis of the parabola parallel to X-axis, then by shifting the origin to (h, k) by translation of axis and using the result $y^2 = 4ax$ we can obtain its equation as $(y - k)^2 = 4a(x-h)$.

3.1.7 Definitions (Chord, focal chord, double ordinate and latus rectum)

Chord : *The line joining two points of a parabola is called a* ***'chord'*** *of a parabola.*

Focal chord : *A chord passing through focus is called a* ***'focal chord'***.

Double ordinate : *A chord through a point P on the parabola, which is perpendicular to the axis of the parabola, is called the* ***'double ordinate'*** *of the point P.*

Latus rectum : *The double ordinate passing through the focus is called the* ***'latus rectum'*** *of the parabola.*

3.1.8 Remark

From $y^2 = 4ax$, for any positive x, $P(x, 2\sqrt{ax})$, $P'(x, -2\sqrt{ax})$ are points on the parabola $y^2 = 4ax$ and PP' is perpendicular to the axis and hence is the double ordinate through P.

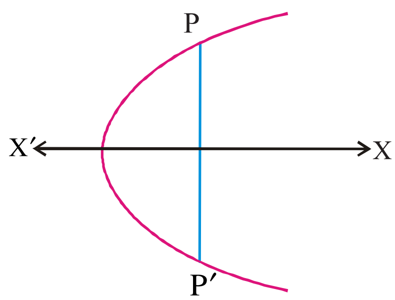


Fig. 3.5

3.1.9 Length of the latus rectum

The equation of the parabola is $y^2 = 4ax$ ... (1)

Let LS L′ be the latus rectum of the parabola (Fig. 3.6)

Let SL = l, then coordinates of L are (a, l)

L lies on the parabola (1)

$\therefore\ l^2 = 4a \cdot a = 4a^2.\quad \therefore\ l = 2a$ and so, LS L′ = 2(SL) = $4a$, which is the length of the latus rectum.

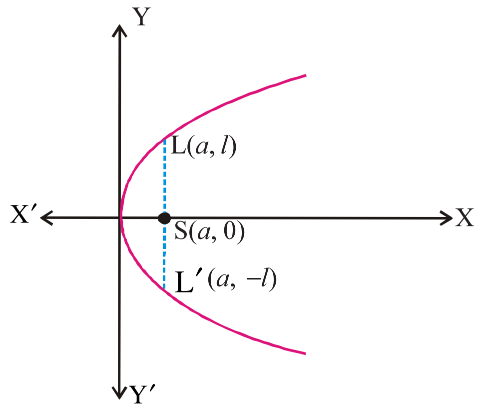


Fig. 3.6

3.1.10 Note : When latus rectum is known, the equation of the parabola is known in its standard form, and the size and shape of the curve are determined accordingly.

3.1.11 Various forms of the parabola

(i) $y^2 = 4ax\ (a > 0)$ (Fig. 3.7)

Focus = $(a, 0)$.

Equation of the directrix $x + a = 0$.

Axis of the parabola $y = 0$.

Vertex = (0, 0).

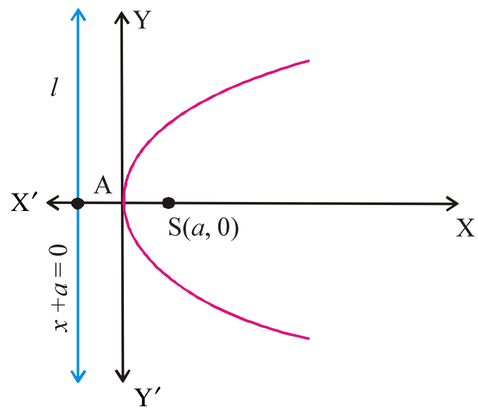


Fig. 3.7

(ii) $y^2 = -4ax (a > 0)$ (Fig. 3.8)

Focus = $(-a, 0)$.

Equation of the directrix $x - a = 0$.

Axis of the parabola : $y = 0$.

Vertex = (0, 0).

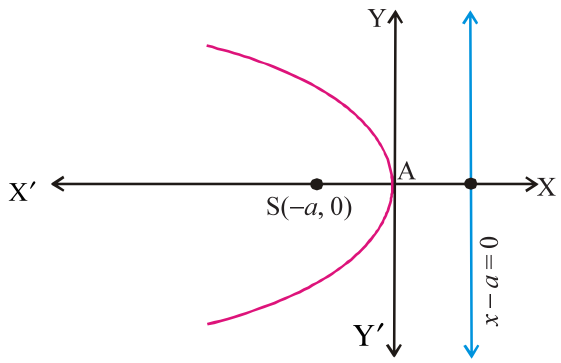


Fig. 3.8

(iii) $x^2 = 4ay (a > 0)$ (Fig. 3.9)

Focus $= (0, a)$.

Equation of the directrix $y + a = 0$.

Axis of the parabola $x = 0$.

Vertex $= (0, 0)$.

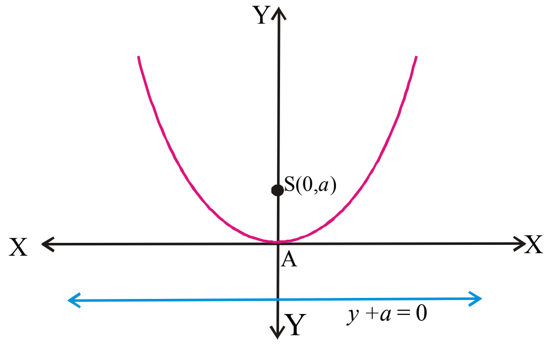


Fig. 3.9

(iv) $x^2 = -4ay (a > 0)$ (Fig. 3.10)

Focus $= (0, -a)$.

Equation of the directrix $y - a = 0$.

Axis of the parabola $x = 0$.

Vertex $= (0, 0)$.

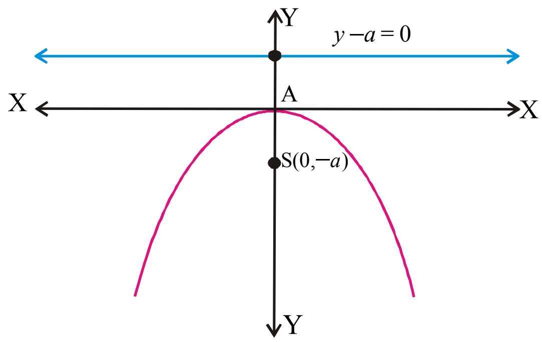


Fig. 3.10

(v) $(y - k)^2 = 4a(x - h)$ $(a > 0)$ (Fig. 3.11)

Focus $= (h + a, k)$.

Equation of the directrix $x - h + a = 0$.

Axis of the parabola $y - k = 0$.

Vertex $= (h, k)$.

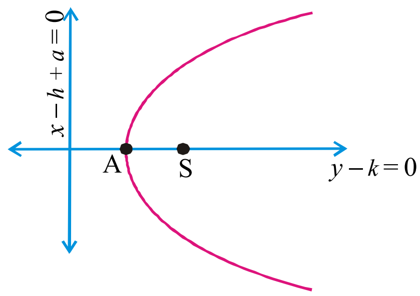


Fig. 3.11

(vi) $(y - k)^2 = -4a(x - h)$ $(a > 0)$ (Fig. 3.12)

Focus $= (h - a, k)$.

Equation of the directrix $x - h - a = 0$.

Axis of the parabola $y - k = 0$.

Vertex $= (h, k)$.

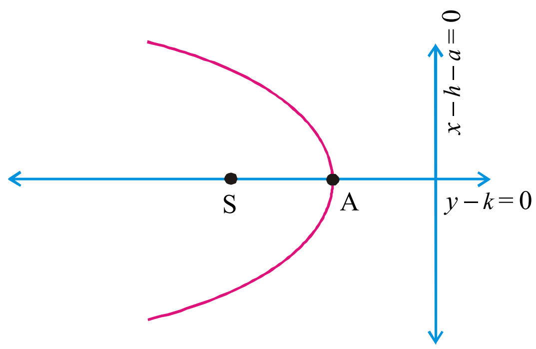


Fig. 3.12

(vii) $(x-h)^2 = 4a(y-k)\ (a>0)$, (Fig. 3.13)

Focus $= (h,\ a+k)$.

Equation of the directrix $y-k+a=0$.

Axis of the parabola $x-h=0$.

Vertex $= (h, k)$.

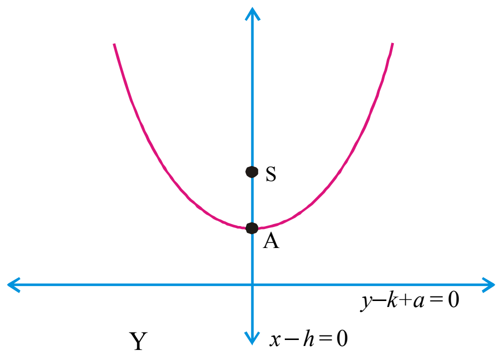


Fig. 3.13

(viii) $(x-h)^2 = -4a(y-k)\ (a>0)$, (Fig. 3.14)

Focus $= (h,\ k-a)$.

Equation of the directrix $y-k-a=0$.

Axis of the parabola $x-h=0$.

Vertex $= (h, k)$.

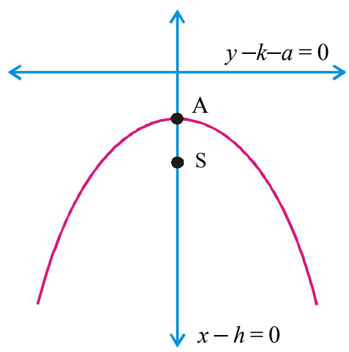


Fig. 3.14

(ix) $(x-\alpha)^2 + (y-\beta)^2 = \dfrac{(\ x+my+n)^2}{^2+m^2}$, (Fig. 3.15)

Focus $= (\alpha, \beta)$.

Equation of the directrix $lx+my+n=0$.

Axis of the parabola $m(x-\alpha) - l(y-\beta) = 0$.

Vertex : Point A in Fig. 3.15.

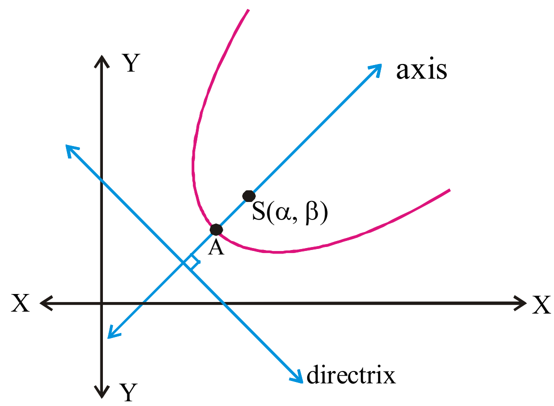


Fig. 3.15

3.1.12 Note

By observing equations of the above figures (i) to (viii), we may conclude that the equation of a parabola

(i) whose axis is parallel to the X-axis is $x = ly^2 + my + n$.

(ii) whose axis is parallel to the Y-axis is $y = lx^2 + mx + n$, where l, m, n are real numbers, l 0.

3.1.13 Definition (Focal distance)

*The distance of a point on the parabola from its focus is called the **'focal distance'** of the point.*

If $P(x_1, y_1)$ is a point on the parabola $y^2 = 4ax$ whose focus is $S(a, 0)$ then from Fig. 3.16

$$\begin{aligned} \text{Focal distance of P} &= SP \\ &= PM \\ &= NZ \\ &= NA + AZ \\ &= x_1 + a. \end{aligned}$$

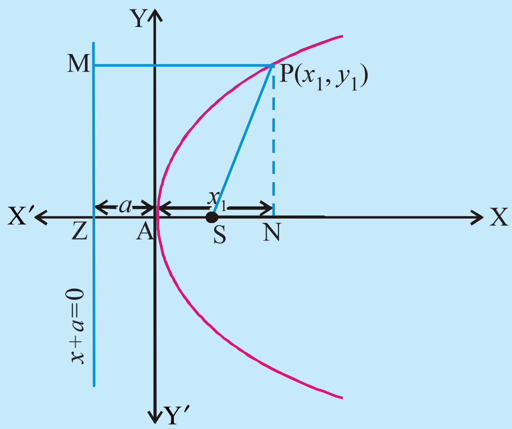


Fig. 3.16

3.1.14 Parametric equations of a parabola

The point $(at^2, 2at)$ satisfies the equation $y^2 = 4ax$ of a parabola for all real values of t. Conversely if $P(x, y)$ is a point on $y^2 = 4ax (a > 0)$ then $x \geq 0$, $a > 0$ there exist a $t \in \mathbf{R}$ such that $x = at^2$ and $y^2 = 4a(at^2) = 4a^2t^2$ then we get $y = 2at$ or $-2at$. Therefore P is of the form $(at^2, 2at)$ or $(a(-t)^2, 2a(-t))$.

Hence the parametric equations of a parabola are $x = at^2$, $y = 2at$. The point $P(at^2, 2at)$ is generally denoted by the point t or $P(t)$ for the sake of brevity.

3.1.15 Notation

Hereafter the following notation will be adapted throughout this chapter.

(i) $S \equiv y^2 - 4ax$

(ii) $S_1 \equiv yy_1 - 2a(x + x_1)$

(iii) $S_{12} \equiv y_1y_2 - 2a(x_1 + x_2)$

(iv) $S_{11} \equiv y_1^2 - 4ax_1$, where (x_1, y_1) and (x_2, y_2) are the points in the plane of the parabola $y^2 = 4ax$.

3.1.16 Parabola and a point in the plane of the parabola

A parabola divides the plane into two disjoint parts, one containing the focus is called the interior of the parabola and the other is called the exterior of the parabola (Fig. 3.17).

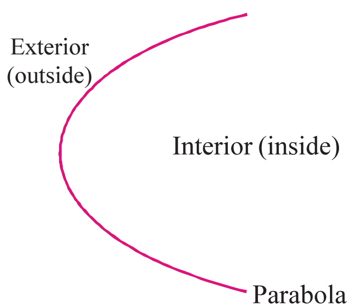


Fig. 3.17

Let $P(x_1, y_1)$ be a point in the plane of the parabola (Fig. 3.18). Draw PM perpendicular to the X-axis to meet the parabola $y^2 = 4ax$ at $Q = (x_1, 2\sqrt{ax_1})$ and $M(x_1, 0)$.

$\therefore\ (PM)^2 = y_1^2,\ (MQ)^2 = 4ax_1.$

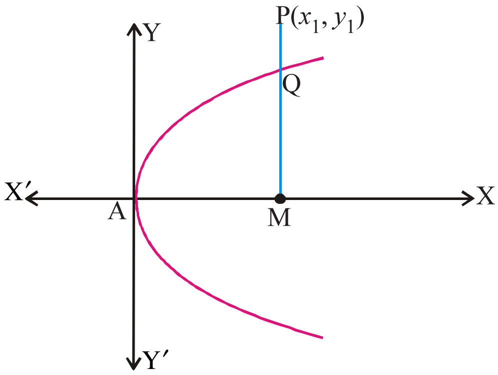


Fig. 3.18

(i) P lies outside the parabola (i.e., P is an external point) $\Leftrightarrow (MP)^2 > (MQ)^2$

$$\Leftrightarrow y_1^2 > 4ax_1 \quad \Leftrightarrow y_1^2 - 4ax_1 > 0 \Leftrightarrow S_{11} > 0.$$

If $x_1 < 0$, then the point P lies in Quadrant II or in Quadrant III in which case the point P clearly lies outside the parabola and $y_1^2 - 4ax_1 > 0$ in this case also.

(ii) P lies on the parabola $\Leftrightarrow MP = MQ \Leftrightarrow (MP)^2 = (MQ)^2$

$$\Leftrightarrow y_1^2 = 4ax_1 \quad \Leftrightarrow y_1^2 - 4ax_1 = 0 \Leftrightarrow S_{11} = 0.$$

(iii) P lies inside the parabola (i.e., P is an internal point) $\Leftrightarrow MP < MQ \Leftrightarrow (MP)^2 < (MQ)^2$

$$\Leftrightarrow y_1^2 < 4ax_1 \Leftrightarrow y_1^2 - 4ax_1 < 0 \Leftrightarrow S_{11} < 0.$$

Thus $P(x_1, y_1)$ lies outside, on or inside the parabola $S \equiv y^2 - 4ax = 0$ according as $S_{11} \gtreqless 0$.

3.1.17 Solved Problems

1. Problem : *Find the coordinates of the vertex and focus, and the equations of the directrix and axes of the following parabolas.*

(i) $y^2 = 16x$ (ii) $x^2 = -4y$ (iii) $3x^2 - 9x + 5y - 2 = 0$ (iv) $y^2 - x + 4y + 5 = 0.$

Solution

(i) $y^2 = 16x$, comparing with $y^2 = 4ax$, we get $4a = 16 \Rightarrow a = 4$.

The coordinates of the vertex = (0, 0).

The coordinates of the focus = $(a, 0) = (4, 0)$.

Equation of the directrix : $x + a =$ i.e., $x + 4 = 0$.

Axis of the parabola $y = 0$.

(ii) $x^2 = -4y$, comparing with $x^2 = -4ay$, we get $4a = 4 \Rightarrow a = 1$.

The coordinates of the vertex = (0, 0).

The coordinates of the focus = $(0, -a) = (0, -1)$.

The equation of the directrix $y - a = 0$ i.e., $y - 1 = 0$.

Equation of the axis $x = 0$.

(iii) $3x^2 - 9x + 5y - 2 = 0.$

$3(x^2 - 3x) = 2 - 5y \Rightarrow 3(x^2 - 2x(\frac{3}{2}) + \frac{9}{4}) = 2 - 5y + \frac{27}{4}.$

$\left(x - \frac{3}{2}\right)^2 = -\frac{5}{3}\left(y - \frac{7}{4}\right)$, comparing with $(x-h)^2 = -4a(y-k)$ we get

$a = \frac{5}{12}, h = \frac{3}{2}, k = \frac{7}{4}.$

$\therefore$ Coordinates of the vertex $= (h, k) = \left(\frac{3}{2}, \frac{7}{4}\right).$

Coordinates of the focus $= (h, k - a) = \left(\frac{3}{2}, \frac{7}{4} - \frac{5}{12}\right) = \left(\frac{3}{2}, \frac{4}{3}\right).$

Equation of the directrix is $y - k - a = 0$ i.e., $6y - 13 = 0.$

Equation of the axis is $x - h = 0$ i.e., $2x - 3 = 0.$

(iv) $y^2 - x + 4y + 5 = 0 \Rightarrow (y-(-2))^2 = (x-1)$, comparing with $(y-k)^2 = 4a(x-h)$, we get $(h, k) = (1, -2)$ and $a = \frac{1}{4}$, coordinates of the vertex $(h, k) = (1, -2)$

coordinates of the focus $(h + a, k) = \left(\frac{5}{4}, -2\right)$

Equation of the directrix $x - h + a = 0$ i.e., $4x - 3 = 0.$

Equation of the axis $y - k = 0.$ i.e., $y + 2 = 0.$

2. Problem : *Find the equation of the parabola whose vertex is (3, −2) and focus is (3, 1).*

Solution : The abcissae of the vertex and focus are equal to 3. Hence the axis of the parabola is $x = 3$, a line parallel to y-axis, focus is above the vertex.

a = distance between focus and vertex = 3.

$\therefore$ Equation of the parabola $(x-3)^2 = 4(3)(y+2)$ i.e., $(x-3)^2 = 12(y+2).$

3. Problem : *Find the coordinates of the points on the parabola $y^2 = 2x$ whose focal distance is $\frac{5}{2}$.*

Solution : Let $P(x_1, y_1)$ be a point on the parabola $y^2 = 2x$ whose focal distance is $\frac{5}{2}$ then $y_1^2 = 2x_1$ and

$x_1 + a = \frac{5}{2} \Rightarrow x_1 + \frac{1}{2} = \frac{5}{2} \Rightarrow x_1 = 2$

$\therefore y_1^2 = 2(2) = 4 \Rightarrow y_1 = \pm 2.$

$\therefore$ The required points are (2, 2) and (2, −2).

4. Problem : *Find the equation of the parabola passing through the points (−1, 2), (1, −1) and (2, 1) and having its axis parallel to the X-axis.*

Solution: Since the axis is parallel to X-axis the equation of the parabola is in the form of $x = ly^2 + my + n$.

Since the parabola passes through (−1, 2), we have

$$-1 = l(2)^2 + m(2) + n \Rightarrow 4l + 2m + n = -1 \quad \text{... (1)}$$

Similarly, since the parabola passes through (1, −1) and (2, 1), we have

$$l - m + n = 1 \quad \text{... (2)}$$

$$l+m+n=2 \quad \ldots (3)$$

Solving (1), (2) and (3) we get $l=-\frac{7}{6}$, $m=\frac{1}{2}$ and $n=\frac{8}{3}$.

Hence the equation of the parabola is $x=-\frac{7}{6}\ y^2+\frac{1}{2}y+\frac{8}{3}$ (or) $7y^2-3y+6x-16=0$.

5. Problem : *A double ordinate of the curve $y^2 = 4ax$ is of length $8a$. Prove that the lines from the vertex to its ends are at right angles.*

Solution : Let $P=(at^2, 2at)$ and $P'=(at^2, -2at)$ be the ends of double ordinate PP'. Then,

$$8a = PP' = \sqrt{0+(4at)^2} = 4at \Rightarrow t=2.$$

$$\therefore\ P=(4a, 4a),\quad P'=(4a,-4a)$$

Slope of $\overline{OP}$ × slope of $\overline{OP'} = \left(\frac{4a}{4a}\right)\left(-\frac{4a}{4a}\right) = -1$

$$\therefore\ \angle POP' = \frac{\pi}{2}.$$

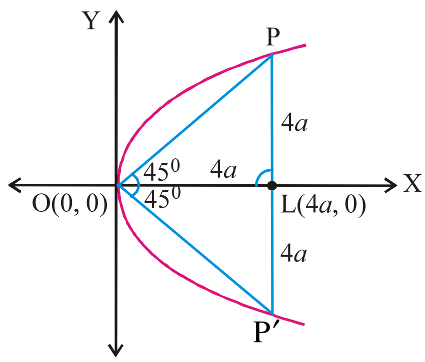


Fig. 3.19

Aliter : Let the double ordinate be $\overline{PP'}$ meeting the axis at L.

$\therefore$ PL = LP′ = $4a$. If $P=(x_1, 4a)$, then

$16a^2 = 4ax_1 \Rightarrow x_1 = 4a$ (see Fig. 3.19).

$\therefore$ OL = $4a$. ΔOLP is right isosceles.

$\angle POL = \frac{\pi}{4}$, similarly $\angle LOP' = \frac{\pi}{4}$. Consequently $\angle POP' = \frac{\pi}{2}$.

6. Problem

(i) *If the coordinates of the ends of a focal chord of the parabola $y^2 = 4ax$ are (x_1, y_1) and (x_2, y_2), then prove that $x_1x_2 = a^2$, $y_1y_2 = -4a^2$.*

(ii) *For a focal chord PQ of the parabola $y^2 = 4ax$, if $SP = l$ and $SQ = l'$ then prove that $\frac{1}{l}+\frac{1}{l'} = \frac{1}{a}$.*

Solution

(i) Let $P(x_1, y_1) = (at_1^2, 2at_1)$ and $Q(x_2, y_2) = (at_2^2, 2at_2)$ be two end points of a focal chord. P, S, Q are collinear.

Slope of $\overleftrightarrow{PS}$ = Slope of $\overleftrightarrow{QS}$

$$\frac{2at_1}{at_1^2-a} = \frac{2at_2}{at_2^2-a}$$

$$t_1t_2^2 - t_1 = t_2t_1^2 - t_2$$

$$t_1t_2(t_2-t_1)+(t_2-t_1)=0$$

$$1+t_1t_2=0 \quad \Rightarrow t_1t_2=-1 \quad \ldots (1)$$

From (1) $\quad x_1x_2 = at_1^2\,at_2^2 = a^2(t_2\,t_1)^2 = a^2$

$$y_1y_2 = 2at_1 2at_2 = 4a^2(t_2\,t_1) = -4a^2.$$

The property holds even if the focal chord is the latus rectum.

(ii) Let $P(at_1^2, 2at_1)$ and $Q(at_2^2, 2at_2)$ be the extremities of a focal chord of the parabola, then $t_1 t_2 = -1$ (from (1))

$$l = SP = \sqrt{(at_1^2 - a)^2 + (2at_1 - 0)^2} = a\sqrt{(t_1^2 - 1)^2 + 4t_1^2} = a(1 + t_1^2)$$

$$l' = SQ = \sqrt{(at_2^2 - a)^2 + (2at_2 - 0)^2} = a\sqrt{(t_2^2 - 1)^2 + 4t_2^2} = a(1 + t_2^2)$$

$$\therefore \quad (l - a)(l' - a) = a^2 t_1^2 t_2^2 = a^2 (t_1 t_2)^2 = a^2 \; [\because t_1 t_2 = -1]$$

$$l l' - a(l + l') = 0 \Rightarrow \frac{1}{l} + \frac{1}{l'} = \frac{1}{a}.$$

7. Problem : *If Q is the foot of the perpendicular from a point P on the parabola $y^2 = 8(x-3)$ to its directrix. S is the focus of the parabola and if SPQ is an equilateral triangle, then find the length of side of the triangle.*

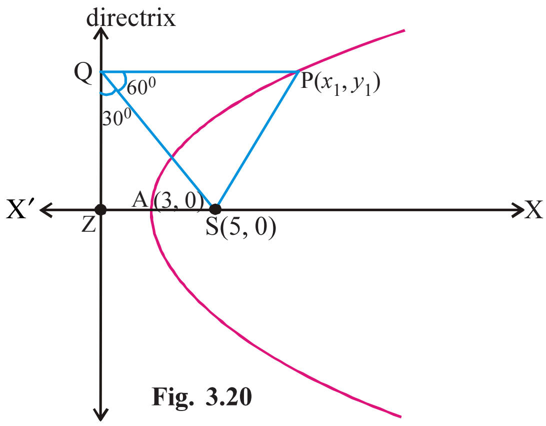


Fig. 3.20

Solution : Given parabola $y^2 = 8(x-3)$, then its vertex A = (3, 0) and Focus = (5, 0) $[4a = 8 \Rightarrow a = 2]$ since PQS is an equilateral triangle

$\underline{|SQP} = 60^o \Rightarrow \underline{|SQZ} = 30^o$ (See Fig. 3.20)

From ΔSZQ we have $\sin 30^o = \dfrac{SZ}{SQ}$.

$\therefore$ Side $SQ = \dfrac{SZ}{\sin 30^o} = 2(SZ) = 2(4) = 8.$

Hence length of each side of the triangle is 8.

8. Problem : *The cable of a uniformely loaded suspension bridge hangs in the form of a parabola. The roadway which is horizontal and 72 mt. long is supported by vertical wires attached to the cable, the longest being 30 mts. and the shortest being 6 mts. Find the length of the supporting wire attached to the road-way 18 mts. from the middle.*

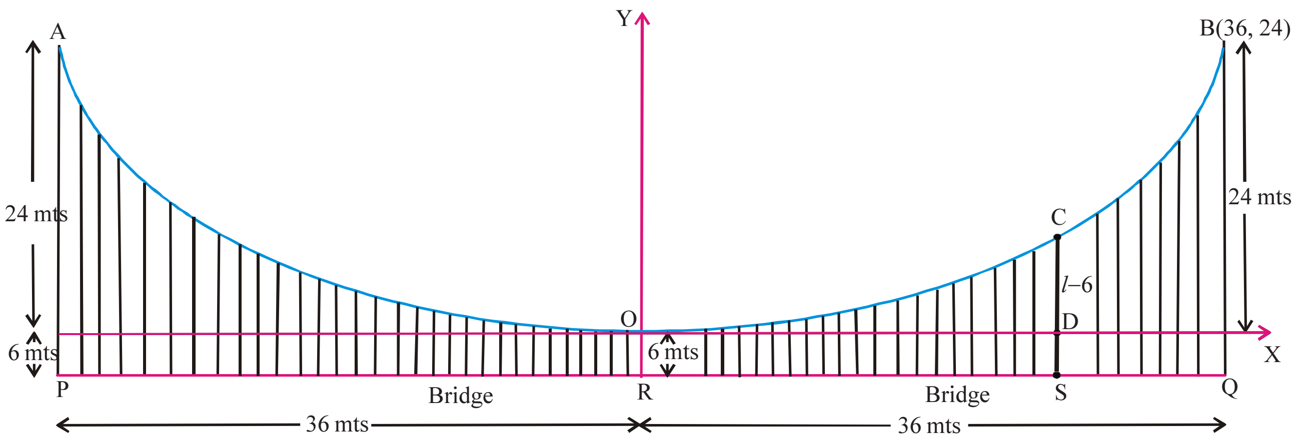


Fig. 3.21

Solution : Let AOB be the cable [O is its lowest point and A, B are the highest points]. Let PRQ be the bridge suspended with PR = RQ = 36 mts (see Fig. 3.21).

PA = QB = 30 mts (longest vertical supporting wires)

OR = 6 mts (shortest vertical supporting wire) [the lowest point of the cable is upright the mid-point R of the bridge]

Therefore, PR = RQ = 36 mts. We take the origin of coordinates at O, X-axis along the tangent at O to the cable and the Y-axis along $\overleftrightarrow{RO}$. The equation of the cable would, therefore, be $x^2 = 4ay$ for some $a > 0$. We get B = (36, 24) and $36^2 = 4a \times 24$.

$$\text{Therefore,} \quad 4a = \frac{36 \times 36}{24} = 54 \text{ mts.}$$

If RS = 18 mts. and SC is the vertical through S meeting the cable at C and the X-axis at D, then SC is the length of the supporting wire required. If SC = l mts, then DC = $(l - 6)$ mts.

As such C = (18, $l - 6$).

Since C is on the cable, $18^2 = 4a\,(l - 6)$

$$\Rightarrow \quad l - 6 = \frac{18^2}{4a} = \frac{18 \times 18}{54} = 6$$

$$\Rightarrow \quad l = 12.$$

Exercise 3(a)

I. 1. Find the vertex and focus of $4y^2 + 12x - 20y + 67 = 0$.

2. Find the vertex and focus of $x^2 - 6x - 6y + 6 = 0$.

3. Find the equations of axis and directrix of the parabola $y^2 + 6y - 2x + 5 = 0$.

4. Find the equations of axis and directrix of the parabola $4x^2 + 12x - 20y + 67 = 0$.

5. Find the equation of the parabola whose focus is S(1, −7) and vertex is A(1, −2).

6. Find the equation of the parabola whose focus is S(3, 5) and vertex is A(1, 3).

7. Find the equation of the parabola whose latus rectum is the line segment joining the points (−3, 2) and (−3, 1).

8. Find the position (interior or exterior or on) of the following points with respect to the parabola $y^2 = 6x$, (i) (6, −6) (ii) (0, 1) (iii) (2, 3)

9. Find the coordinates of the points on the parabola $y^2 = 8x$ whose focal distance is 10.

10. If (1/2, 2) is one extremity of a focal chord of the parabola $y^2 = 8x$. Find the coordinates of the other extremity.

11. Prove that the point on the parabola $y^2 = 4ax$, $(a > 0)$ nearest to the focus is its vertex.
12. A comet moves in a parabolic orbit with the sun as focus. When the comet is 2×10^7 K.M from the sun, the line from the sun to it makes an angle $\frac{\pi}{2}$ with the axis of the orbit. Find how near the comet comes to the sun.

II. 1. Find the locus of the point of trisection of double ordinate of a parabola $y^2 = 4ax$, $(a > 0)$.

2. Find the equation of the parabola whose vertex and focus are on the positive x-axis at a distance 'a' and 'a'' from the origin respectively.
3. If L and L$'$ are the ends of the latus rectum of the parabola $x^2 = 6y$, find the equations of OL and OL$'$ where 'O' is the origin. Also find the angle between them.
4. Find the equation of the parabola whose axis is parallel to x-axis and which passes through the points $(-2, 1)$, $(1, 2)$ and $(-1, 3)$.
5. Find the equation of the parabola whose axis is parallel to y-axis and which passes through the points $(4, 5)$, $(-2, 11)$ and $(-4, 21)$

III. 1. Find the equation of the parabola whose focus is $(-2, 3)$ and directrix is the line $2x + 3y - 4 = 0$. Also find the length of the latus rectum and the equation of the axis of the parabola.

2. Prove that the area of the triangle inscribed in the parabola $y^2 = 4ax$ is $\frac{1}{8a}|(y_1 - y_2)(y_2 - y_3)(y_3 - y_1)|$ sq. units where y_1, y_2, y_3 are the ordinates of its vertices.
3. Find the coordinates of the vertex and focus, the equation of the directrix and axis of the following parabolas.

   (i) $y^2 + 4x + 4y - 3 = 0$ (ii) $x^2 - 2x + 4y - 3 = 0$

3.2 Equation of tangent and normal at a point on the parabola

In this section, the condition for a straight line to be a tangent to a given parabola is obtained. The Cartesian and parametric equations of the tangent and the normal at a given point on the parabola are derived.

3.2.1 Point of intersection of the parabola $y^2 = 4ax$, $(a > 0)$ and the line $y = mx + c$, $(m \neq 0)$.

Let $y^2 = 4ax$, $(a > 0)$ ...(1)

and the straight line $y = mx + c$ be given ...(2)

The coordinates of the point of the intersection of the straight line and the parabola satisfy both the equations (1) and (2) and, therefore, can be found by solving them. Substituting the values of y from (2) in (1), we have

$$(mx + c)^2 = 4ax, \quad \text{i.e.,} \quad m^2x^2 + 2x(mc - 2a) + c^2 = 0 \quad \text{...(3)}$$

This is a quadratic equation in x and therefore has two roots which are distinct real (Fig.3.22(i)) equal (Fig. 3.22(ii)) or imaginary (Fig. 3.22(iii)) according as the discriminant of equation (3) is positive, zero (or) negative respectively.

The ordinates of the points of intersection y_1, y_2 can be obtained by substituting x_1, x_2 for x in $y = mx + c$.

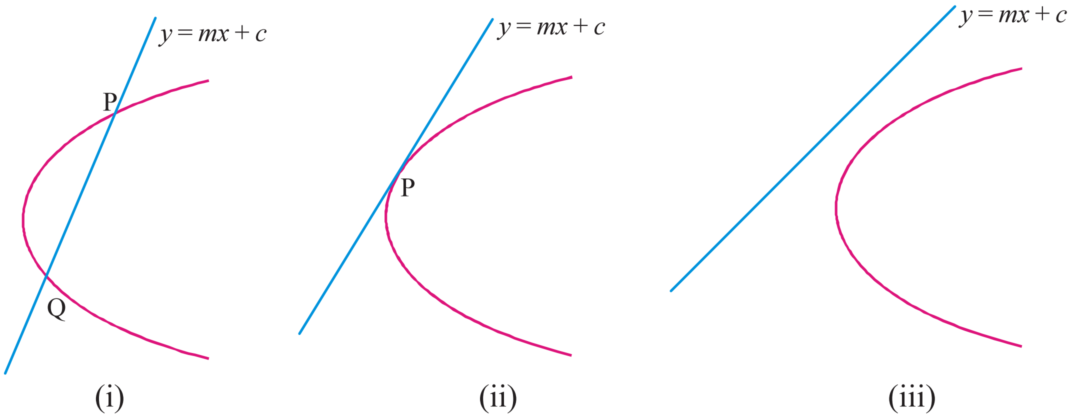


Fig .3.22

3.2.2 Theorem : *The condition for a straight line $y = mx + c$ ($m \neq 0$) to be a tangent to the parabola $y^2 = 4ax$ is $cm = a$ or $c = a/m$.*

Proof : The 'x' coordinates of the points of intersection of the line $y = mx + c$ and the given parabola are given by the equation (3) of 3.2.1 i.e.,

$$m^2x^2 + 2x(mc - 2a) + c^2 = 0 \qquad \text{... (1)}$$

The given line will touch the parabola $\Leftrightarrow$ the two points coincide.

$\Leftrightarrow$ discriminant of (1) is zero

$\Leftrightarrow 4(mc - 2a)^2 - 4m^2c^2 = 0$

$\Leftrightarrow 16a\,(a - mc) = 0$

$\Leftrightarrow a - mc = 0 \Leftrightarrow a = mc$ (or) $c = \dfrac{a}{m}$.

3.2.3 Note

(i) When $m = 0$, the line $y = c$ is parallel to the axis of the parabola $y^2 = 4ax$, i.e., X-axis. Further $y = c \Rightarrow x = \dfrac{c^2}{4a}$.

$\therefore$ The straight line intersects the parabola at the point $\left(\dfrac{c^2}{4a}, c\right)$, (Fig. 3.23).

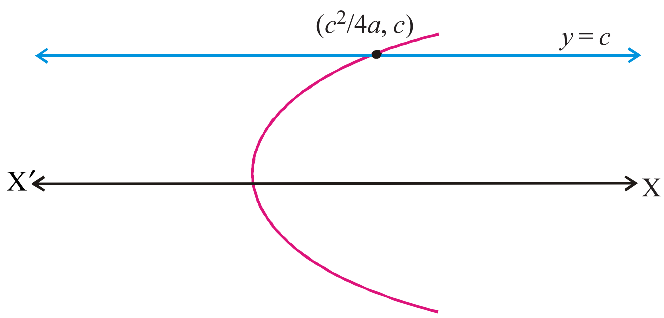


Fig. 3.23

(ii) we have seen $y = mx + c, (m \neq 0)$ is a tangent to the parabola $y^2 = 4ax$ when $c = \frac{a}{m}$. Hence $y = mx + \frac{a}{m}$ is always a tangent to the parabola $y^2 = 4ax$ at $\left(\frac{a}{m^2}, \frac{2a}{m}\right) = \left(\frac{c}{m}, 2c\right)$ when $m \neq 0$.

$\left[\because\right.$ from (3) of 3.2.1, the abscissa $x_1 = \frac{-2(mc-2a)}{2m^2} = \frac{-(a-2a)}{m^2} = \frac{a}{m^2}$ and ordinate $\left. y_1 = \frac{ma}{m^2} + \frac{a}{m} = \frac{2a}{m}. \right]$

(iii) If $m \neq 0$ and $c = 0$, then the line $y = mx$ is non vertical and passes through the origin which intersects the parabola in two points (0, 0) and $\left(\frac{4a}{m^2}, \frac{4a}{m}\right)$. Hence it is not a tangent

$\left[\because \text{ from (3) of 3.2.1, } m^2x^2 - 4ax = 0 \Rightarrow x = 0 \text{ or } \frac{4a}{m^2} \text{ then } y = 0 \text{ or } \frac{4a}{m}\right]$

(iv) Observe that for a parabola $y^2 = 4ax$, there is one and only one tangent, parallel to Y - axis (i.e., Y - axis it self) and there is no tangent parallel to X-axis.

(v) Let $P(x_0, y_0)$ be a point other than origin, on the parabola $y^2 = 4ax$. If the tangent at P makes an angle θ with y-axis, then $\theta \neq \frac{\pi}{2}$. We write $t = \tan\theta$. Then slope of the tangent $= \frac{2a}{y_0} = \cot\theta = \frac{1}{t}$.

Hence $y_0 = 2at$ and $4ax_0 = 4a^2t^2$ gives $x_0 = at^2$ (Fig. 3.24).

If, however, $P(x_0, y_0)$ is the origin, then $x_0 = at^2, y_0 = 2at$ where $t = 0$.

So, any point $P(x_0, y_0)$ on the parabola $y^2 = 4ax$ can always be written in the form $(at^2, 2at)$ for some $t \in \mathbf{R}$.

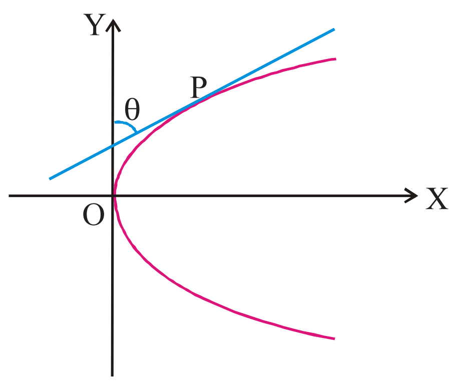


Fig. 3.24

3.2.4 Theorem : *Two tangents can be drawn from an external point (x_1, y_1) to the parabola $y^2 = 4ax$.*

Proof : Let $P(x_1, y_1)$ be an external point to the parabola $y^2 = 4ax$ then

$$S_{11} \equiv y_1^2 - 4ax_1 > 0 \qquad \text{... (1)}$$

We have $y = mx + \frac{a}{m}$ is a tangent to the parabola $y^2 = 4ax$ for all non zero values of m.

If it passes through the point (x_1, y_1) then $y_1 = mx_1 + \frac{a}{m}$ or $m^2x_1 - my_1 + a = 0$ and its discriminant $y_1^2 - 4ax_1 > 0$ [from (1)]. The equation being a quadratic in m, has two distinct real roots, say, m_1 and m_2. Then $y = m_1x + \frac{a}{m_1}$ and $y = m_2x + \frac{a}{m_2}$ are the two distinct tangents through (x_1, y_1).

3.2.5 Theorem : *The equation of the chord joining the points (x_1, y_1) and (x_2, y_2) on $S = 0$ is $S_1 + S_2 = S_{12}$.*

Proof : Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two points on the parabola $S \equiv y^2 - 4ax = 0$, then $S_{11} = 0$ and $S_{22} = 0$. Consider the first degree equation $S_1 + S_2 = S_{12}$

i.e., $\{yy_1 - 2a(x + x_1)\} + \{yy_2 - 2a(x + x_2)\} = y_1y_2 - 2a(x_1 + x_2)$

i.e., $4ax - (y_1 + y_2)y + y_1y_2 = 0$ which represents a straight line.

Clearly (x_1, y_1) and (x_2, y_2) satisfies the equation(1) $\quad (\because y_1^2 - 4ax_1 = 0 = y_2^2 - 4ax_2)$

$\therefore$ $S_1 + S_2 = S_{12}$ is a straight line passing through $P(x_1, y_1)$ and $Q(x_2, y_2)$.

$\therefore$ The equation of the chord PQ is $S_1 + S_2 = S_{12}$.

3.2.6 Theorem : *The equation of the tangent at $P(x_1, y_1)$ to the parabola $S = 0$ is $S_1 = 0$.*

Proof : Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two points on the parabola $S \equiv y^2 - 4ax = 0$ then $S_{11} = 0$ and $S_{22} = 0$.

By Theorem 3.2.5, the equation of the chord PQ is $S_1 + S_2 = S_{12}$ ...(1)

The chord PQ becomes the tangent at P when Q approaches P

(i.e., (x_2, y_2) approaches to (x_1, y_1)).

$\therefore$ The equation of the tangent at P is obtained by taking limits as (x_2, y_2) tends to (x_1, y_1) on either sides of (1).

So, the equation of the tangent at P given by $\underset{Q \to P}{Lt}(S_1 + S_2) = \underset{Q \to P}{Lt} S_{12}$.

i.e., $S_1 + S_1 = S_{11}$ $\quad [\because S_2 \to S_1, S_{12} \to S_{11} \text{ as } (x_2, y_2) \to (x_1, y_1)]$

$\therefore$ $2S_1 = 0 \Rightarrow S_1 = 0.$

$\therefore$ The equation of the tangent to the parabola $S \equiv y^2 - 4ax = 0$ at $P(x_1, y_1)$ is

$S_1 \equiv yy_1 - 2a(x + x_1) = 0$.

3.2.7 Theorem : *The equation of the normal at $P(x_1, y_1)$ on the parabola $S = 0$ is* $\boldsymbol{(y - y_1) = -\frac{y_1}{2a}(x - x_1)}$.

Proof : By Theorem 3.2.6, the equation of the tangent to the parabola $y^2 - 4ax = 0$ at $P(x_1, y_1)$ is $S_1 \equiv yy_1 - 2a(x + x_1) = 0$.

$\therefore$ Slope of the tangent at P is $\frac{2a}{y_1}$.

$\therefore$ Slope of the normal at P is $-\frac{y_1}{2a}$.

Hence equation of the normal at $P(x_1, y_1)$ is $(y - y_1) = -\frac{y_1}{2a}(x - x_1)$.

3.2.8 Theorem (Parametric form)

(i) *Equation of the tangent to the parabola $y^2 = 4ax$ at a point 't' is $x - yt + at^2 = 0$.*

(ii) *Equation of the normal to the parabola $y^2 = 4ax$ at a point 't' is $y + xt = 2at + at^3$.*

Proof : Let P(t) be a point on the parabola $y^2 = 4ax$ then $P = (at^2, 2at)$.

(i) We have equation of the tangent at $P(x_1, y_1)$, is $yy_1 - 2a(x + x_1) = 0$, then replacing (x_1, y_1) by $(at^2, 2at)$, the equation of tangent is $2at\, y - 2a(x + at^2) = 0$ i.e., $x - yt + at^2 = 0$.

(ii) We have equation of the normal at $P(x_1, y_1)$ is $(y - y_1) = -\frac{y_1}{2a}(x - x_1)$ then replacing (x_1, y_1) by $(at^2, 2at)$, the equation of the normal is

$$(y - 2at) = \frac{-2at}{2a}(x - at^2)$$

$$\text{i.e., } y + xt = 2at + at^3.$$

3.2.9 Number of normals through a given point

(i) The equation of the normal to the parabola $y^2 = 4ax$ at $(at^2, 2at)$ is $y + xt = 2at + at^3$, if this line passes through (x_1, y_1), then $y_1 + x_1 t = 2at + at^3$ is $at^3 + t(2a - x_1) - y_1 = 0$. This is a cubic equation in 't' and has, at most three real roots.

Hence the number of normals through a given point (x_1, y_1) to a parabola $y^2 = 4ax$ is either 1 or 2 or 3 accordingly as the number of distinct real roots of the cubic equation

$at^3 + (2a - x_1)t - y_1 = 0$ is 1 or 2 or 3.

Criterion for the number of normals

Write $H=\frac{2a-x_1}{3a}$, $G=-\frac{y_1}{a}$ and $\Delta=G^2+4H^3$ if $x_1=2a$ and $y_1=0$ then the number of normals $=1$.

Assume either $x_1 \neq 2a$ or $y_1 \neq 0$

(i) If $\Delta>0$ then the number of normals is 1

(ii) If $\Delta=0$ then the number of normals is 2

(iii) If $\Delta<0$ then the number of normals is 3

The proof of the above is beyond the scope of this book.

(ii) The equation of the tangent 't' is $yt=x+at^2$. Hence slope of the normal at t is

$m=-t \Rightarrow t=-m$, substituting in the equation of the normal at t (i.e., $y+xt=2at+at^3$) we get $y - mx = -2am - am^3$ is $y = mx - 2am - am^3$.

$\therefore$ The equation of the normal to the parabola $y^2=4ax$, having slope m, is

$y = mx - 2am - am^3 = m(x - 2a - am^2)$.

3.2.10 Solved Problems

1. Problem : *Find the condition for the straight line $lx + my + n = 0$ to be a tangent to the parabola $y^2 = 4ax$ and find the coordinates of the point of contact.*

Solution : Let the line $lx + my + n = 0$ be a tangent to the parabola $y^2 = 4ax$ at $(at^2, 2at)$. Then the equation of the tangent at $P(t)$ is $x - yt + at^2 = 0$ then it represents the given line $lx + my + n = 0$, then

$$\frac{l}{1} = \frac{m}{-t} = \frac{n}{at^2} \Rightarrow t = \frac{-m}{l} \text{ and } t = \frac{-n}{ma}$$

$$\therefore -t = \frac{m}{l} = \frac{n}{ma} \Rightarrow m^2 a = nl$$

and the point of contact is $P(at^2, 2at) = \left(\frac{am^2}{l^2}, \frac{-2am}{l}\right)$ or $\left(\frac{n}{l}, \frac{-2am}{l}\right)$.

2. Problem : *Show that the straight line $7x + 6y = 13$ is a tangent to the parabola $y^2 - 7x - 8y + 14 = 0$ and find the point of contact.*

Solution : Equation of the given line is $7x + 6y = 13$, equation of the given parabola is

$y^2 - 7x - 8y + 14 = 0$.

By eliminating x, we get the ordinates of the points of intersection of line and parabola adding the equations $y^2 - 2y + 1 = 0$.

i.e., $(y-1)^2 = 0 \Rightarrow y = 1, 1.$

$\therefore$ The given line is tangent to the given parabola.

If $y = 1$ then $x = 1$ hence the point of contact is $(1, 1)$.

3. Problem : *Prove that the normal chord at the point other than origin whose ordinate is equal to its abscissa subtends a right angle at the focus.*

Solution : Let the equation of the parabola be $y^2 = 4ax$ and $P(at^2, 2at)$ be any point ... (1)

on the parabola for which the abscissa is equal to the ordinate. i.e., $at^2 = 2at \Rightarrow t = 0$ or $t = 2$. But $t \neq 0$. Hence the point $(4a, 4a)$ at which the normal is

$$y + 2x = 2a(2) + a(2)^3 \quad \text{(or)} \quad y = (12a - 2x) \qquad \text{... (2)}$$

substituting the value of $y = 12a - 2x$ in (1) we get $(12a - 2x)^2 = 4ax$ (or)

$$x^2 - 13ax + 36a^2 = (x - 4a)(x - 9a) = 0 \Rightarrow x = 4a, 9a$$

corresponding values of y are $4a$ and $-6a$.

Hence the other points of intersection of that normal at $P(4a, 4a)$ to the given parabola is $Q(9a, -6a)$, we have $S(a, 0)$.

Slope of the $\overleftrightarrow{SP} = m_1 = \dfrac{4a - 0}{4a - a} = \dfrac{4}{3}$.

Slope of the $\overleftrightarrow{SQ} = m_2 = \dfrac{-6a - 0}{9a - a} = -\dfrac{3}{4}$.

Clearly $m_1 m_2 = -1$, so that $\overleftrightarrow{SP} \perp \overleftrightarrow{SQ}$.

4. Problem : *From an external point P, tangent are drawn to the parabola $y^2 = 4ax$ and these tangents make angles θ_1, θ_2 with its axis, such that $\tan\theta_1 + \tan\theta_2$ is a constant b. Then show that P lies on the line $y = bx$.*

Solution : Let the coordinates of P be (x_1, y_1) and the equation of the parabola $y^2 = 4ax$. Any tangent to the parabola is $y = mx + \dfrac{a}{m}$, if this passes through (x_1, y_1) then

$$y_1 = mx_1 + \frac{a}{m} \quad \text{i.e.,} \quad m^2 x_1 - my_1 + a = 0 \qquad \text{... (1)}$$

Let the roots of (1) be m_1, m_2. Then $m_1 + m_2 = \dfrac{y_1}{x_1} \Rightarrow \tan\theta_1 + \tan\theta_2 = \dfrac{y_1}{x_1}$.

[$\because$ The tangents make angles θ_1, θ_2 with its axis (x-axis) then their slopes $m_1 = \tan\theta_1$ and $m_2 = \tan\theta_2$].

$$\therefore b = \frac{y_1}{x_1} \Rightarrow y_1 = bx_1.$$

$\therefore P(x_1, y_1)$ lies on the line $y = bx$.

5. Problem : *Show that the common tangent to the parabola $y^2 = 4ax$ and $x^2 = 4by$ is*
$xa^{1/3} + yb^{1/3} + a^{2/3}b^{2/3} = 0$.

Solution : The equations of the parabolas are $y^2 = 4ax$ ... (1)

and $x^2 = 4by$ ... (2)

Equation of any tangent to (1) is of the form $y = mx + \frac{a}{m}$. ... (3)

If the line (3) is a tangent to (2) also, the points of intersection of (2) and (3) coincide.

Substituting the value of y from (3) in (2), we get $x^2 = 4b\left(mx + \frac{a}{m}\right)$ i.e., $mx^2 - 4bm^2x - 4ab = 0$ which should have equal roots. Therefore its discriminant is zero. Hence

$$16b^2m^4 - 4m(-4ab) = 0$$

$$16b(bm^4 + am) = 0$$

$$m(bm^3 + a) = 0. \quad \text{But } \; m \neq 0.$$

$\therefore\ m = -a^{1/3}/b^{1/3}$ substituting in (3) the equation of the common tangent

becomes $y = -\left(\frac{a}{b}\right)^{1/3} x + \frac{a}{(-\frac{a}{b})^{1/3}}$ (or) $a^{1/3}x + b^{1/3}y + a^{2/3}\,b^{2/3} = 0$.

6. Problem : *Prove that the area of the triangle formed by the tangents at $(x_1, y_1), (x_2, y_2)$ and (x_3, y_3) to the parabola $y^2 = 4ax\ (a > 0)$ is $\frac{1}{16a}\left|(y_1 - y_2)(y_2 - y_3)(y_3 - y_1)\right|$ sq. units.*

Solution : Let $D(x_1, y_1) = (at_1^2, 2at_1)$, $E(x_2, y_2) = (at_2^2, 2at_2)$ and $F(x_3, y_3) = (at_3^2, 2at_3)$ be three points on the parabola $y^2 = 4ax\ (a > 0)$.

The equation of the tangents at D, E and F are, respectively

$$t_1y = x + at_1^2 \qquad ... (1)$$

$$t_2y = x + at_2^2 \qquad ... (2)$$

$$t_3y = x + at_3^2 \qquad ... (3)$$

$(1) - (2) \Rightarrow (t_1 - t_2)y = a(t_1 - t_2)(t_1 + t_2) \Rightarrow y = a(t_1 + t_2)$ substituting in (1) we get $x = at_1t_2$.

$\therefore$ The point of intersection of the tangents at D and E is $P(at_1t_2, a(t_1 + t_2))$.

Similarly, the points of intersection of tangents at E, F and at F, D are $Q(at_2t_3, a(t_2 + t_3))$ and $R(at_3t_1, a(t_3 + t_1))$ respectively.

$$\text{Area of } \Delta PQR = \text{Absolute value of } \frac{1}{2}\begin{vmatrix} at_1t_2 & a(t_1 + t_2) & 1 \\ at_2t_3 & a(t_2 + t_3) & 1 \\ at_1t_3 & a(t_1 + t_3) & 1 \end{vmatrix}$$

$$= \text{Absolute value of } \frac{a^2}{2}\begin{vmatrix} t_1t_2 & t_1+t_2 & 1 \\ t_2t_3 & t_2+t_3 & 1 \\ t_1t_3 & t_1+t_3 & 1 \end{vmatrix}$$

$$= \text{Absolute value of } \frac{a^2}{2}\begin{vmatrix} t_1(t_2-t_3) & (t_2-t_3) & 0 \\ t_3(t_2-t_1) & (t_2-t_1) & 0 \\ t_1t_3 & t_1+t_3 & 1 \end{vmatrix}$$

$$= \text{Absolute value of } \frac{a^2}{2}(t_2-t_3)(t_2-t_1)\begin{vmatrix} t_1 & 1 & 0 \\ t_3 & 1 & 0 \\ t_1t_3 & t_1+t_3 & 1 \end{vmatrix}$$

$$= \frac{a^2}{2}\left|(t_2-t_3)(t_2-t_1)(t_1-t_3)\right|$$

$$= \frac{1}{16a}\left|2a(t_1-t_2)\ 2a(t_2-t_3)\ 2a(t_3-t_1)\right|$$

$$= \frac{1}{16a}\left|(y_1-y_2)\ (y_2-y_3)\ (y_3-y_1)\right| \text{ sq.units.}$$

7. Problem : *Prove that the two parabolas $y^2 = 4ax$ and $x^2 = 4by$ intersect (other than the origin) at an angle of* $Tan^{-1}\left[\frac{3a^{1/3}b^{1/3}}{2(a^{2/3}+b^{2/3})}\right]$ (see Fig. 3.25).

Solution : Without loss of generality we assume $a > 0$ and $b > 0$.

Let P(x, y) be the point of intersection of the parabolas other than the origin.

Then

$$y^4 = 16a^2x^2$$
$$= 16a^2(4by)$$
$$= 64a^2by$$
$$\therefore\ y[y^3 - 64a^2b] = 0$$
$$\Rightarrow\ y^3 - 64a^2b = 0$$
$$\Rightarrow\ y = (64a^2b)^{1/3} \quad [\because y > 0]$$
$$= 4a^{2/3}b^{1/3}$$

Also from $y^2 = 4ax$, $\quad x = \frac{16a^{4/3}b^{2/3}}{4a}$

$$= 4a^{1/3}b^{2/3}$$

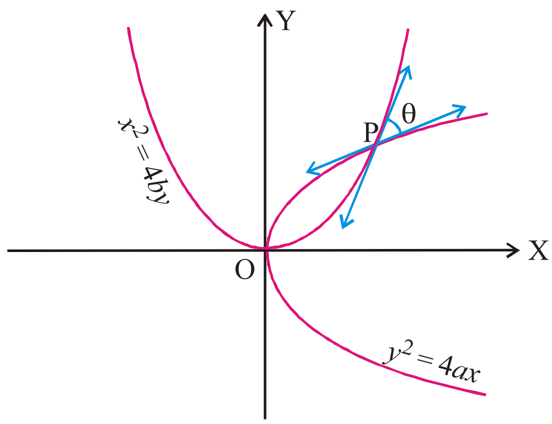


Fig. 3.25

$\therefore$ P = $(4a^{1/3}b^{2/3}, 4a^{2/3}b^{1/3})$

Differentiating both sides of $y^2 = 4ax$ w.r.t. 'x', we get

$$\frac{dy}{dx} = \frac{2a}{y}$$

$$\therefore \left[\frac{dy}{dx}\right]_P = \frac{2a}{4a^{2/3}b^{1/3}} = \frac{1}{2}\left(\frac{a}{b}\right)^{1/3}.$$

If m_1 be the slope of the tangent at P to $y^2 = 4ax$, then

$$m_1 = \frac{1}{2}\left(\frac{a}{b}\right)^{1/3}$$

Similarly, we get $m_2 = 2\left(\frac{a}{b}\right)^{1/3}$ where m_2 is the slope of the tangent at P to $x^2 = 4by$.

If θ is the acute angle between the tangents to the curves at P, then

$$\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right| = \frac{3a^{1/3}b^{1/3}}{2(a^{2/3} + b^{2/3})} \text{ so that } \theta = \text{Tan}^{-1}\left[\frac{3a^{1/3}b^{1/3}}{2(a^{2/3} + b^{2/3})}\right].$$

8. Problem : *Prove that the orthocenter of the triangle formed by any three tangents to a parabola lies on the directrix of the parabola.*

Solution : Let $y^2 = 4ax$ be the parabola and A = $(at_1^2, 2at_1)$, B = $(at_2^2, 2at_2)$, C = $(at_3^2, 2at_3)$ be any three points on it.

Now we consider the triangle PQR formed by the tangents to the parabola at A, B, C where

P = $(at_1t_2, a(t_1+t_2))$, Q = $(at_2t_3, a(t_2+t_3))$ and R = $(at_3t_1, a(t_3+t_1))$.

Equation of $\overleftrightarrow{QR}$ (i.e., the tangent at C) is

$$x - t_3 y + at_3^2 = 0.$$

Therefore, the attitude through P of triangle PQR is

$$t_3 x + y = at_1 t_2 t_3 + a(t_1 + t_2) \quad \text{... (1)}$$

Similarly, the attitude through Q is

$$t_1 x + y = at_1 t_2 t_3 + a(t_2 + t_3) \quad \text{... (2)}$$

Solving (1) and (2), we get $(t_3 - t_1)x = a(t_1 - t_3)$ i.e., $x = -a$.

Therefore, the orthocenter of the triangle PQR, with abscissa as $-a$, lies on the directrix of the parabola.

Exercise 3(b)

I. 1. Find the equations of the tangent and normal to the parabola $y^2 = 6x$ at the positive end of the latus rectum.

2. Find the equation of the tangent and normal to the parabola $x^2 - 4x - 8y + 12 = 0$ at $\left(4, \frac{3}{2}\right)$.

3. Find the value of k if the line $2y = 5x + k$ is a tangent to the parabola $y^2 = 6x$.

4. Find the equation of the normal to the parabola $y^2 = 4x$ which is parallel to $y - 2x + 5 = 0$.

5. Show that the line $2x - y + 2 = 0$ is a tangent to the parabola $y^2 = 16x$. Find the point of contact also.

6. Find the equation of tangent to the parabola $y^2 = 16x$ inclined at an angle 60^o with its axis and also find the point of contact.

II. 1. Find the equations of tangents to the parabola $y^2 = 16x$ which are parallel and perpendicular respectively to the line $2x - y + 5 = 0$, also find the coordinates of their points of contact.

2. If $lx + my + n = 0$ is a normal to the parabola $y^2 = 4ax$, then show that $al^3 + 2alm^2 + nm^2 = 0$.

3. Show that the equation of common tangents to the circle $x^2 + y^2 = 2a^2$ and the parabola $y^2 = 8ax$ are $y = \pm(x + 2a)$.

4. Prove that tangents at the extremities of a focal chord of a parabola intersect at right angles on the directrix.

5. Find the condition for the line $y = mx + c$ to be a tangent to the parabola $x^2 = 4ay$.

6. Three normals are drawn from $(k, 0)$ to the parabola $y^2 = 8x$ one of the normal is the axis and the remaining two normals are perpendicular to each other, then find the value of k.

7. Show that the locus of point of intersection of perpendicular tangents to the parabola $y^2 = 4ax$ is the directrix $x + a = 0$.

8. Two parabolas have the same vertex and equal length of latus rectum such that their axes are at right angle. Prove that the common tangents touch each at the end of latus rectum.

9. Show that the foot of the perpendicular from focus to the tangent of the parabola $y^2 = 4ax$ lies on the tangent at vertex.

10. Show that the tangent at one extremity of a focal chord of a parabola is parallel to the normal at the other extremity.

III. 1. The normal at a point t_1 on $y^2 = 4ax$ meets the parabola again in the point t_2. Then prove that $t_1t_2 + t_1^2 + 2 = 0$.

2. From an external point P tangents are drawn to the parabola $y^2 = 4ax$ and these tangents make angles θ_1, θ_2 with its axis, such that $\cot\theta_1 + \cot\theta_2$ is a constant 'd'. Then show that all such P lie on a horizontal line.

3. Show that the common tangents to the circle $2x^2 + 2y^2 = a^2$ and the parabola $y^2 = 4ax$ intersect at the focus of the parabola $y^2 = -4ax$.

4. The sum of the ordinates of two points on $y^2 = 4ax$ is equal to the sum of the ordinates of two other points on the same curve. Show that the chord joining the first two points is parallel to the chord joining the other two points.

5. If a normal chord a point 't' on the parabola $y^2 = 4ax$ subtends a right angle at vertex, then prove that $t = \pm\sqrt{2}$.

Key Concepts

- ❖ In a parabola eccentricity $e = 1$
- ❖ Equation in standard form $y^2 = 4ax, (a > 0)$. Focus $(a, 0)$, equation of directrix $x + a = 0$, axis $y = 0$, and length of latus rectum '$4a$'.
- ❖ Its equation when its axis is parallel to the x-axis is $x = ly^2 + my + n$ and when axis is parallel to the y-axis is $y = lx^2 + mx + n$.
- ❖ Focal distance of a point $P(x_1, y_1)$ on the parabola $y^2 = 4ax, (a > 0)$ is $x_1 + a$.
- ❖ Parametric equations $x = at^2, y = 2at$.
- ❖ $P(x_1, y_1)$ lies outside, on, or inside the parabola $S \equiv y^2 - 4ax = 0$ according as $S_{11} \gtreqless 0$.
- ❖ $y = mx + c, (m \neq 0)$ is a tangent to the parabola $y^2 = 4ax$ when $c = \frac{a}{m}$.
- ❖ $y = mx + \frac{a}{m}, \ (m \neq 0)$ is always a tangent to the parabola $y^2 = 4ax$ at $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$.
- ❖ A horizontal line cannot be a tangent to the parabola $y^2 = 4ax$.

- Equation of the tangent at the point (x_1, y_1) on the parabola $S = 0$ is $S_1 = 0$.
- Equation of the normal at the point (x_1, y_1) on the parabola $S = 0$ is $(y - y_1) = \frac{-y_1}{2a}(x - x_1)$.
- Equation of the tangent at a point 't' on the parabola $y^2 = 4ax$ is $x - yt + at^2 = 0$.
- Equation of the normal at a point 't' on the parabola $y^2 = 4ax$ is $y + xt = 2at + at^3$.

Historical Note

Menaechmus, (380 - 320 BC) an associate of *Plato* and a pupil of *Eudoxus*, invented the conic sections.

The conic sections were named and studied as long ago as when *Apollonius* of "Perga" undertook a systematic study of their properties. The names for the conic sections : ellipse, parabola and hyperbola were given by *Apollonius*. Book II of *Apollonius* treatise on conic sections deals with properties of asymptotes and conjugate hyperbolas. *Kepler* was the first to notice that planetary orbits were ellipses. *Galileo* (1564 - 1642) proved that the path of a projectile is a parabola. *Newton* was then able to derive the shape of orbits mathematically using calculus under the assumption that the gravitational force goes as the inverse square of distance.

Answers

Exercise 3(a)

I. 1. Vertex $\left(-\frac{7}{2}, \frac{5}{2}\right)$, Focus $\left(-\frac{17}{4}, \frac{5}{2}\right)$

2. Vertex $\left(3, -\frac{1}{2}\right)$, Focus $(3, 1)$

3. Equation of the axis is $y + 3 = 0$, equation of the directrix is $2x + 5 = 0$.

4. axis $2x + 3 = 0$, directrix $20y - 33 = 0$

5. $(x - 1)^2 = -20(y + 2)$

6. $x^2 - 2xy + y^2 - 12x - 20y + 68 = 0$

7. $(2y - 3)^2 = (4x + 13)$ or $(2y - 3)^2 = -(4x + 11)$

8. (i) on the parabola (ii) outside the parabola (iii) inside the parabola.

9. $(8, \pm 8)$ 10. $(8, -8)$ 12. 10^7 k.m.

II. 1. $9y^2 = 4ax$

2. $y^2 = 4(a' - a)(x - a)$

3. $x + 2y = 0,\ x - 2y = 0,\ \pi\text{-}\mathrm{Tan}^{-1}(4/3)$

4. $5y^2 + 2x - 21y + 20 = 0$

5. $x^2 - 4x - 2y + 10 = 0$

III. 1. $9x^2 - 12xy + 4y^2 + 68x - 54y + 153 = 0$, length of the latus rectum $= \frac{2}{\sqrt{13}}$. Equation of the axis of the parabola is $3x - 2y + 12 = 0$.

3. (i) Vertex $\left(\frac{7}{4}, -2\right)$, Focus $\left(\frac{3}{4}, -2\right)$, Directrix $4x - 11 = 0$, axis $y + 2 = 0$

(ii) Vertex $(1, 1)$, Focus $(1, 0)$, Directrix $y = 2$, axis $x = 1$

Exercise 3(b)

I. 1. Tangent $2x - 2y + 3 = 0$, Normal $2x + 2y - 9 = 0$

2. Tangent $x - 2y - 1 = 0$, Normal $4x + 2y - 19 = 0$

3. $k = \frac{6}{5}$

4. $2x - y - 12 = 0$

5. $(1, 4)$

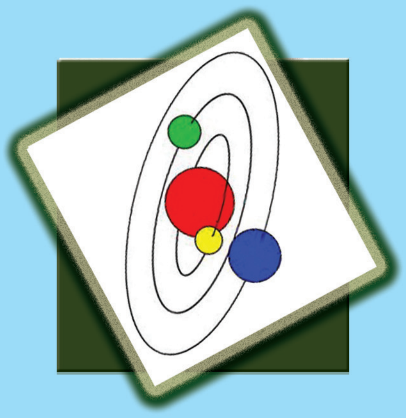
6. $3x - \sqrt{3}y + 4 = 0 \left(\frac{4}{3}, \frac{8}{\sqrt{3}}\right)$.

II. 1. $2x - y + 2 = 0$, point of contact $(1, 4)$

$x + 2y + 16 = 0$, point of contact $(16, -16)$

5. $am^2 + c = 0$

6. $k = 6$



Chapter 4

Ellipse

"trinaabhi cakramajara manarvam yeenee maa visva bhuvanaani tasthuh"

"The elliptical path through which all the celestial bodies move, is imperishable and unslackened"

-Rigvē da

Introduction

We study the ellipse in this chapter. We also discuss, about the standard form of equation of ellipse, conditions for a line to be a tangent to the ellipse, chord of contact, parameteric equations of an ellipse, in the chapter.

4.1 Equation of ellipse in standard form, parametric equations

In this section, we study the equation of an ellipse in the standard form and also its parameteric equations.

Girard Desargues
(ca. 1593 - 1662)

__Desargues,__ a French mathematician who gave remarkably original writings on the conic sections. He was besides being a mathematician, an engineer, an architect and one time French army officer. In geometry his contributions have been phenomenal; fundamental theorems on involution, harmonic ranges, homology, poles and polars and principles of per-spectivity in projective geometry are some of his works. The fundamental two triangle theorem bears his name.

4.1.1 Definition (Ellipse)

A conic with eccentricity less than unity is called an ellipse. Hence an ellipse is the locus of a point whose distances from a fixed point and a fixed straight line are in constant ratio 'e' which is less than unity. The fixed point and the fixed straight line are called the focus and the directrix of the ellipse respectively.

4.1.2 Equation of ellipse in standard form

Let S be a focus, the line l be the corresponding directrix and e be the eccentricity. Let Z be the foot of the perpendicular from S on the directrix. Let A and A$'$ be the points which divide $\overline{SZ}$ in the ratio $e:1$, internally and externally, respectively, (Fig. 4.1).

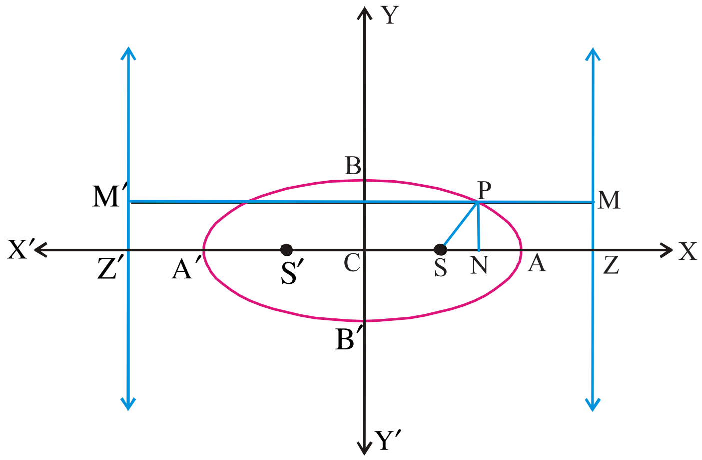


Fig. 4.1

Consider C, mid point of $\overline{AA'}$ as origin, consider the line CZ extended as X-axis and a line perpendicular to it at C as Y-axis.

Let CA = a = CA$'$ so that A = $(a, 0)$ and A$'$ = $(-a, 0)$.

But $\dfrac{SA}{AZ} = e = \dfrac{SA'}{A'Z} \Rightarrow SA = e(AZ)$ and $SA' = e(A'Z)$

$\therefore \quad CA - CS = e(CZ - CA) \Rightarrow a - CS = e(CZ - a)$ ... (1)

$CS + CA' = e(CA' + CZ) \Rightarrow CS + a = e(a + CZ)$ ... (2)

Adding (1) and (2) above, we get $2a = 2e(CZ)$ or $(CZ) = \dfrac{a}{e}$.

$\therefore$ Equation of the directrix is $x = \dfrac{a}{e}$. ... (3)

Subtracting (1) from (2), we get $2(CS) = 2ea \Rightarrow CS = ae$.

$\therefore$ Coordinates of focus S are $(ae, 0)$.

Now let P(x, y) be a point on the ellipse and PM be the perpendicular distance from P to the directrix. Then by the definition SP $= e$(PM).

$\therefore \quad (SP)^2 = e^2 (PM)^2$

i.e., $\quad (x - ae)^2 + y^2 = e^2\left(x^2 + \frac{a^2}{e^2} - \frac{2ax}{e}\right) \qquad \left[\because PM = x - \frac{a}{e}\right]$

i.e., $\quad x^2(1 - e^2) + y^2 = a^2(1 - e^2)$

$$\frac{x^2}{a^2} + \frac{y^2}{a^2(1-e^2)} = 1.$$

Since $0 < e < 1 \Rightarrow 1 - e^2 > 0 \Rightarrow a^2(1 - e^2) > 0$.

$\therefore$ We can choose a real number $b > 0$ such that $a^2(1 - e^2) = b^2$.

$$\therefore \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \; (a>0, b>0) \qquad ...(4)$$

we have shown that coordinates of P must satisfy (4) if P satisfies the geometric condition SP $= e$(PM) conversely, if x, y satisfy the algebraic equation (4) with $b^2 = a^2(1 - e^2)$ and $0 < e < 1$, then

$$y^2 = b^2\left(1 - \frac{x^2}{a^2}\right) = b^2\left(\frac{a^2 - x^2}{a^2}\right) = \frac{a^2(1-e^2)(a^2 - x^2)}{a^2} = (1-e^2)(a^2 - x^2).$$

$$\therefore \quad SP = \sqrt{(x-ae)^2 + y^2} = \sqrt{x^2 + a^2e^2 - 2aex + (1-e^2)\,(a^2 - x^2)}$$

$$\therefore \quad SP = \sqrt{(xe)^2 - 2(xe)a + a^2} = |xe - a| = e\left|x - \frac{a}{e}\right| = e(PM).$$

If P satisfies the algebraic condition then P satisfies the geometric condition and vice versa.

Thus the locus of P is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, the equation of ellipse in the standard form.

Now let S' be the image of S and $Z'M'$ be the image of ZM with respect to Y-axis, taking S' as focus and $Z'M'$ as corresponding directrix, it can be seen that the corresponding equation of ellipse is also $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Hence for every ellipse, there are two foci and two corresponding directrices.

we have $b^2 = a^2(1 - e^2)$ and $0 < e < 1 \Rightarrow b^2 < a^2 \Rightarrow b < a$.

4.1.3 Nature of the curve

Equation of the ellipse in standard form is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, (a > b > 0)$ ... (1)

(i) Point of intersection with coordinate axes :

If $y = 0$, then $x = \pm a$ i.e., the curve intersect X-axis at A$(a, 0)$ and A$'(-a, 0)$.
hence A A$' = 2a$.

If $x=0$, then $y=\pm b$ i.e., the curve intersect Y-axis at B$(0, b)$ and B$'(0, -b)$. hence B B$' = 2b$.

(ii) From (1) we have $y = \pm \frac{b}{a}\sqrt{a^2 - x^2}$ and $x = \pm \frac{a}{b}\sqrt{b^2 - y^2}$. ... (2)

From (2), y is real $\Leftrightarrow a^2 - x^2 \geq 0 \Leftrightarrow -a \leq x \leq a \Leftrightarrow |x| \leq a$.

From (2), x is real $\Leftrightarrow b^2 - y^2 \geq 0 \Leftrightarrow -b \leq y \leq b \Leftrightarrow |y| \leq b$.

$\therefore$ Corresponding to every real value of x, with $|x| \leq a$, there are two real values of y, equal in magnitude but opposite in sign. Similarly corresponding to every real value of y with $|y| \leq b$, there are two real values of x, equal in magnitude but opposite in sign. Hence ellipse is symmetric about both the axes.

(iii) The curve lies inside the rectangle bounded by the lines $x = a$, $x = -a$, $y = b$, $y = -b$ (see Fig. 4.2)

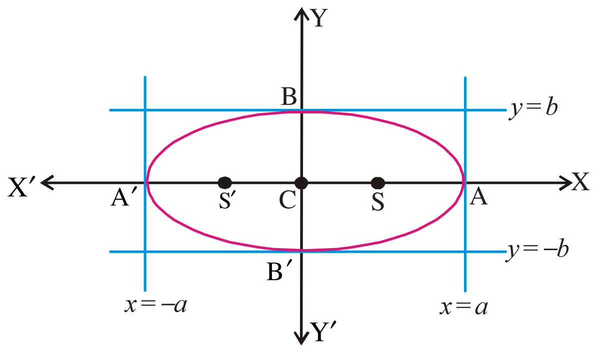


Fig. 4.2

(iv) The trace of the curve $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in first quadrant is shown in Fig. 4.2(a).

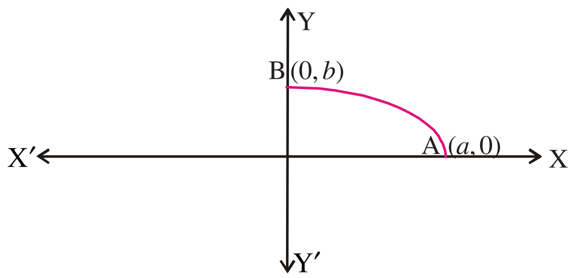


Fig. 4.2(a)

Since the curve is symmetric about both axes, the complete trace of the curve is shown in Fig. 4.3.

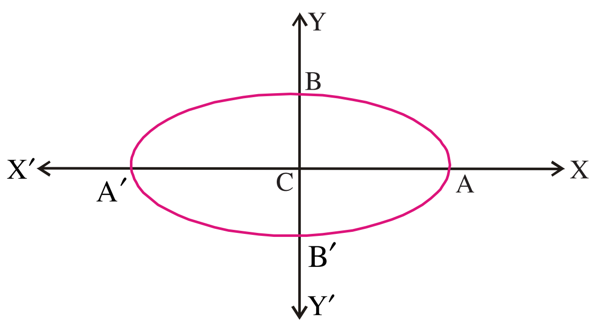


Fig. 4.3

(v) Any chord through C(0, 0) of the ellipse is bisected at the point C, for the points (x, y), $(-x, -y)$ simultaneously lie on the curve. The centre of an ellipse is defined as the point of intersection of its axes of symmetry. Therefore the centre of the ellipse is the point C.

4.1.4 Definition (Major and Minor axes)

The line segment AA′ and BB′ of lengths 2a and 2b respectively are called axes of ellipse. If $a > b$, AA′ is called major axis and BB′ is called minor axis and vice versa if $a < b$. The extremities of the major axis of the ellipse are called the vertices of that ellipse.

4.1.5 Various forms of the ellipse

If $a = b$, then the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is a circle ($x^2+y^2=a^2$) with centre at origin and having radius 'a' and we are familiar with circles. We assumed $a \neq b$ and in the following discussion, we describe different forms of the ellipse.

(i) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ $(a > b > 0)$ (Fig. 4.4)

Major axis	along X-axis
Length of major axis (AA′)	$2a$
Minor axis	along Y-axis
Length of minor axis (BB′)	$2b$
Centre	C = (0, 0)
Foci	S = $(ae, 0)$, S′ = $(-ae, 0)$
Equation of the directrices	$x = a/e$ $x = -a/e$
Eccentricity	$e = \sqrt{\frac{a^2-b^2}{a^2}}$

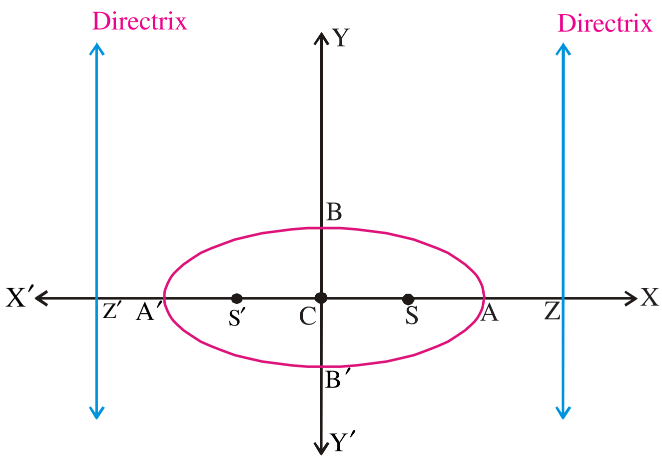


Fig. 4.4

(ii) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \, (0<a<b)$ (Fig. 4.5)

Major axis	along Y-axis
Length of major axis (B B′)	$2b$
Minor axis	along X-axis
Length of minor axis (A A′)	$2a$
Centre	$C=(0,0)$
Foci	$S=(0,be)$ $S'=(0,-be)$
Equation of the directrices	$y=b/e$ $y=-b/e$
Eccentricity	$e=\sqrt{\frac{b^2-a^2}{b^2}}$

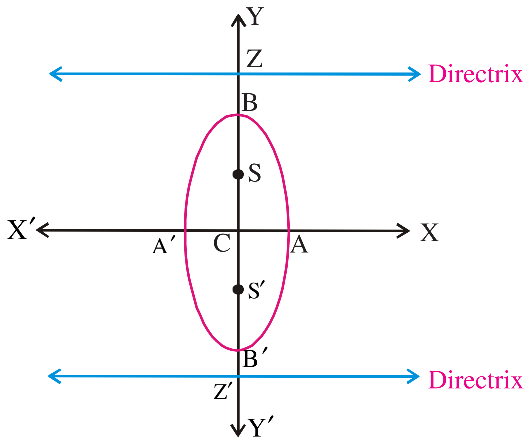


Fig. 4.5

4.1.6 Centre not at the origin

If the centre is at (h,k) and the axes of the ellipse are parallel to the X-and Y - axis, then by shifting the origin to (h,k) by translation of axes and using the results (i) and (ii) above, the following results (iii) and (iv) can be obtained.

(iii) $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1, \; (a>b>0)$ (Fig. 4.6)

Major axis	along $y=k$
Length of major axis (AA′)	$2a$
Minor axis	along $x=h$
Length of minor axis (B B′)	$2b$
Centre	$C=(h,k)$
Foci	$S=(h+ae,k)$ $S'=(h-ae,k)$
Equation of the directrices	$x=h+a/e$ $x=h-a/e$
Eccentricity	$e=\sqrt{\frac{a^2-b^2}{a^2}}$

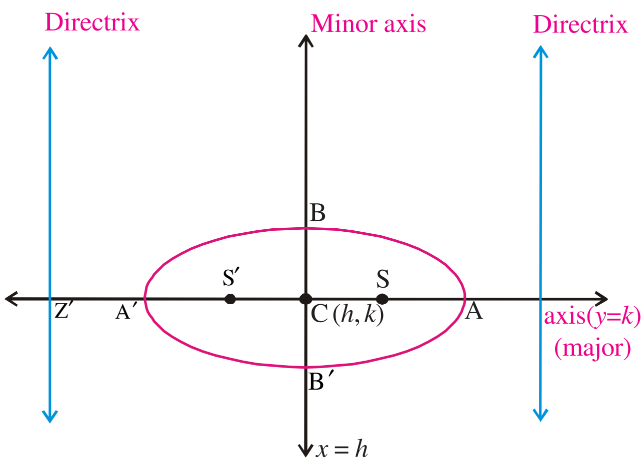


Fig. 4.6

(iv) $\dfrac{(x-h)^2}{a^2}+\dfrac{(y-k)^2}{b^2}=1,\ (0<a<b)$, (Fig. 4.7)

Major axis	along $x=h$
Length of the major axis (B B′)	$2b$
Minor axis	along $y=k$
Length of the minor axis (AA′)	$2a$
Centre	$C=(h,k)$
Foci	$S=(h,k+be)$ $S'=(h,k-be)$
Equation of the directrices	$y=k+b/e$ $y=k-b/e$
Eccentricity	$e=\sqrt{\dfrac{b^2-a^2}{b^2}}$

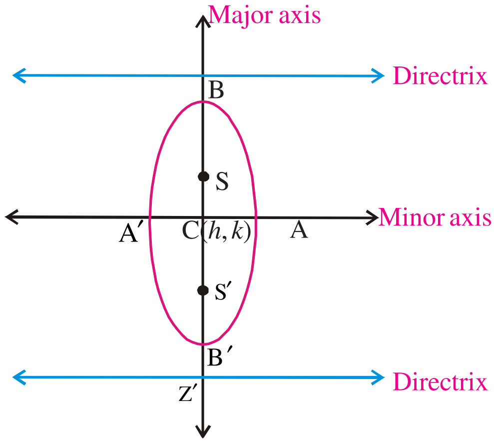


Fig. 4.7

4.1.7 Definitions (Chord, Focal chord, Latus rectum)

Chord : *A line segment joining two points on the ellipse is called a **'chord'** of the ellipse.*

Focal chord : *A chord passing through one of the foci is called a **'Focal chord'**.*

Latus rectum : *A focal chord perpendicular to the major axis of the ellipse is called a **'Latus rectum'**. An ellipse has two Latera Recta.*

4.1.8 Length of the latus rectum

Let L, L′ be the ends of the latus rectum passing through the one of the foci S(ae, 0) of the ellipse (Fig. 4.8)

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ (a>b) \qquad \ldots (1)$$

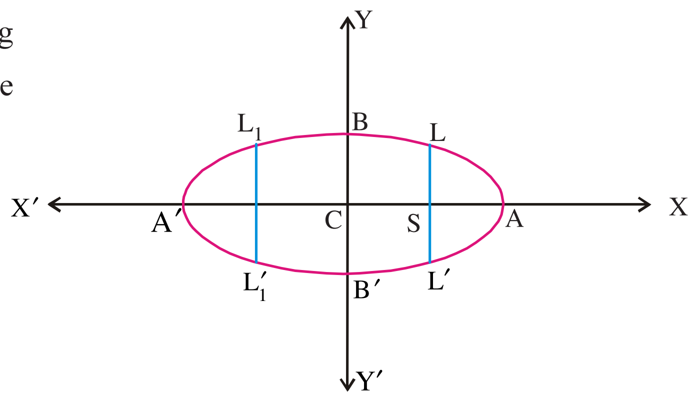


Fig. 4.8

Since LL' is perpendicular to x-axis, the x coordinate of L and L' are equal to 'ae'. This $L=(ae, y_1)$ is on (1), we have

$$\frac{(ae)^2}{a^2}+\frac{y_1^2}{b^2}=1 \Rightarrow \frac{y_1^2}{b^2}=(1-e^2) \Rightarrow y_1^2=b^2(1-e^2)=b^2\left(\frac{b^2}{a^2}\right), \qquad [\because b^2=a^2(1-e^2)]$$

$$\therefore\ y_1=\pm\frac{b^2}{a}.$$

Hence $L=\left(ae, \frac{b^2}{a}\right)$ and $L'=\left(ae, -\frac{b^2}{a}\right)$.

$\therefore$ Length of the latus rectum $LL'=\frac{2b^2}{a}$.

4.1.9 Note

1. The coordinates of the four ends of the latera recta of the ellipse. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, (a>b)$ are

   $L=\left(ae, \frac{b^2}{a}\right)$, $L'=\left(ae, -\frac{b^2}{a}\right)$ and $L_1=\left(-ae, \frac{b^2}{a}\right)$, $L_1'=\left(-ae, -\frac{b^2}{a}\right)$, (Fig.4.8).

2. Length of the latus rectum of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, (b>a)$ is $\frac{2a^2}{b}$ and the coordinates of the four ends of the latera recta are $L\left(\frac{a^2}{b}, be\right)$, $L'\left(-\frac{a^2}{b}, be\right)$ and $L_1\left(\frac{a^2}{b}, -be\right)$ $L_1'\left(-\frac{a^2}{b}, -be\right)$.

3. (i) The equation of the latus rectum of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, (a>b)$ through S is $x=ae$ and through S' is $x=-ae$.

   (ii) The equation of the latus rectum of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, (b>a)$ through S is $y=be$ and through S' is $y=-be$.

4.1.10 Theorem : *If $P(x, y)$ is any point on the ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, (a>b)$. whose foci are S and S' then $SP+S'P$ is a constant.

Proof : The equation of the ellipse is given as $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, (a>b)$. ... (1)

Let S, S' be the foci and ZM, $Z'M'$ be the corresponding directrices. Join SP and $S'P$. Draw PL perpendicular to x-axis and $M'MP$ perpendicular to the two directrices (Fig. 4.9).

By the definition of the ellipse SP = ePM = e(LZ).

$\therefore$ SP $= e(\text{CZ} - \text{CL}) = e\left(\dfrac{a}{e} - x\right)$

$\therefore$ SP $= a - xe$

S′P $= e\text{PM}' = e(\text{LZ}')$

$= e(\text{CL} + \text{CZ}')$

$= e\left(x + \dfrac{a}{e}\right)$

$= a + xe$

$\therefore$ SP + S′P $= a - xe + a + xe$.

$\therefore$ SP + S′P $= 2a$ (constant) = Length of the major axis.

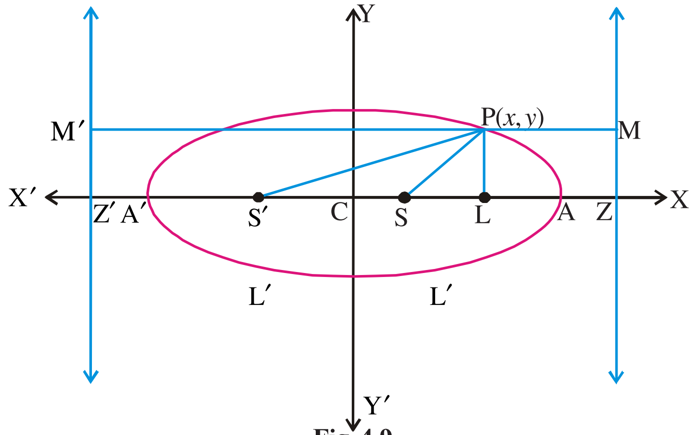


Fig. 4.9

4.1.11 Note (Constructing an ellipse)

There are several methods of constructing an ellipse. One of these uses the fact that SP + S′P = $2a$ (Constant) directly. The two ends of string of length $2a$ are held fixed at the foci S and S′ and a pencil draws the curve as it is held tight against the string (Fig. 4.10).

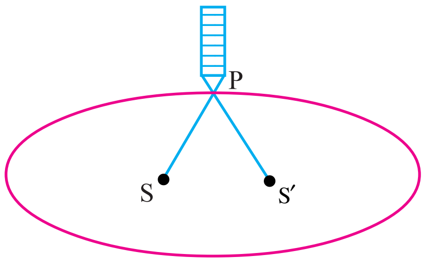


Fig. 4.10

Hence, an ellipse is the locus of a point the sum of whose distances from two fixed points is a constant k, provided the distance between the fixed points is less than k.

4.1.12 Definition (Auxiliary circles)

The circle described on the major axis of an ellipse as diameter is called 'auxiliary circle' of the ellipse.

The equation of the 'auxiliary circle' of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 (a > b)$ is $x^2 + y^2 = a^2$.

(see Fig. 4.10(a)).

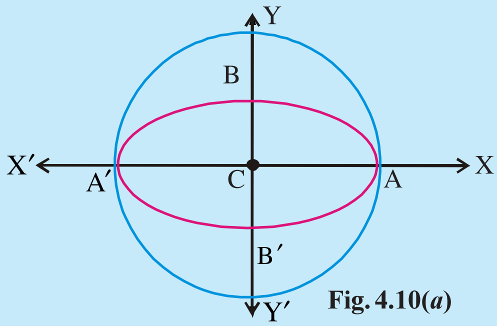


Fig. 4.10(*a*)

4.1.13 Eccentric angle and parametric equation

Let P be any point on the ellipse. Draw PN perpendicular to the major axis and produce it to meet the auxiliary circle at Q. Then angle ACQ is called the 'eccentric angle' of the point P. Let us denote the angle as θ. If P starts from A and moves along the ellipse in the anti-clock wise direction and comes once again at A, then θ will vary from 0 to 2π.

Let the coordinates of P be (x, y). Then $x = a\cos\theta$.

[$\because$ from ΔCNQ, $\cos\theta = \frac{x}{a}$ where CQ is the radius of the auxiliary circle (Fig. 4.11)]

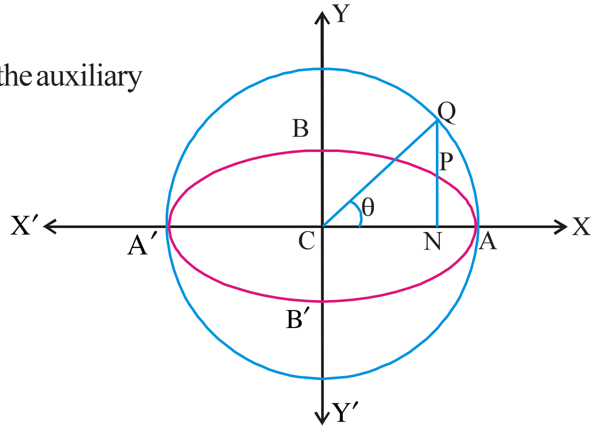


Fig. 4.11

Since P lies on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$,

we have $\frac{a^2\cos^2\theta}{a^2} + \frac{y^2}{b^2} = 1$

$\Rightarrow y^2 = b^2(1 - \cos^2\theta) = b^2\sin^2\theta$

$\Rightarrow y = \pm b\sin\theta$.

$\therefore$ The coordinates of P are of the form $(a\cos\theta, b\sin\theta)$ or $(a\cos\alpha, b\sin\alpha)$ where $\alpha = 2\pi - \theta$. The point $(a\cos\theta, b\sin\theta)$ is for the sake of brevity, called the point θ and is denoted by P(θ).

If we put $x = a\cos\theta, y = b\sin\theta$ in the equation of the ellipse, the equation is satisfied for all values of θ. Hence the pair of equations $x = a\cos\theta$, $y = b\sin\theta$ together yield the single equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. These two equations are known as the parametric equations of the ellipse and θ is called the parameter.

4.1.14 Notation

We denote $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1$ by S throughout this chapter. Thus the equation of the ellipse in standard form is S = 0. Further, we use the following notation similar to the notation given in the chapter on circle.

$$S_1 \equiv \frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1, \quad S_{12} \equiv \frac{x_1x_2}{a^2} + \frac{y_1y_2}{b^2} - 1, \quad S_{11} \equiv \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1.$$

4.1.15 Ellipse and a point in the plane of the ellipse

An ellipse divides the XY-plane into two disjoint regions, one containing the foci, called the interior region of the ellipse and the other is called the exterior region of the ellipse.

Let $P(x_1, y_1)$ be a point in the plane of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $|x_1| \le a$. ... (1)

Draw PN, perpendicular to the major axis of the ellipse (1), which meets the ellipse in Q. Then $N(x_1, 0)$

$$Q = \left(x_1, \frac{b}{a}\sqrt{a^2 - x_1^2}\right) \text{ (or)}$$

$$\left(x_1, -\frac{b}{a}\sqrt{a^2 - x_1^2}\right) \text{ and } PN = |y_1|$$

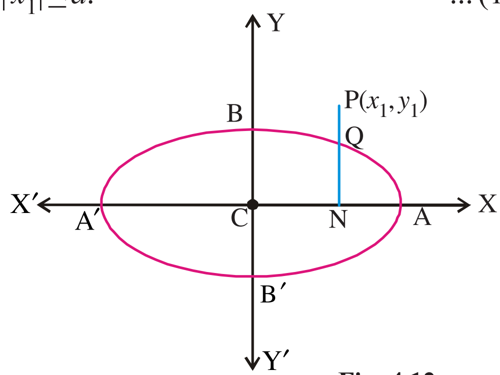


Fig. 4.12

consider $\dfrac{(PN)^2-(QN)^2}{b^2} = \dfrac{y_1^2 - \dfrac{b^2}{a^2}(a^2-x_1^2)}{b^2} = \dfrac{x_1^2}{a^2}+\dfrac{y_1^2}{b^2}-1 \equiv S_{11}$. ... (2)

Now

(i) P lies outside the ellipse $\Leftrightarrow$ PN > QN $\Leftrightarrow (PN)^2-(QN)^2>0$

$$\Leftrightarrow S_{11} \equiv \frac{(PN)^2-(QN)^2}{b^2} > 0$$

(ii) P lies on the ellipse $\Leftrightarrow$ PN = QN $\Leftrightarrow (PN)^2-(QN)^2=0$

$$\Leftrightarrow S_{11} \equiv \frac{(PN)^2-(QN)^2}{b^2} = 0$$

(iii) P lies inside the ellipse $\Leftrightarrow$ PN < QN $\Leftrightarrow (PN)^2<(QN)^2$

$$\Leftrightarrow S_{11} \equiv \frac{(PN)^2-(QN)^2}{b^2} < 0.$$

If $|x_1|>a$, then the point $P(x_1, y_1)$ clearly lies outside the ellipse and $\dfrac{x_1^2}{a^2}+\dfrac{y_1^2}{b^2}-1>0$ in this case also.

Thus the point P lies outside, on or inside the ellipse S = 0 according as S_{11} is positive, zero or negative i.e., $S_{11} \gtreqless 0$.

4.1.16 Solved Problems

1. Problem : *Find the eccentricity, coordinates of foci, Length of latus rectum and equations of directrices of the following ellipses.*

(i) $9x^2+16y^2-36x+32y-92=0$

(ii) $3x^2+y^2-6x-2y-5=0$

Solutions

(i) given ellipse $9x^2+16y^2-36x+32y-92 = 0$

$$9(x^2-4x+4)+16(y^2+2y+1) = 92+36+16$$

$$9(x-2)^2+16(y+1)^2 = 144$$

$$\frac{(x-2)^2}{16}+\frac{(y-(-1))^2}{9}=1,$$

comparing with $\dfrac{(x-h)^2}{a^2}+\dfrac{(y-k)^2}{b^2}=1$, we get

$$a^2=16,\ b^2=9,\ (h,k)=(2,-1) \Rightarrow a=4,\ b=3 \Rightarrow a>b.$$

$$\therefore\quad e=\sqrt{\frac{a^2-b^2}{a^2}}=\sqrt{\frac{16-9}{16}}=\frac{\sqrt{7}}{4}.$$

$\therefore$ Foci $=(h \pm ae, k) = (2 \pm \frac{4\sqrt{7}}{4}, -1)$.

$\therefore$ Foci are $(2 \pm \sqrt{7}, -1)$.

Length of latus rectum $= \frac{2b^2}{a} = \frac{2(9)}{4} = \frac{9}{2}$.

Equations of directrices $x = h \pm \frac{a}{e} = 2 \pm \frac{4\times 4}{\sqrt{7}}$.

i.e., Equations of directrices are $\sqrt{7}x = (2\sqrt{7} \pm 16)$.

(ii) Given ellipse

$$3x^2 + y^2 - 6x - 2y - 5 = 0$$

$$3(x^2 - 2x + 1) + (y^2 - 2y + 1) = 5 + 3 + 1$$

$$3(x-1)^2 + (y-1)^2 = 9$$

$$\frac{(x-1)^2}{3} + \frac{(y-1)^2}{9} = 1,$$

comparing with $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$, we get

$a^2 = 3,\ b^2 = 9,\ (h, k) = (1, 1) \Rightarrow a = \sqrt{3};\ b = 3 \Rightarrow b > a$

$$\therefore\ e = \sqrt{\frac{b^2 - a^2}{b^2}} = \sqrt{\frac{9-3}{9}} = \sqrt{\frac{6}{9}} = \sqrt{\frac{2}{3}} = \frac{\sqrt{2}}{\sqrt{3}}.$$

Foci $= (h, k \pm be) = \left(1, 1 \pm 3.\frac{\sqrt{2}}{\sqrt{3}}\right) = (1, 1 \pm \sqrt{6})$.

Length of the latus rectum $= \frac{2a^2}{b} = \frac{2(3)}{3} = 2$.

Equations of directrices $y = k \pm \frac{b}{e} = 1 \pm \frac{3\sqrt{3}}{\sqrt{2}}$

i.e., $\sqrt{2}y = (\sqrt{2} \pm 3\sqrt{3})$.

2. Problem : *Find the equation of the ellipse referred to its major and minor axes as the coordinate axes X, Y-respectively with latus rectum of length 4 and distance between foci* $4\sqrt{2}$.

Solution : Let the equation of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\ (a > b)$.

Length of the latus rectum $= \frac{2b^2}{a} = 4 \Rightarrow b^2 = 2a$.

Distance between foci $S = (ae, 0)$ $S' = (-ae, 0)$ is $2ae = 4\sqrt{2} \Rightarrow ae = 2\sqrt{2}$.

Now $b^2 = a^2(1-e^2) \Rightarrow 2a = a^2 - (ae)^2 = a^2 - 8 \Rightarrow a^2 - 2a - 8 = 0.$

(or) $(a-4)(a+2) = 0 \Rightarrow a = 4, \ (\because a > 0).$

$b^2 = 2a = 8$

$\therefore$ Equation of the ellipse $\frac{x^2}{16} + \frac{y^2}{8} = 1$ (or) $x^2 + 2y^2 = 16.$

3. Problem : *If the length of the latus rectum is equal to half of its minor axis of an ellipse in the standard form, then find the eccentricity of the ellipse.*

Solution : Let $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $(a > b)$ be the ellipse in its standard form.

Given that the length of the latus rectum $= \frac{1}{2}$ (minor axis)

$$\frac{2b^2}{a} = \frac{1}{2}(2b) \Rightarrow 2b = a$$

$$\therefore \quad 4b^2 = a^2 \Rightarrow 4a^2(1-e^2) = a^2$$

$$\therefore \quad 1 - e^2 = \frac{1}{4} \Rightarrow e^2 = \frac{3}{4} \Rightarrow e = \frac{\sqrt{3}}{2}.$$

4. Problem : *If θ_1, θ_2 are the eccentric angles of the extremeties of a focal chord (other than the vertices) of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $(a > b)$ and e its eccentricity. Then show that*

(i) $e \cos \frac{(\theta_1 + \theta_2)}{2} = \cos \frac{(\theta_1 - \theta_2)}{2}.$

(ii) $\frac{e+1}{e-1} = \cot\left(\frac{\theta_1}{2}\right) \cot\left(\frac{\theta_2}{2}\right).$

Solution : (i) Let $P(\theta_1), Q(\theta_2)$ be two extremeties of a focal chord of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \ (a > b).$$

$\therefore \quad P = (a\cos\theta_1, b\sin\theta_1), \ (\theta_1 \neq 0)$

$Q = (a\cos\theta_2, \ b\sin\theta_2), (\theta_2 \neq \pi)$

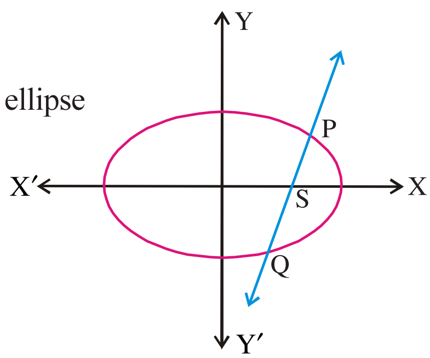


Fig. 4.13

and focus $S = (ae, 0)$. But PQ is focal chord

hence P, S, Q are collinear, (Fig. 4.13)

$\therefore$ Slope of $\overleftrightarrow{PS}$ = Slope of $\overleftrightarrow{SQ}$

$$\frac{b \sin \theta_1}{a(\cos\theta_1 - e)} = \frac{b \sin \theta_2}{a(\cos\theta_2 - e)}$$

$$\therefore \sin\theta_1 \cos\theta_2 - e\sin\theta_1 = \cos\theta_1 \sin\theta_2 - e\sin\theta_2$$

$$\sin\theta_1 \cos\theta_2 - \cos\theta_1 \sin\theta_2 = e(\sin\theta_1 - \sin\theta_2)$$

$$\Rightarrow \sin(\theta_1 - \theta_2) = e(\sin\theta_1 - \sin\theta_2)$$

$$\therefore\ 2\sin\left(\frac{\theta_1-\theta_2}{2}\right)\cos\left(\frac{\theta_1-\theta_2}{2}\right) = 2e\cos\left(\frac{\theta_1+\theta_2}{2}\right)\sin\left(\frac{\theta_1-\theta_2}{2}\right)$$

$$\therefore\ e\cos\left(\frac{\theta_1+\theta_2}{2}\right) = \cos\frac{\theta_1-\theta_2}{2}. \qquad \left(\because\ \sin\left(\frac{\theta_1-\theta_2}{2}\right) \neq 0\right)$$

(ii) From (i) we have $\dfrac{e}{1} = \dfrac{\cos\left(\dfrac{\theta_1-\theta_2}{2}\right)}{\cos\left(\dfrac{\theta_1+\theta_2}{2}\right)}$

$$\therefore\ \frac{e+1}{e-1} = \frac{\cos\left(\frac{\theta_1}{2}-\frac{\theta_2}{2}\right)+\cos\left(\frac{\theta_1}{2}+\frac{\theta_2}{2}\right)}{\cos\left(\frac{\theta_1}{2}-\frac{\theta_2}{2}\right)-\cos\left(\frac{\theta_1}{2}+\frac{\theta_2}{2}\right)} = \frac{2\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}}{2\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}}$$

$$\therefore\ \frac{e+1}{e-1} = \cot\frac{\theta_1}{2}\cot\frac{\theta_2}{2}.$$

5. Problem : *C is the centre, AA' and BB' are major and minor axes of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. If (PN) is the ordinate of a point P on the ellipse then show that* $\dfrac{(PN)^2}{(A'N)\,(AN)} = \dfrac{(BC)^2}{(CA)^2}$.

Solution : Let $P(\theta) = (a\cos\theta, b\sin\theta)$ be a point on the ellipse

$$S \equiv \frac{x^2}{a^2}+\frac{y^2}{b^2} = 1 \text{ (Fig. 4.14)}$$

$\therefore$ PN $= b\sin\theta$, CN $= a\cos\theta$, CA $=$ CA$'$ $= a$, CB $=$ CB$'$ $= b$.

$$\text{L.H.S.} = \frac{(PN)^2}{(A'N)\,(AN)} = \frac{(PN)^2}{(CA'+CN)\,(CA-CN)}$$

$$= \frac{(b\sin\theta)^2}{(a+a\cos\theta)\,(a-a\cos\theta)} = \frac{b^2\sin^2\theta}{a^2(1+\cos\theta)\,(1-\cos\theta)}$$

$$= \frac{b^2\sin^2\theta}{a^2(1-\cos^2\theta)} = \frac{b^2\sin^2\theta}{a^2\sin^2\theta} = \frac{b^2}{a^2} = \frac{(BC)^2}{(CA)^2} = \text{R.H.S.}$$

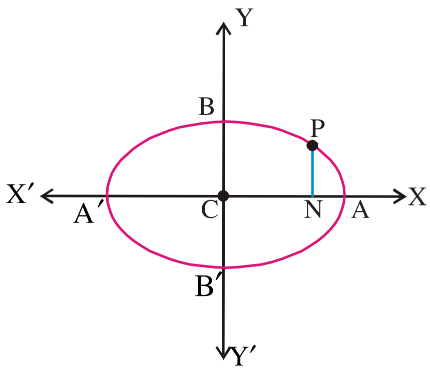


Fig. 4.14

6. Problem : *S and T are the foci of an ellipse and B is one end of the minor axis. If STB is an equilateral triangle, then find the eccentricity of the ellipse.*

Solution : Let $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ $(a>b)$ be an ellipse whose foci are S and T, B is an end of the minor axis such that STB is equilateral triangle, then SB = ST = TB. We have S$(ae, 0)$, T = $(-ae, 0)$ and B$(0, b)$. Consider SB = ST $\Rightarrow$ $(SB)^2 = (ST)^2 \Rightarrow (ae)^2 + b^2 = 4a^2e^2$

$$\therefore\ a^2e^2 + a^2(1-e^2) = 4a^2e^2 \qquad [\because\ b^2 = a^2(1-e^2)]$$

$$e^2 = \frac{1}{4}$$

$\therefore$ Eccentricity of the ellipse is $\frac{1}{2}$.

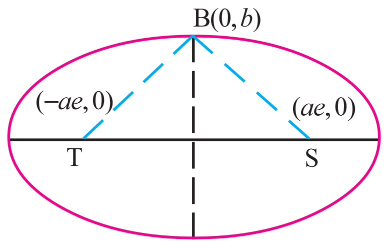


Fig. 4.15

7. Problem : *Show that among the points on the ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ $(a>b)$, $(-a, 0)$ *is the farthest point and (a, 0) is the nearest point from the focus (ae, 0).*

Solution : Let P = (x, y) be any point on the ellipse so that $-a \le x \le a$ and S = $(ae, 0)$ be the focus.

Since (x, y) is on the ellipse,

$$y^2 = \frac{b^2}{a^2}(a^2-x^2)$$

$$= (1-e^2)(a^2-x^2) \qquad [\because b^2 = a^2(1-e^2)] \qquad ....(1)$$

Then we know that

$$SP^2 = (x-ae)^2 + y^2$$

$$= (x-ae)^2 + (1-e^2)(a^2-x^2) \qquad \text{[from (1)]}$$

$$= -2xae + a^2 + e^2x^2$$

$$= [a-ex]^2$$

$$\therefore\ SP = |a-ex|$$

we have $-a \le x \le a$

$$\Rightarrow\ -ae \le xe \le ae$$

$$\Rightarrow\ -ae-a \le xe-a \le ae-a \qquad ....(2)$$

$$\therefore\ ex-a < 0$$

$$\therefore\ SP = a-ex \qquad ....(3)$$

From (2) and (3)

$$ae + a \geq SP \geq a - ae$$

$$\Rightarrow \quad a - ae \leq SP \leq ae + a$$

$\therefore$ Max $SP = ae + a$ when $P = (-a, 0)$

and Min $SP = a - ae$ when $P = (a, 0)$

Hence the nearest point is $(a, 0)$,

and the farthest one is $(-a, 0)$.

8. Problem : *The orbit of the Earth is an ellipse with eccentricity* $\frac{1}{60}$ *with the Sun at one of its foci, the major axis being approximately 186 × 10⁶ miles in length. Find the shortest and longest distance of the Earth from the Sun.*

Solution : We take the orbit of the Earth to be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \ (a > b)$.

Since the major axis is 186×10^6 miles,

$$2a = 186 \times 10^6 \text{ miles}$$

$$\therefore \quad a = 93 \times 10^6 \text{ miles.}$$

If e be the eccentricity of the orbit, $e = \frac{1}{60}$.

We know, the longest and shortest distances of the Earth from the Sun are respectively $a + ae$ and $a - ae$(problem 7)

Here, the longest distance $= 93 \times 10^6 \times \left(1 + \frac{1}{60}\right)$ miles

$= 9455 \times 10^4$ miles.

and the shortest distance $= 93 \times 10^6 \times \left(1 - \frac{1}{60}\right)$ miles

$= 9145 \times 10^4$ miles.

Exercise 4(a)

I. 1. Find the equation of the ellipse with focus at $(1, -1)$, $e = 2/3$ and directrix as $x + y + 2 = 0$.

2. Find the equation of the ellipse in the standard form whose distance between foci is 2 and the length of latus rectum is 15/2.

3. Find the equation of the ellipse in the standard form such that distance between foci is 8 and distance between directrices is 32.

4. Find the eccentricity of the ellipse (in standard form), if its length of the latus rectum is equal to half of its major axis.

5. The distance of a point on the ellipse $x^2 + 3y^2 = 6$ from its centre is equal to 2. Find the eccentric angles.

6. Find the equation of ellipse in the standard form, if it passes through the points $(-2, 2)$ and $(3, -1)$.

7. If the ends of major axis of an ellipse are $(5, 0)$ and $(-5, 0)$. Find the equation of the ellipse in the standard form if its focus lies on the line $3x - 5y - 9 = 0$.

8. If the length of the major axis of an ellipse is three times the length of its minor axis then find the eccentricity of the ellipse.

II. 1. Find the length of major axis, minor axis, latus rectum, eccentricity, coordinates of centre, foci and the equations of directrices of the following ellipse.

(i) $9x^2 + 16y^2 = 144$ (ii) $4x^2 + y^2 - 8x + 2y + 1 = 0$

(iii) $x^2 + 2y^2 - 4x + 12y + 14 = 0$

2. Find the equation of the ellipse in the form $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$, given the following data.

(i) centre $(2, -1)$, one end of the major axis $(2, -5)$, $e = \frac{1}{3}$

(ii) centre $(4, -1)$, one end of the major axis $(-1, -1)$ and passes through $(8, 0)$

(iii) centre $(0, -3)$, $e = 2/3$, semi minor axis 5

(iv) centre $(2, -1)$, $e = 1/2$, length of latus rectum 4

3. Find the radius of the circle passing through the foci of an ellipse $9x^2 + 16y^2 = 144$ and having least radius.

4. A man running on a race course notices that the sum of the distances of the two flag posts from him is always 10 m. and the distance between the flag posts is 8 m. Find the equation of the race course traced by the man.

III. 1. A line of fixed length $(a + b)$ moves so that its ends are always on two fixed perpendicular straight lines. Prove that a marked point on the line, which divides this line into portions of length 'a' and 'b' describes an ellipse and also find the eccentricity of the ellipse when $a = 8$, $b = 12$.

2. Prove that the equation of the chord joining the points 'α' and 'β' on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $\frac{x}{a}\cos\left(\frac{\alpha+\beta}{2}\right) + \frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right) = \cos\left(\frac{\alpha-\beta}{2}\right)$.

4.2 Equation of tangent and normal at a point on the ellipse

In this section, the relation between an ellipse and a straight line in its plane is discussed. The condition for a straight line to be a tangent to a given ellipse is obtained. The cartesian and parametric equations of the tangent and the normal at a given point on the ellipse are also derived.

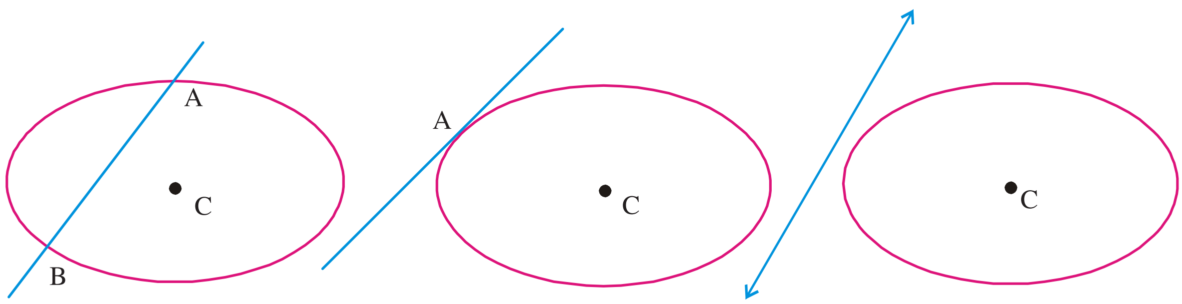
4.2.1 Points of intersection of the ellipse and a straight line

Let $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ be the ellipse and the line $y=mx+c$ be given. The x coordinates of the point of intersection of given ellipse and line are given by the quadratic equation in x obtained by eliminating y.

i.e., $$\frac{x^2}{a^2}+\frac{(mx+c)^2}{b^2}=1$$

i.e., $$x^2(a^2m^2+b^2)+2a^2\,mcx+a^2(c^2-b^2)=0 \qquad ...(1)$$

This quadratic equation in x, has two roots (say x_1 and x_2) distinct real (Fig.4.16(i)), coinciding (Fig.4.16(ii)) or imaginary (Fig. 4.16(iii)) according as the discriminant of equation (1) is positive, zero (or) negative respectively. The ordinates of the points of intersection, y_1, y_2 can be obtained by substituting x_1, x_2 for x in $y=mx+c$.



(i) (ii) (iii)

Fig. 4.16

Note that any straight line that intersects the ellipse at only one point (touches) is tangent to the ellipse.

4.2.2 Theorem : *The condition for a straight line $y = mx + c$ to be a tangent to the ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $c^2=a^2m^2+b^2$.

Proof: The x coordinates of the points of intersection of the line $y=mx+c$ and the ellipse are given by (eq(1) of 4.2.1)

$$(a^2m^2+b^2)\,x^2+2a^2cmx+a^2\,(c^2-b^2)=0 \qquad ...(1)$$

The line will touch the ellipse iff the two points are coincident.

$\Leftrightarrow$ discriminent of (1) is zero.

$$\Leftrightarrow \quad 4a^4 c^2m^2 - 4(a^2m^2 + b^2)\, a^2\, (c^2 - b^2) = 0$$

$$\Leftrightarrow \quad c^2 = a^2m^2 + b^2 \Leftrightarrow \quad c = \pm\sqrt{a^2m^2 + b^2}.$$

4.2.3 Note

(i) In view of the Theorem 4.2.2, the equation of any tangent to the ellipse $S = 0$ can be taken as $y = mx \pm \sqrt{a^2m^2 + b^2}$.

(ii) For every real value of m, there are two parallel tangents to the ellipse as shown in Fig. 4.17.

(iii) The points of contact of these tangents are

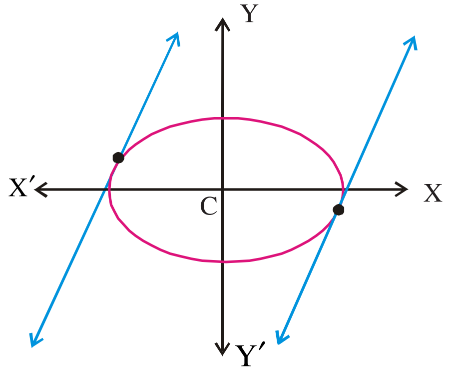


Fig. 4.17

$$\left(\frac{-a^2m}{\sqrt{a^2m^2 + b^2}}, \frac{b^2}{\sqrt{a^2m^2 + b^2}}\right) = \left(\frac{-a^2m}{c}, \frac{b^2}{c}\right) \text{ and}$$

$$\left(\frac{a^2m}{\sqrt{a^2m^2 + b^2}}, \frac{-b^2}{\sqrt{a^2m^2 + b^2}}\right) = \left(\frac{a^2m}{c}, \frac{-b^2}{c}\right) \text{ where } c^2 = a^2\, m^2 + b^2.$$

4.2.4 Theorem : *The equation of the chord joining two points* (x_1, y_1) *and* (x_2, y_2) *on the ellipse* $S = 0$ *is* $S_1 + S_2 = S_{12}$.

Proof : Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two points on the ellipse $S = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$ then $S_{11} = 0$ and $S_{22} = 0$. Consider the first degree equation $S_1 + S_2 = S_{12}$ which represents a straight line.

Substituting (x_1, y_1) it becomes $S_{11} + S_{12} = S_{12} \Rightarrow 0 + S_{12} = S_{12}$.

$\therefore$ (x_1, y_1) satisfies the equation $S_1 + S_2 = S_{12}$.

Similarly (x_2, y_2) also satisfies the equation $S_1 + S_2 = S_{12}$.

$\therefore$ Equation of the chord PQ will be $S_1 + S_2 = S_{12}$.

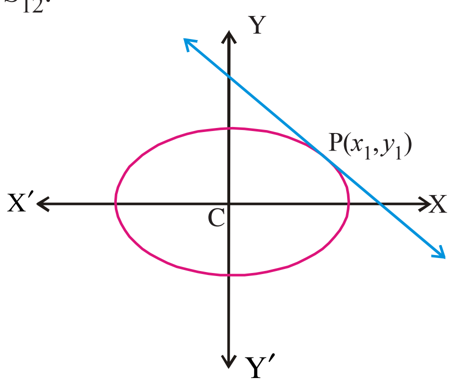


Fig. 4.18

4.2.5 Theorem : *The equation of the tangent at* $P(x_1, y_1)$ *to the ellipse* $S = 0$ *is* $S_1 = 0$.

Proof : Let $P(x_1\, y_1)$ and $Q(x_2\, y_2)$ be two points on $S = 0$ then $S_{11} = 0$ and $S_{22} = 0$.

By Theorem 4.2.4 the equation of the chord PQ is

$$S_1 + S_2 = S_{12}. \qquad \text{... (1)}$$

Chord PQ becomes the tangent at P when Q approaches to P that is $(Q(x_2, y_2)$ approaches to $P(x_1, y_1))$ (Fig. 4.18)

Therefore the equation of the tangent at P obtained by taking limits as (x_2, y_2) tends to $(x_1 y_1)$ on either side of (1) (Fig. 4.18)

So the equation of tangent at P given by $\underset{Q\to P}{Lt}(S_1 + S_2) = \underset{Q\to P}{Lt} S_{12}$.

i.e., $S_1 + S_1 = S_{11}$ $[\because S_2 \to S_1$ and $S_{12} \to S_{11}$ as $(x_2 y_2) \to (x_1 y_1)]$

i.e., $2S_1 = 0$

i.e., $S_1 = 0$.

4.2.6 Theorem : *The equation of the normal at $P(x_1, y_1)$ to the ellipse* $S \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$ is $\frac{a^2 x}{x_1} - \frac{b^2 y}{y_1} = a^2 - b^2$ $(x_1 \neq 0,\ y_1 \neq 0)$.

Proof: By the Theorem 4.2.5, the equation of the tangent to the ellipse S = 0 at $P(x_1, y_1)$ is $S_1 \equiv \frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1 = 0$.

$\therefore$ Slope of the tangent at P $= \frac{-x_1/a^2}{y_1/b^2} = \frac{-b^2 x_1}{a^2 y_1}$.

$\therefore$ Slope of the normal at P $= \frac{a^2 y_1}{b^2 x_1}$.

Hence the equation of the normal at $P(x_1 y_1)$ is $(y - y_1) = \frac{a^2 y_1}{b^2 x_1}(x - x_1)$.

Simplifying this, we get $\frac{a^2 x}{x_1} - \frac{b^2 y}{y_1} = a^2 - b^2$.

4.2.7 Note

(i) If $x_1 = 0$ and $y_1 \neq 0$ then the normal at $P(x_1 y_1) = (0, y_1) = (0, \pm b)$ is the Y - axis.

(ii) If $y_1 = 0$ and $x_1 \neq 0$ then the normal at $P(x_1, y_1) = (x_1, 0) = (\pm a, 0)$ is the X-axis.

(iii) Equation of the tangent at $P(\theta)$ on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$.

We know that the equation of the tangent at $P(x_1, y_1)$ to the ellipse S = 0 is

$S_1 \equiv \frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1 = 0$. Replacing (x_1, y_1) by $P(\theta) = (a\cos\theta, b\sin\theta)$, we get

$$\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1.$$

(iv) Equation of the normal at $P(\theta)$ to the ellipse S = 0 is $\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2$ when $\theta \neq 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$.

The equation of the normal at $P(x_1, y_1)$ to the ellipse $S = 0$.

$\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2$ $(x_1 \neq 0, y_1 \neq 0)$. Replacing (x_1, y_1) by $P(\theta) = (a\cos\theta, b\sin\theta)$

we get $\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2 (\theta \neq 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2})$.

(v) When $\theta = 0$ or π, the equation of the normal is $y = 0$.

When $\theta = \frac{\pi}{2}$ or $\frac{3\pi}{2}$, the equation of the normal is $x = 0$.

4.2.8 Theorem : *At most four normals can be drawn from a given point to an ellipse.*

Proof : Equation of the normal at the point $P(\theta)$ on the ellipse $S = 0$ is $\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2$. If this passes through the point (x_1, y_1) then $\frac{ax_1}{\cos\theta} - \frac{by_1}{\sin\theta} = a^2 - b^2$ ... (1)

The equation will give different values of θ for which the normal passes through (x_1, y_1).

Equation (1) can be written as $ax_1\left(\frac{1+\tan^2\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}\right) - by_1\left(\frac{1+\tan^2\frac{\theta}{2}}{2\tan\frac{\theta}{2}}\right) = a^2 - b^2$. After simplification we get $by_1\tan^4\frac{\theta}{2} + 2(ax_1 + a^2e^2)\tan^3\frac{\theta}{2} + 2(ax_1 - a^2e^2)\tan\frac{\theta}{2} - by_1 = 0.$

This equation in $\tan\frac{\theta}{2}$ is satisfied by atmost four values of $\tan\frac{\theta}{2}$. If we consider one of these values as α_1, $\tan\frac{\theta}{2} = \alpha_1$, $\theta = 2\,\mathrm{Tan}^{-1}(\alpha_1)$ and the general value of $\theta = 2n\pi + 2\,\mathrm{Tan}^{-1}(\alpha_1)$, ($n$ is an integer) which gives the same point on the ellipse as θ.

$\therefore$ Corresponding to one value of $\tan\frac{\theta}{2}$, we get one point on the ellipse.

Hence there will be at most four normals to the ellipse passing through a point.

4.2.9 Solved Problems

1. Problem : *Find the equation of tangent and normal to the ellipse $9x^2 + 16y^2 = 144$ at the end of the latus rectum in the first quadrant.*

Solution : Given ellipse $9x^2 + 16y^2 = 144$,

i.e., $\frac{x^2}{16} + \frac{y^2}{9} - 1 = 0$ (comparing with $S = 0$; $a^2 = 16, b^2 = 9$)

$$\therefore \quad e = \sqrt{\frac{a^2 - b^2}{a^2}} = \sqrt{\frac{16-9}{16}} = \frac{\sqrt{7}}{4}.$$

and end of the latus rectum in the first quadrant is $P\left(ae, \frac{b^2}{a}\right) = \left(\sqrt{7}, \frac{9}{4}\right)$

$\therefore$ Equation of the tangent at P is $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$

i.e., $\frac{x(\sqrt{7})}{16} + \frac{y}{9}\left(\frac{9}{4}\right) = 1$

i.e., $\sqrt{7}x + 4y = 16$

Equation of normal at P is $\frac{a^2 x}{x_1} - \frac{b^2 y}{y_1} = a^2 - b^2$

i.e., $\frac{16x}{\sqrt{7}} - \frac{9y}{\frac{9}{4}} = 16 - 9$

i.e., $\frac{16x}{\sqrt{7}} - 4y = 7$

i.e., $16x - 4\sqrt{7}y = 7\sqrt{7}.$

2. Problem : *If a tangent to the ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $(a > b)$ *meets its major axis and minor axis at M and N respectively then prove that* $\frac{a^2}{(\text{CM})^2} + \frac{b^2}{(\text{CN})^2} = 1$ *where C is the centre of the ellipse.*

Solution : Let $P(\theta) = (a\cos\theta, b\sin\theta)$ be a point on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Then the equation of the tangent at $P(\theta)$ is $\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$

i.e., $\frac{x}{\frac{a}{\cos\theta}} + \frac{y}{\frac{b}{\sin\theta}} = 1$

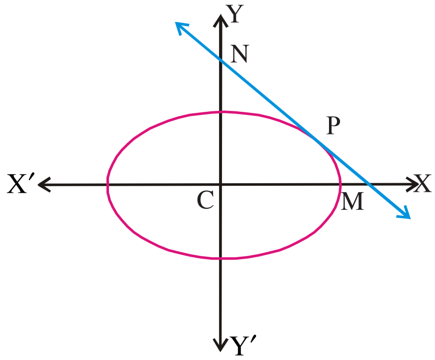


Fig. 4.19

a meets major axis (X-axis) and minor axis (Y-axis) at M and N respectively (Fig. 4.19).

$\therefore$ $\text{CM} = \frac{a}{\cos\theta}$, $\text{CN} = \frac{b}{\sin\theta}$

$$\Rightarrow \frac{a}{CM} = \cos\theta, \ \frac{b}{CN} = \sin\theta$$

$$\therefore \frac{a^2}{(CM)^2} + \frac{b^2}{(CN)^2} = \cos^2\theta + \sin^2\theta = 1.$$

3. Problem : *Find the condition for the line*

(i) *$lx + my + n = 0$ to be a tangent to the ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

(ii) *$lx + my + n = 0$ to be a normal to the ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Solution

(i) Let $lx + my + n = 0$ be a tangent at $P(\theta) = (a\cos\theta, b\sin\theta)$ on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

$\therefore$ Equation of the tangent at $P(\theta)$ is $\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$ is same as the line $lx + my = -n$,

then comparing the coefficients

$$\frac{\cos\theta}{al} = \frac{\sin\theta}{bm} = \frac{-1}{n} \Rightarrow \cos\theta = -\frac{al}{n}, \ \sin\theta = \frac{-bm}{n}, \text{ squaring and adding}$$

we get $1 = \frac{a^2l^2}{n^2} + \frac{b^2m^2}{n^2} = 1 \Rightarrow a^2l^2 + b^2m^2 = n^2$.

(ii) Let $lx + my + n = 0$ be a normal at $P(\theta) = (a\cos\theta, b\sin\theta)$ on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

$\therefore$ Equation of the normal at $P(\theta)$ is $\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2$ is same as the line

$lx + my = -n$ then comparing the coefficients.

$$\frac{l\cos\theta}{a} = \frac{-m\sin\theta}{b} = \frac{-n}{a^2 - b^2} \Rightarrow \cos\theta = \frac{-an}{l(a^2 - b^2)}, \ \sin\theta = \frac{bn}{m(a^2 - b^2)}.$$

Squaring and adding we get $1 = \frac{a^2n^2}{l^2(a^2 - b^2)^2} + \frac{b^2n^2}{m^2(a^2 - b^2)^2}$

i.e., $\frac{a^2}{l^2} + \frac{b^2}{m^2} = \frac{(a^2 - b^2)^2}{n^2}$.

4. Problem : *If the normal at one end of a latus rectum of the ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ *passes through one end of the minor axis, then show that* $e^4+e^2=1$ *[e is the eccentricity of the ellipse]*

Solution : Let L be the one end of the latus rectum of $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Then the coordinates of $L=\left(ae, \frac{b^2}{a}\right)$.

Hence equation of the normal at L is

$$\frac{a^2x}{ae}-\frac{b^2y}{b^2/a}=a^2-b^2$$

$$\frac{ax}{e}-ay=a^2-b^2$$

is a line passes through the one end $B'=(0,-b)$

or minor axis of $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ as shown in Fig. 4.20.

$$\frac{a(0)}{e}-a(-b)=a^2-b^2$$

$$ab=a^2-a^2(1-e^2)$$

$$ab=a^2e^2 \Rightarrow e^2=\frac{b}{a} \Rightarrow e^4=\frac{b^2}{a^2}=\frac{a^2(1-e^2)}{a^2}=1-e^2 \Rightarrow e^4+e^2=1.$$

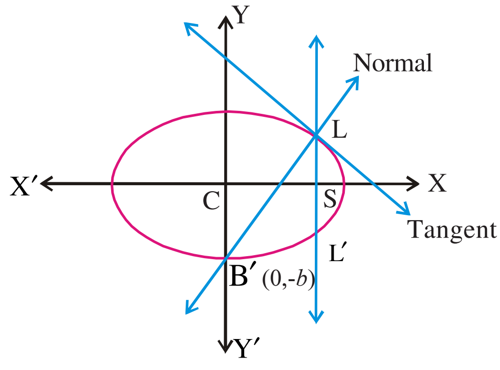


Fig. 4.20

5. Problem : *If PN is the ordinate of a point P on the ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ *and the tangent at P meets the X-axis at T then show that (CN)(CT) =* a^2 *where C is the centre of the ellipse.*

Solution : Let $P(\theta)=(a\cos\theta, b\sin\theta)$ be a point on the ellipse

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Then the equation of the tangent at $P(\theta)$ is

$\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$ or $\frac{x}{\frac{a}{\cos\theta}}+\frac{y}{\frac{b}{\sin\theta}}=1$ meets the X-axis at T

x-intercept $(CT)=\frac{a}{\cos\theta}$ and the ordinate of P is $PN=b\sin\theta$ then its absicca $CN=a\cos\theta$. (see Fig. 4.21)

$$\therefore (CN)\cdot(CT)=(a\cos\theta)\left(\frac{a}{\cos\theta}\right)=a^2.$$

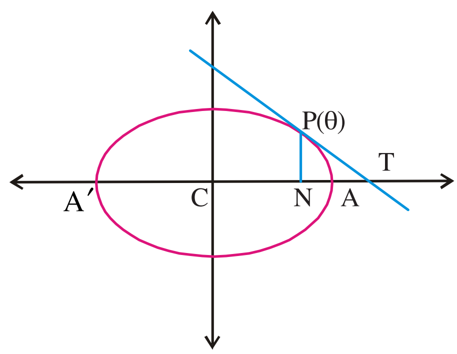


Fig. 4.21

6. Problem : *Show that the points of intersection of the perpendicular tangents to an ellipse lie on a circle.*

Solution : Let the equation of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\,(a>b)$. Any tangent to it in the slope intercept form is

$$y = mx \pm \sqrt{a^2m^2+b^2} \qquad \ldots (1)$$

Let the perpendicular tangents intersect at $P(x_1, y_1)$.

$\therefore$ P lies on (1) for some real m, i.e., $y_1 = mx_1 \pm \sqrt{a^2m^2+b^2}$.

$\therefore$ $(y_1 - mx_1)^2 = a^2m^2 + b^2$.

or

$(x_1^2 - a^2)m^2 - 2x_1y_1m + (y_1^2 - b^2) = 0$ being a quadratic equation in 'm', has two roots say m_1 and m_2 then m_1, m_2 are the slopes of tangents from P to the ellipse

$$\therefore\ m_1m_2 = \left(\frac{y_1^2-b^2}{x_1^2-a^2}\right)$$

$$\therefore\ -1 = \left(\frac{y_1^2-b^2}{x_1^2-a^2}\right)$$ [$\because$ The tangents are perpendicular to each other so that $m_1m_2 = -1$]

i.e., $x_1^2 + y_1^2 = a^2 + b^2$.

If, however, one of the perpendicular tangents is vertical, then such pair of perpendicular tangents intersect at one of the points $(\pm a, \pm b)$ and any of these points satisfies $x^2+y^2=a^2+b^2$.

$\therefore$ The point of intersection of perpendicular tangents to the ellipse S = 0 lies on the circle $x^2+y^2=a^2+b^2$.

4.2.10 Note

The circle $x^2+y^2=a^2+b^2$ is called the **'Director circle'** of the ellipse S = 0. i.e., the centre of the director circle is the centre of the ellipse and its radius is equal to $\sqrt{a^2+b^2}$.

7. Problem : *If a circle is concentric with the ellipse, find the inclination of their common tangent to the major axis of the ellipse.*

Solution : Let the circle be $x^2+y^2=r^2$ and the ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with $a>b$.

The major axis of the ellipse is, therefore, the X-axis.

If $r<b<a$, then the circle lies wholly inside the ellipse making no common tangent possible.

Also, if $b<a<r$, the two curves cannot have a common tangent (the ellipse lies wholly inside the circle)

Therefore, $b \le r \le a$.

Case (i) : $b<r<a$ (see the Fig. 4.22)

Let one of the common tangents to the curves make angle 'θ' with the positive direction of the X-axis.

Let the equation of the tangent to the circle be $x\cos\alpha + y\sin\alpha = r$, where α is the angle made by the radius through the point of contact with the positive direction of the X-axis.

$$\therefore \theta = \frac{\pi}{2} + \alpha \quad \text{or} \quad \theta = \alpha - \frac{\pi}{2}.$$

Since $x\cos\alpha + y\sin\alpha = r$ touches the ellipse also, we have

$$a^2\cos^2\alpha + b^2\sin^2\alpha = r^2.$$

$$\therefore\ a^2\cos^2\left(\theta - \frac{\pi}{2}\right) + b^2\sin^2\left(\theta - \frac{\pi}{2}\right) = r^2 \quad \text{or}$$

$$a^2\cos^2\left(\frac{\pi}{2} + \theta\right) + b^2\sin^2\left(\frac{\pi}{2} + \theta\right) = r^2$$

$$\therefore\ a^2\sin^2\theta + b^2\cos^2\theta = r^2$$

$$\Rightarrow a^2\left(\frac{1-\cos 2\theta}{2}\right) + b^2\left(\frac{1+\cos 2\theta}{2}\right) = r^2$$

$$\Rightarrow (a^2+b^2) + (b^2-a^2)\cos 2\theta = 2r^2$$

$$\Rightarrow \cos 2\theta = \frac{a^2+b^2-2r^2}{a^2-b^2}$$

$$\therefore\ \theta = \frac{1}{2}\cos^{-1}\left[\frac{a^2+b^2-2r^2}{a^2-b^2}\right].$$

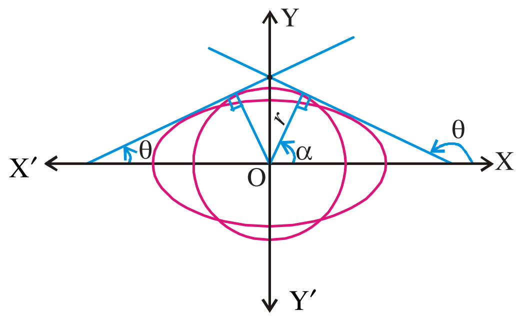


Fig. 4.22

Case(ii) : When $r=a$.

The circle touches the ellipse at the ends of the major axis of the ellipse so that the common tangents would be $x=\pm a$, so that $\theta = \frac{\pi}{2}$.

Case(iii) : When $r = b$

The circle touches the ellipse at the ends of the minor axis of the ellipse, so that the common tangents would be $y = \pm b$, making $\theta = 0$.

Exercise 4(b)

I. 1. Find the equation of tangent and normal to the ellipse $x^2 + 8y^2 = 33$ at $(-1, 2)$.

2. Find the equation of tangent and normal to the ellipse $x^2 + 2y^2 - 4x + 12y + 14 = 0$ at $(2, -1)$.

3. Find the equation of the tangents to $9x^2 + 16y^2 = 144$, which makes equal intercepts on the coordinate axis.

4. Find the coordinates of the points on the ellipse $x^2 + 3y^2 = 37$ at which the normal is parallel to the line $6x - 5y = 2$.

5. Find the value of k if $4x + y + k = 0$ is a tangent to the ellipse $x^2 + 3y^2 = 3$.

6. Find the condition for the line $x \cos \alpha + y \sin \alpha = p$ to be a tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

II. 1. Find the equations of tangent and normal to the ellipse $2x^2 + 3y^2 = 11$ at the point whose ordinate is 1.

2. Find the equations to the tangents to the ellipse $x^2 + 2y^2 = 3$ drawn from the point $(1, 2)$ and also find the angle between these tangents.

3. Find the equation of the tangents to the ellipse $2x^2 + y^2 = 8$ which are

(i) parallel to $x - 2y - 4 = 0$ (ii) perpendicular to $x + y + 2 = 0$

(iii) which makes an angle $\frac{\pi}{4}$ with x-axis.

4. A circle of radius 4, is concentric with the ellipse $3x^2 + 13y^2 = 78$. Prove that a common tangent is inclined to the major axis at an angle $\frac{\pi}{4}$.

III. 1. Show that the foot of the perpendicular drawn from the centre on any tangent to the ellipse lies on the curve $(x^2 + y^2)^2 = a^2x^2 + b^2y^2$.

2. Show that the locus of the feet of the perpendiculars drawn from foci to any tangent of the ellipse is the auxiliary circle.

3. The tangent and normal to the ellipse $x^2+4y^2=4$ at a point $P(\theta)$ on it meets the major axis in Q and R respectively. If $0<\theta<\frac{\pi}{2}$ and $QR=2$ then show that $\theta=\text{Cos}^{-1}\left(\frac{2}{3}\right)$.

Key Concepts

- Equation of ellipse in standard from is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\ (a>b)$. Centre $(0,0)$ and having foci $(\pm ae, 0)$ directrix $x=\pm\frac{a}{e}$ and eccentricity $e=\sqrt{\frac{a^2-b^2}{a^2}}$.
- If P is any point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\ (a>b)$ and foci are S and S′ then $SP+S'P=2a$.
- The equation of the ‘auxiliary circle of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\ (a>b)$ is $x^2+y^2=a^2$.
- $x=a\cos\theta,\ y=b\sin\theta$ are called the parametric equations of the ellipse $S=0$ and θ is called the parameter.
- If $P(x_1, y_1)$ is a point on the plane of the ellipse, then P lies outside, on or inside the ellipse $S=0$ according as S_{11} is positive, zero or negative.
- The condition for a straight line $y=mx+c$ be a tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is $c^2=a^2m^2+b^2$.
- $y=mx\pm\sqrt{a^2m^2+b^2}$ is always a tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at $\left(-\frac{a^2m}{c}, \frac{b^2}{c}\right)$ and $\left(\frac{a^2m}{c}, -\frac{b^2}{c}\right)$ respectively $(c\neq 0, c^2=a^2m^2+b^2)$.
- The equation of the tangent at $P(x_1, y_1)$ to the ellipse $S=0$ is $S_1=0$.
- The equation of the normal at $P(x_1, y_1)$ to the ellipse $S=0$ is $\frac{a^2x}{x_1}-\frac{b^2y}{y_1}=a^2-b^2$ $(x_1\neq 0, y_1\neq 0)$.
- Equation of the tangent at $P(\theta)$ on the ellipse $S=0$ is $\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$.
- Equation of the normal at $P(\theta)$ on the ellipse $S=0$ is $\frac{ax}{\cos\theta}-\frac{by}{\sin\theta}=a^2-b^2$ when $\theta\neq 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$.
- Equation of the director circle of the ellipse $S=0$ is $x^2+y^2=a^2+b^2$.

Historical Note

The name of the curve, ellipse finds a place in ***Rigvēda***. The ellipse was first studied by *Menaechmus*. *Euclid* wrote about the curve and it was given its present name by *Apollonius* (262-190 B.C.). The focus and directrix of an ellipse were first considered by *Pappus* (290-350). In 1602 *A.D., Kepler* discovered that the orbit of Mars was elliptical with Sun at one focus. Infact, the word focus was introduced first by *Kepler* in 1609. There is no exact formula for the length of an ellipse in elementary functions. *Srinivasa Ramanujan* gave the formula : $\pi[3(a+b)-\sqrt{(3a+b)(a+3b)}]$ which is a close approximation of the actual length.

Desargues wrote in 1639 a treatise on conic sections which later was recognised an a classic in the early development of synthetic projective geometry.

Answers

Exercise 4(a)

I. 1. $7x^2 + 7y^2 - 4xy - 26x + 10y + 10 = 0$

2. $\frac{x^2}{16}+\frac{y^2}{15} = 1$ 3. $\frac{x^2}{64}+\frac{y^2}{48} = 1$ 4. $e = \frac{1}{\sqrt{2}}$

5. $\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$ 6. $3x^2 + 5y^2 = 32$ 7. $16x^2 + 25y^2 = 400$

8. $\frac{2\sqrt{2}}{3}$

II. 1.

	Lengths of		Latus	e	centre	Foci	Equation of
	Major axis	Minor axis	Rectum				directrices
(i)	8	6	9/2	$\frac{\sqrt{7}}{4}$	(0, 0)	$(\pm\sqrt{7}, 0)$	$\sqrt{7}x = \pm 16$
(ii)	4	2	1	$\frac{\sqrt{3}}{2}$	(1, −1)	$(1, -1\pm\sqrt{3})$	$\sqrt{3}y + \sqrt{3} \pm 4 = 0$
(iii)	$4\sqrt{2}$	4	$2\sqrt{2}$	$\frac{1}{\sqrt{2}}$	(2, −3)	(4, −3), (0, −3)	$x = 6, x = -2$

2. (i) $9(x-2)^2 + 8(y+1)^2 = 128$ (ii) $(x-4)^2 + 9(y+1)^2 = 25$

(iii) $\frac{x^2}{45} + \frac{(y+3)^2}{25} = 1$ or $\frac{x^2}{25} + \frac{(y+3)^2}{45} = 1$

(iv) $9(x-2)^2 + 12(y+1)^2 = 64$ or $12(x-2)^2 + 9(y+1)^2 = 64$,

3. $\sqrt{7}$

4. $\frac{x^2}{25} + \frac{y^2}{9} = 1$

III. 1. $\frac{\sqrt{5}}{3}$

Exercise 4(b)

I. 1. $x - 16y + 33 = 0$, $16x + y + 14 = 0$

2. $y + 1 = 0$, $x - 2 = 0$

3. $x + y \pm 5 = 0$

4. $(5, 2)$ $(-5, -2)$

5. $k = \pm 7$

6. $p^2 = a^2 \cos^2 \alpha + b^2 \sin^2 \alpha$.

II. 1. Tangents $4x + 3y - 11 = 0$ and $4x - 3y + 11 = 0$

Normals $3x - 4y - 2 = 0$ and $3x + 4y + 2 = 0$

2. $x - 2y + 3 = 0$, $5x + 2y - 9 = 0$, $\text{Tan}^{-1}(12)$

3. (i) $x - 2y \pm 6 = 0$, (ii) $x - y \pm 2\sqrt{3} = 0$, (iii) $y = x \pm 2\sqrt{3}$

Chapter 5

Hyperbola

"The knowledge of which geometry aims is the knowledge of the eternal"

- Plato

Introduction

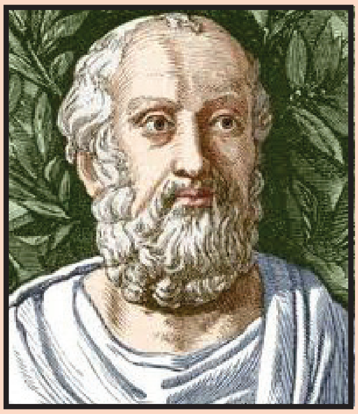
We defined that a hyperbola is a conic in which the eccentricity is greater than unity. Thus a hyperbola is the locus of a point that moves so that the ratio of the distance from a fixed point to its distance from a fixed straight line is greater than 1. The fixed point is called focus, the fixed straight line is called directrix.

5.1 Equation of hyperbola in standard form - Parametric equations

In this section we study the equation of a hyperbola in the standard form and also study its parametric equations.

5.1.1 Equation of a Hyperbola in the standard form

Let S be the focus, ZM be the corresponding directrix and SZ be the perpendicular from S on the directrix. We can divide SZ both internally and externally in the ratio $e : 1$, $(e > 1)$; let the points of division be A and A' as shown in the Figure 5.1. Let C the midpoint of AA'. Now take CZ as the axis of X and the perpendicular at C as Y-axis.



Plato
(427 - 347 B.C.)

Plato *was a disciple of Socrates. Arithmetic was one among various subjects of his study. He was particularly interested in the mysticism of numbers.*

He appreciated so highly the value of geometry. He ran an Academy and at its entrance he displayed a sentence "Let no one ignorant of geometry enter my doors."

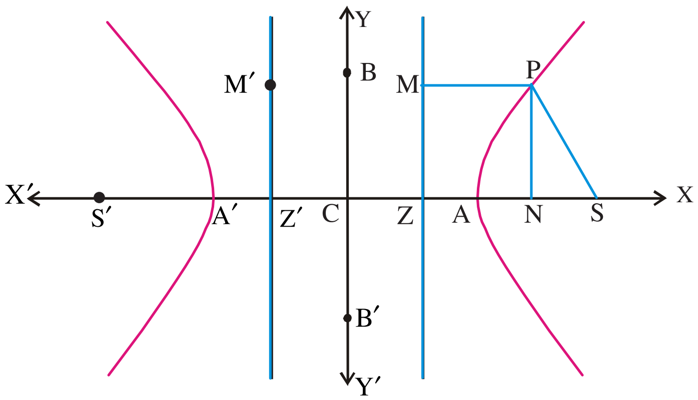


Fig. 5.1

Let AA' be $2a$

$SA = eAZ$, $SA' = eA'Z$ adding

$SA + SA' = e(AZ + A'Z)$

$CS - CA + CS + CA' = e(AA')$

$2(CS) - a + a = e(2a) \Rightarrow 2(CS) = 2(ae) \Rightarrow CS = ae.$

Hence focus $S = (ae, 0)$

$SA' - SA = e(A'Z - AZ) = e(CZ + A'C - CA + CZ)$

$\therefore \quad AA' = e(2(CZ) + a - a) = 2e(CZ)$

$2a = 2eCZ$

$CZ = \frac{a}{e}.$

$\therefore$ Equation of the directrix is $x = \frac{a}{e}.$

Let $P(x, y)$ be any point on the hyperbola, PM, PN be the perpendiculars from P upon the directrix and X-axis respectively.

Thus $SP = e(PM) \Rightarrow (SP)^2 = e^2(PM)^2$

$(x - ae)^2 + y^2 = e^2\left(x - \frac{a}{e}\right)^2$

$\therefore \quad x^2(e^2 - 1) - y^2 = a^2(e^2 - 1)$

$\therefore \quad \frac{x^2}{a^2} - \frac{y^2}{a^2(e^2 - 1)} = 1$, Put $b^2 = a^2(e^2 - 1) > 0 \qquad [\because e > 1]$

then we obtain the equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$ ... (1)

We have shown that the coordinates of P must satisfy the algebraic condition (1) when P satisfies the geometric condition $SP = e(PM)$.

Conversely, if x, y satisfy the algebraic equation (1) with $b^2 = a^2(e^2 - 1)$ and

$e > 1$ then $y^2 = b^2\left(\frac{x^2}{a^2} - 1\right) = \frac{b^2}{a^2}(x^2 - a^2) = (e^2 - 1)(x^2 - a^2)$ ... (2)

$\therefore$ SP $= \sqrt{(x-ae)^2+y^2} = \sqrt{x^2+a^2e^2-2aex+(e^2-1)(x^2-a^2)}$ [$\because$ from (2)]

$= \sqrt{(xe)^2-2(xe)(a)+a^2} = \sqrt{(xe-a)^2} = |xe-a| = e\left|x-\frac{a}{e}\right| = e(\text{PM}).$

$\therefore$ P satisfies the geometric condition, when P satisfies the algebraic condition (1).

Thus the locus of P is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, the equation of the hyperbola in the standard form.

Now let S$'$ be image of S and Z$'$M$'$ be the image of ZM w.r.t. Y-axis. Taking S$'$ as focus and Z$'$M$'$ as directrix, it can be seen that the corresponding equation of hyperbola is also $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. Hence for every hyperbola, there are two foci and two corresponding directrices.

We have $b^2 = a^2(e^2-1)$ and $e>1 \Rightarrow e=\sqrt{\frac{a^2+b^2}{a^2}}$.

5.1.2 Trace of the curve

The equation of hyperbola in standard form is

$$S \equiv \frac{x^2}{a^2}-\frac{y^2}{b^2}-1=0 \qquad \text{... (1)}$$

where $a>0, b>0$ and $b^2=a^2(e^2-1)$.

(i) Put $y=0$ in the equation (1) then we get $x=\pm a$.

$\therefore$ The hyperbola cuts the X-axis at A$(a, 0)$ and A$'(-a, 0)$.

(ii) Put $x=0$ in equation (1). Then we get $y=\pm\sqrt{-b^2}$ does not exist in the cartesian plane. Hence, the curve does not intersect the y-axis.

(iii) Equation of the curve may be written as $y=\pm\frac{b}{a}\sqrt{x^2-a^2}$ then y is real

$\Leftrightarrow x^2-a^2\geq 0 \Leftrightarrow x\leq -a$ or $x\geq a$.

i.e., the curve does not exist between the vertical lines $x=a$ and $x=-a$ further from $x=\pm\frac{a}{b}\sqrt{y^2+b^2}$ then x is real for all values of y and hence each horizontal line $y=k$ intersects the hyperbola at exactly two points. Also $x \longrightarrow \pm\infty$ when $y \longrightarrow \pm\infty$ i.e., the curve is unbounded.

(iv) For any value of x belonging $\mathbf{R} \setminus (-a, a)$, we have two values of $y=\pm\frac{b}{a}\sqrt{x^2-a^2}$ equal but opposite in sign.

$\therefore$ The curve is symmetric about the X-axis (Fig. 5.1).

(v) For each real y, we have two values $x=\pm\frac{a}{b}\sqrt{y^2+b^2}$ equal but opposite in sign

$\therefore$ The curve is symmetric about Y-axis.

$\therefore$ The curve consists of two symmetrical branches each extending to infinity in two directions as shown in Fig. 5.1.

(vi) The $\overline{AA'}$ along the x-axis is called transverse axis of the hyperbola. B and B′ are taken on y-axis such that $BC = B'C = b = a\sqrt{e^2 - 1}$. The $\overline{BB'}$ is called the conjugate axis. Notice that the curve does not meet its conjugate axis.

(vii) As in the ellipse, the symmetry of the curve about its axis shown that it has two foci [S(ae, 0), S′($-ae$, 0)] and two directrices $x = \pm\frac{a}{e}$.

(viii) C is called the centre of the hyperbola. It is the point of intersection of the transverse and conjugate axis. It can be shown that C bisects every chord of the hyperbola that passes through it.

5.1.3 Theorem : *The difference of the focal distances of any point on the hyperbola is constant.*

Proof : Let P(x,y) be any point on the hyperbola whose centre is the origin C foci are S, S′ directrices are $\overleftrightarrow{ZM}$ and $\overleftrightarrow{Z'M'}$ as shown in Fig. 5.2.

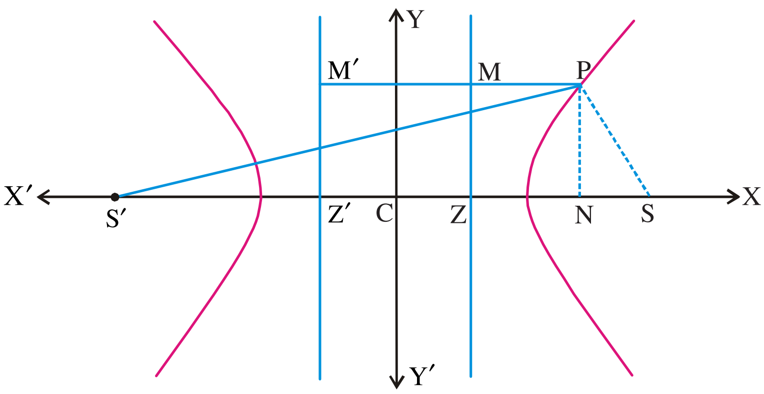


Fig. 5.2

Let PN, PM, PM′ be the perpendiculars drawn from P upon x-axis and the two directrices respectively.

Now $\quad SP = e(PM) = e(NZ) = e(CN - CZ).$

$$\therefore \quad SP = e\left(x - \frac{a}{e}\right) = ex - a.$$

$$\text{and} \quad S'P = e(PM') = e(NZ') = e(CN + CZ') = e\left(x + \frac{a}{e}\right) = ex + a.$$

$$\therefore \quad S'P - SP = 2a.$$

By the above theorem, the hyperbola is sometimes defined as the locus of a point, the difference of whose distances from two fixed points is constant.

5.1.4 Notation

We denote the expression $\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1$ by S throughout this chapter. Thus the equation of a hyperbola in standard form is S = 0. As usual, we use the following notation.

$$S_1 \equiv \frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1.$$

$$S_{12} \equiv \frac{x_1x_2}{a^2} - \frac{y_1y_2}{b^2} - 1.$$

$$S_{11} \equiv \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1.$$

5.1.5 Definition (Rectangular Hyperbola)

If in a hyperbola the length of the transverse axis $(2a)$ *is equal to the length of the conjugate axis* $(2b)$, *the hyperbola is called a rectangular hyperbola.*

Its equation is $x^2 - y^2 = a^2 \qquad [\because\ a = b]$

In this case $e^2 = \frac{a^2+b^2}{a^2} = \frac{2a^2}{a^2} = 2 \Rightarrow e = \sqrt{2}.$

The eccentricity of a rectangular hyperbola is $\sqrt{2}$.

5.1.6 Definition (Auxiliary Circle)

The circle described on the transverse axis of a hyperbola as diameter is called the auxiliary circle of the hyperbola.

The equation of the auxiliary circle of $S = 0$ is $x^2 + y^2 = a^2$.

5.1.7 Parametric equations

Let the equation of the hyperbola be $S = 0$, then the equation of the auxiliary circle is $x^2 + y^2 = a^2$.

Let $P(x, y)$ be any point on the hyperbola and C be the centre. Let M be the projection of P on the transverse axis. Draw the tangent QM to the auxiliary circle from M (Fig. 5.3). If $\angle MCQ = \theta$ then

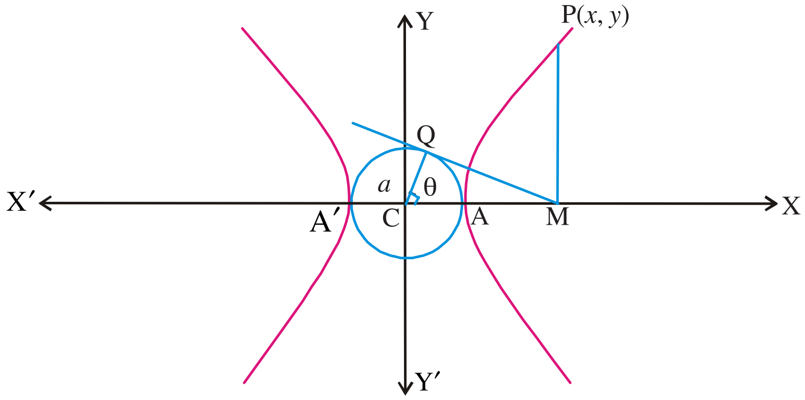


Fig. 5.3

the x coordinate of the point $P = CM = x = a\sec\theta$. Substituting this value of x in $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, we have $\frac{a^2\sec^2\theta}{a^2} - \frac{y^2}{b^2} = 1 \Rightarrow \frac{y^2}{b^2} = \tan^2\theta \Rightarrow y = \pm b\tan\theta.$

Therefore, any point on the hyperbola can be expressed as $(a \sec\theta, b\tan\theta)$ or $(a \sec(2\pi - \theta),\ b\tan(2\pi - \theta))$ for some θ in $[0, 2\pi)$.

Conversely, for any θ in $[0, 2\pi) \setminus \left\{\frac{\pi}{2}, \frac{3\pi}{2}\right\}$, the point $(a\sec\theta, b\tan\theta)$ clearly satisfies $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and so, it lies on the hyperbola. For this reason, the equations $x = a\sec\theta,\ y = b\tan\theta$ $(0 \le \theta < 2\pi,\ \theta \ne \frac{\pi}{2}$ and $\theta \ne \frac{3\pi}{2})$ are taken as the parametric equations of the hyperbola and θ is called the parameter. The point $P(a\sec\theta, b\tan\theta)$ is for the sake of brevity, called the point θ and is denoted by $P(\theta)$ some times.

5.1.8 Definition (Conjugate hyperbola)

The hyperbola whose transverse and conjugate axis are respectively the conjugate and transverse axis of a given hyperbola, is called the conjugate hyperbola of the given hyperbola.

The equation of the hyperbola conjugate to $S \equiv \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0$ is $S' \equiv \frac{x^2}{a^2} - \frac{y^2}{b^2} + 1 = 0$.

For $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$,
(i) The transverse axis lies along X-axis and its length is $2a$.
(ii) The conjugate axis lies along Y-axis and its length is $2b$.

For $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$,
(i) The transverse axis lies along Y-axis and its length is 2b.
(ii) The conjugate axis lies along X-axis and its length is $2a$.

$\therefore$ The hyperbola $S' = 0$ is called the conjugate hyperbola of $S = 0$. Also $S = 0$ is called the conjugate hyperbola of $S' = 0$. Thus each is called the conjugate of the other.

5.1.9 Various forms of the hyperbola

Let $S \equiv \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0$ be a hyperbola and $S' \equiv \frac{x^2}{a^2} - \frac{y^2}{b^2} + 1 = 0$ be its conjugate hyperbola (see Fig. 5.4).

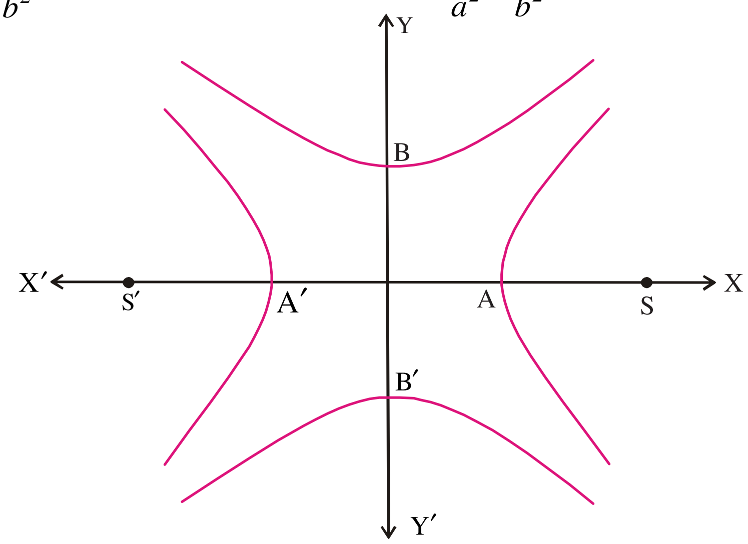


Fig. 5.4

Hyperbola $S \equiv \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0$	Conjugate hyperbola $S' \equiv \frac{x^2}{a^2} - \frac{y^2}{b^2} + 1 = 0$
1. Transverse axis is along X-axis ($y = 0$) length of the transverse axis is $2a$.	Transverse axis is along Y-axis ($x = 0$) length of the transverse axis is $2b$.
2. Conjugate axis is along Y-axis ($x = 0$) length of the conjugate axis is $2b$.	Conjugate axis is along X-axis ($y = 0$) length of the conjugate axis is $2a$.
3. Coordinates of the centre (0, 0).	Coordinates of the centre (0, 0).
4. Coordinates of the foci $(\pm ae, 0)$.	Coordinates of the foci $(0, \pm be)$.
5. Equation of the directrices $x = \pm \frac{a}{e}$.	Equation of the directrices $y = \pm \frac{b}{e}$.
6. Eccentricity $e = \sqrt{\frac{a^2 + b^2}{a^2}}$.	Eccentricity $e' = \sqrt{\frac{a^2 + b^2}{b^2}}$.

5.1.10 Centre not at origin

If the centre is at (h, k) and the axes of the hyperbola are parallel to the coordinate axis, then by shifting the origin (h, k) by translation of axis and using above properties of $S = 0$ and $S' = 0$ the following results can be obtained.

Hyperbola $S \equiv \frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$	Conjugate hyperbola $S' \equiv \frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = -1$
1. Transverse axis is along $y = k$, length of the transverse axis is $2a$.	Transverse axis is along $x = h$, length of the transverse axis is $2b$.
2. Conjugate axis is along $x = h$ length of conjugate axis is $2b$.	Conjugate axis is along $y = k$, length of conjugate axis is $2a$.
3. Coordinates of the centre (h, k).	Coordinates of the centre (h, k).
4. Coordinates of the foci $(h \pm ae, k)$.	Coordinates of the foci $(h, k \pm be)$.
5. Equations of the directrices $x = h \pm \frac{a}{e}$.	Equations of the directrices $y = k \pm \frac{b}{e}$.
6. Eccentricity $e = \sqrt{\frac{a^2 + b^2}{a^2}}$.	Eccentricity $e' = \sqrt{\frac{a^2 + b^2}{b^2}}$.

5.1.11 Note

To find the foci, centre, equations of the directrices, etc. for a rectangular hyperbola $x^2 - y^2 = a^2$ (or) $y^2 - x^2 = a^2$ replace 'b' with 'a' and $e = \sqrt{2}$ in 5.1.9 and 5.1.10 as the length of the transverse axis is equal to the length of the conjugate axis.

5.2 Equation of Tangent and Normal at a point on the hyperbola

The concepts of focal chord, latus rectum, tangent, normal at a point on a hyperbola are defined analogously as in the case of an ellipse. In the following, we list out certain important properties of a hyperbola. The reader can easily supplement the proofs of these, which are similar to those of an ellipse.

1. A point $P(x_1, y_1)$ in the plane of the hyperbola $S = 0$ lies inside the hyperbola (i.e., in the region not containing the centre) if $S_{11} > 0$, lies out side (i.e., in the region containing the centre) if $S_{11} < 0$ and on the curve if $S_{11} = 0$.
2. The end of the latera recta are $\left(\pm ae, \pm\frac{b^2}{a}\right)$ and the length of the latus rectum is $\frac{2b^2}{a}$.
3. The equation of the tangent at $P(x_1, y_1)$ is $S_1 = 0$.
4. The equation of the tangent at 'θ' is $\frac{x}{a}\sec\theta - \frac{y}{b}\tan\theta = 1 \left(\theta \neq \frac{\pi}{2}, \frac{3\pi}{2}\right)$.
5. The equation of the normal at $P(x_1, y_1)$ is $\frac{a^2x}{x_1} + \frac{b^2y}{y_1} = a^2 + b^2$ $(y_1 \neq 0)$ which is always the case except at vertices. At vertices x-axis is the normal.
6. The equation of the normal at 'θ' is $\frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a^2 + b^2$ $(\theta \neq 0, \pi)$.
7. The condition for a straight line $y = mx + c$ to be a tangent to the hyperbola $S = 0$ is $c^2 = a^2m^2 - b^2$.

   Hence the equation of a tangent to the hyperbola in slope form may be taken as $y = mx \pm \sqrt{a^2m^2 - b^2}$.

For any real value of m with $m^2 > \frac{b^2}{a^2}$ there are two parallel tangents to the hyperbola. Note that any horizontal line cannot be a tangent to the hyperbola. Two vertical tangents are $x = \pm a$.

5.2.1 Asymptotes of a curve

Definition 1 : *A non vertical line with equation $y = mx + c$ is called an asymptote of the graph of $y = f(x)$ if the difference of $f(x)$ and $mx + c$ is non-zero and tends to '0' as $x \to \infty$ or $x \to -\infty$.*

Definition 2 : *A vertical line $x = a$, is called a vertical asymptote of the graph $y = f(x)$ if $|f(x)| \to \infty$ as $x \to a$ from the left or from the right.*

1. **Example :** $y = x$ is an asymptote of the curve $y = x + \frac{1}{x}$.

   $[\because f(x) - x = \frac{1}{x} \to 0$ as $x \to \infty$ or $x \to -\infty]$

2. **Example :** y axis $(x = 0)$ is a vertical asymptote of the curve $y = x + \frac{1}{x}$.

   $[\because |f(x)| = \left|x + \frac{1}{x}\right| = x + \frac{1}{x}$ if $x > 0$, $|f(x)| \to \infty$ as $x \to 0^+$ also

   $|f(x)| = -\left(x + \frac{1}{x}\right)$ if $x < 0$ and therefore $|f(x)| \to \infty$ as $x \to 0^-]$

Asymptotes of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Because of the symmetry of the hyperbola about both the axes, we consider the portion of the curve in the first quadrant whose equation is $y = \pm\frac{b}{a}\sqrt{x^2 - a^2}$, $(x \geq a)$.

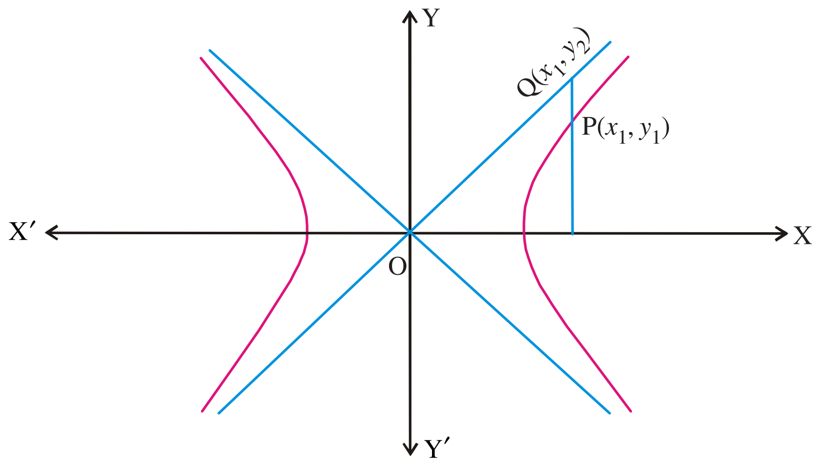


Fig. 5.5

If P(x_1, y_1) is a point on this branch of the hyperbola and (x_1, y_2) is a point where ordinate through P meets the line $y = \frac{b}{a}x$, then $0 < (y_2 - y_1) = \frac{b}{a}(x_1 - \sqrt{x_1^2 - a^2}) = \frac{ab}{x_1 + \sqrt{x_1^2 - a^2}} \leq \frac{ab}{x_1}$ and $\frac{ab}{x_1} \to 0$ as $x_1 \to \infty$ therefore $y_2 - y_1 \to 0$ as $x_1 \to \infty$.

Therefore the line $y = \frac{b}{a}x$ is an asymptote of the hyperbola. By considering the portion of the curve $y = -\frac{b}{a}\sqrt{x_1^2 - a^2}$, $(x \leq -a)$ it can be similarly seen that $y = -\frac{b}{a}x$ is another as asymptote for this hyperbola.

5.2.2 Note

(i) From the above equation of asymptotes, it is clear that they pass through the centre of the hyperbola and the axes of the hyperbola are the angle bisectors of the angle between the asymptotes.

(ii) Let $S \equiv \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0$ be hyperbola. Then $S' \equiv \frac{x^2}{a^2} - \frac{y^2}{b^2} + 1 = 0$ is its conjugate hyperbola and $A \equiv \frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$ is the combined equation of its asymptotes.

$$S + S' = 2A.$$

(iii) Even if the equation of a hyperbola is not in the standard form, by suitable rotation and translation of the coordinate axes, it is possible to transform the equation of the curve into the standard form and accordingly, the combined equation of its pair of asymptotes. Therefore, we observe that the equation to a hyperbola and the combined equation to its asymptotes differs only by a constant. (Notice that the equation $A = 0$ differs from the equation $S = 0$ by a constant and the equation $S' = 0$ differs from $A = 0$ by exactly the same number that $A = 0$ differs from $S = 0$).

5.2.3 Solved Problems

1. Problem : *Find the centre, eccentricity, foci, directrices and the length of the latus rectum of the following hyperbolas.* (i) $4x^2 - 9y^2 - 8x - 32 = 0$ (ii) $4(y+3)^2 - 9(x-2)^2 = 1$.

Solution

(i) $4(x^2 - 2x) - 9y^2 = 32$

$4(x^2 - 2x + 1) - 9y^2 = 32 + 4$

$4(x-1)^2 - 9y^2 = 36$

$$\frac{(x-1)^2}{9} - \frac{y^2}{4} = 1$$

$\therefore$ Centre of the hyperbola is the point (1, 0).

The semi-transverse axis $a = 3$, the semi-conjugate axis $b = 2$.

$$\therefore\ e = \sqrt{\frac{a^2+b^2}{a^2}} = \sqrt{\frac{9+4}{9}} = \frac{\sqrt{13}}{3}.$$

Coordinates of the foci are $= \left(1 \pm 3\frac{\sqrt{13}}{3}, 0\right) = (1 \pm \sqrt{13}, 0)$.

The equations of directrices $x = 1 \pm \dfrac{3\times 3}{\sqrt{13}} \Rightarrow x = 1 \pm \dfrac{9}{\sqrt{13}}$.

Length of latus rectum $= \dfrac{2(4)}{3} = \dfrac{8}{3}$.

(ii) The given equation of the hyperbola $4(y+3)^2 - 9(x-2)^2 = 1$. It can be written as

$$\frac{[y-(-3)]^2}{1/4} - \frac{(x-2)^2}{1/9} = 1$$

Centre of the hyperbola is the point (2, –3).

The semi-transverse axis $b = \dfrac{1}{2}$, the semi-conjugate axis $a = \dfrac{1}{3}$.

$$\therefore\ e = \sqrt{\frac{a^2+b^2}{b^2}} = \sqrt{\frac{\frac{1}{9}+\frac{1}{4}}{\frac{1}{4}}} = \sqrt{\frac{13}{9}} = \frac{\sqrt{13}}{3}.$$

Coordinates of the foci are $(h, k \pm be) = \left(2, -3 \pm \frac{1}{2}\frac{\sqrt{13}}{3}\right) = \left(2, -3 \pm \frac{\sqrt{13}}{6}\right)$.

The equations of directrices $y = k \pm \dfrac{b}{e} = -3 \pm \dfrac{1}{2}\dfrac{3}{\sqrt{13}} \Rightarrow y = -3 \pm \dfrac{3}{2\sqrt{13}}$.

Length of latus rectum $= \dfrac{2(1/9)}{1/2} = \dfrac{4}{9}$.

2. Problem : *If e, e_1 are the eccentricities of a hyperbola and its conjugate hyperbola prove that* $\dfrac{1}{e^2} + \dfrac{1}{e_1^2} = 1$.

Solution : Let e, e_1 be the eccentricities of the hyperbola $S \equiv \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} - 1 = 0$ and its conjugate hyperbola is $S' \equiv \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} + 1 = 0$ respectively, then

$$e = \sqrt{\frac{a^2+b^2}{a^2}}, \quad e_1 = \sqrt{\frac{a^2+b^2}{b^2}} \quad (5.1.10)$$

$$\frac{1}{e^2} + \frac{1}{e_1^2} = \frac{a^2}{a^2+b^2} + \frac{b^2}{a^2+b^2} = \frac{a^2+b^2}{a^2+b^2} = 1$$

3. Problem : (i) *If the line $lx + my + n = 0$ is a tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ then, show that $a^2l^2 - b^2m^2 = n^2$.*

(ii) *If the $lx + my + n = 0$ is a normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ then show that $\frac{a^2}{l^2} - \frac{b^2}{m^2} = \frac{(a^2+b^2)^2}{n^2}$.*

Solution

(i) Let the line $L \equiv lx + my + n = 0$ be a tangent to the hyperbola $S \equiv \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0$ at $P(\theta)$.

Then the equation of the tangent to $S = 0$ at $P(\theta)$ is $S_1 \equiv \frac{x}{a}\sec\theta - \frac{y}{b}\tan\theta - 1 = 0$. Since $L = 0$ and $S_1 = 0$ represents same line, comparing coefficients, $\frac{\sec\theta}{al} = -\frac{\tan\theta}{bm} = -\frac{1}{n}$.

$$\therefore \sec\theta = -\frac{al}{n} \quad \text{and} \quad \tan\theta = \frac{bm}{n}$$

$$\therefore 1 = \sec^2\theta - \tan^2\theta = \frac{a^2l^2}{n^2} - \frac{b^2m^2}{n^2} \text{ (or) } a^2l^2 - b^2m^2 = n^2.$$

(ii) Let the line $L = lx + my + n = 0$ be a normal to the hyperbola $S \equiv \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0$ at $P(\theta)$.

Then the equation of the normal to $S = 0$ at $P(\theta)$ is $\frac{ax}{\sec\theta} + \frac{by}{\tan\theta} - (a^2+b^2) = 0$, represents same as $L = 0$ then, comparing coefficients.

$$\frac{l}{a/\sec\theta} = \frac{m}{b/\tan\theta} = \frac{n}{-(a^2+b^2)} \Rightarrow \frac{l\sec\theta}{a} = \frac{m\tan\theta}{b} = \frac{-n}{a^2+b^2}.$$

$$\therefore \sec\theta = \frac{-na}{l(a^2+b^2)}, \quad \tan\theta = \frac{-nb}{m(a^2+b^2)}$$

But $$1 = \sec^2\theta - \tan^2\theta = \frac{n^2a^2}{l^2(a^2+b^2)^2} - \frac{n^2b^2}{m^2(a^2+b^2)^2}$$

i.e., $$\frac{a^2}{l^2} - \frac{b^2}{m^2} = \frac{(a^2+b^2)^2}{n^2}.$$

4. Problem : *Find the equations of the tangents to the hyperbola $3x^2 - 4y^2 = 12$ which are* (i) *parallel and* (ii) *perpendicular to the line $y = x - 7$.*

Solution : Equation of the given hyperbola can be written as $\frac{x^2}{4} - \frac{y^2}{3} = 1$ so that $a^2 = 4$, $b^2 = 3$. Equation of the given line $y = x - 7$ and its slope is '1'.

(i) Slope of the tangents which are parallel to the given line is 1 (i.e., $m = 1$). Equations of the tangents are $y = mx \pm \sqrt{a^2m^2 - b^2} = x \pm \sqrt{4-3}$ i.e., $y = x \pm 1$.

(ii) Slope of the tangents which are perpendicular to the given line is (-1) (i.e., $m=-1$)

Equations of tangents which are perpendicular to the given line are

$$y = (-1)x \pm \sqrt{4(-1)^2 - 3} \qquad \text{i.e.,} \quad x + y = \pm 1.$$

5. Problem : *Prove that the point of intersection of two perpendicular tangents to the hyperbola* $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ *lies on the circle* $x^2 + y^2 = a^2 - b^2$.

Solution : Let $P(x_1, y_1)$ be the point of intersection of two perpendicular tangents to the hyperbola $S \equiv \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0$. The equation of any tangents to $S = 0$ in the slope intercept form is $y = mx \pm \sqrt{a^2m^2 - b^2}$. If it passes through $P(x_1, y_1)$ then $y_1 - mx_1 = \pm\sqrt{a^2m^2 - b^2}$, squaring both sides

$y_1^2 + m^2x_1^2 - 2mx_1y_1 = a^2m^2 - b^2$

(or) $\quad (x_1^2 - a^2)m^2 - 2x_1y_1m + (y_1^2 + b^2) = 0 \qquad \ldots (1)$

(1) is a quadratic equation in 'm' and therefore, gives two values of m say m_1, m_2 which are slopes of tangents from (x_1, y_1).

$$\therefore \; m_1m_2 = \text{product of roots of (1)} = \frac{y_1^2 + b^2}{x_1^2 - a^2}.$$

$$\therefore \; \frac{y_1^2 + b^2}{x_1^2 - a^2} = -1 \quad [\because \text{ tangents are perpendicular } \Rightarrow m_1m_2 = -1]$$

i.e., $\quad x_1^2 + y_1^2 = a^2 - b^2$.

Hence point $P(x_1, y_1)$ lies on the circle $x^2 + y^2 = a^2 - b^2$.

Note that the circle $x^2 + y^2 = a^2 - b^2$ is called the **director circle** of the hyperbola $S = 0$.

6. Problem : *A circle cuts the rectangular hyperbola* $xy = 1$ *in the points* (x_r, y_r), $r = 1, 2, 3, 4$. *Prove that* $x_1x_2x_3x_4 = y_1y_2y_3y_4 = 1$.

Solution : Let the circle be $x^2 + y^2 = a^2$.

Since $\left(t, \frac{1}{t}\right)$ $(t \neq 0)$ lies on $xy = 1$, the points of intersection of the circle and the hyperbola are given by

$$t^2 + \frac{1}{t^2} = a^2$$

$$\Rightarrow t^4 - a^2t^2 + 1 = 0$$

$$\Rightarrow t^4 + 0.t^3 - a^2t^2 + 0.t + 1 = 0.$$

If t_1, t_2, t_3 and t_4 are the roots of the above biquadratic, then $t_1t_2t_3t_4 = 1$.

If $(x_r, y_r) = \left(t_r, \frac{1}{t_r}\right)$, $r = 1, 2, 3, 4$, then $x_1x_2x_3x_4 = t_1t_2t_3t_4 = 1$,

and $y_1y_2y_3y_4 = \frac{1}{t_1t_2t_3t_4} = 1$.

7. Problem : *If four points be taken on a rectangular hyperbola such that the chords joining any two points is perpendicular to the chord joining the other two, and if α, β, γ and δ be the inclinations to either asymptote of the straight lines joining these points to the centre, prove that* $\tan\alpha\ \tan\beta\ \tan\gamma\ \tan\delta = 1$.

Solution : Let the equation of the rectangular hyperbola be $x^2 - y^2 = a^2$. By rotating the X-axis and the Y-axis about the origin through an angle $\frac{\pi}{4}$ in the clockwise sense, the equation $x^2 - y^2 = a^2$ can be transformed to the form $xy = c^2$.

Let $\left(ct_r, \frac{c}{t_r}\right)$, $r = 1, 2, 3, 4$ ($t_r \neq 0$) be four points on the curve. Let the chord joining $A = \left(ct_1, \frac{c}{t_1}\right)$, $B = \left(ct_2, \frac{c}{t_2}\right)$ be perpendicular to the chord joining $C = \left(ct_3, \frac{c}{t_3}\right)$ and $D = \left(ct_4, \frac{c}{t_4}\right)$.

The slope of $\overleftrightarrow{AB}$ is $\dfrac{\frac{c}{t_1} - \frac{c}{t_2}}{ct_1 - ct_2} = \dfrac{-1}{t_1t_2}$ [No chord of the hyperbola can be vertical]

Similarly slope of $\overleftrightarrow{CD}$ is $-\dfrac{1}{t_3t_4}$,

Since $\overleftrightarrow{AB} \perp \overleftrightarrow{CD}$, $\left(-\dfrac{1}{t_1t_2}\right)\left(-\dfrac{1}{t_3t_4}\right) = -1$

$$\Rightarrow t_1t_2t_3t_4 = -1 \qquad \text{... (1)}$$

We know the coordinate axes are the asymptotes of the curve, If $\overleftrightarrow{OA}$, $\overleftrightarrow{OB}$, $\overleftrightarrow{OC}$ and $\overleftrightarrow{OD}$ make angles α, β, γ and δ with the positive direction of the X-axis, then $\tan\alpha$, $\tan\beta$, $\tan\gamma$ and $\tan\delta$ are their respective slopes. [O, the origin is the centre, None of $\overleftrightarrow{OA}$, $\overleftrightarrow{OB}$, $\overleftrightarrow{OC}$ and $\overleftrightarrow{OD}$ is vertical]

$\therefore\ \tan\alpha = \dfrac{\frac{c}{t_1} - 0}{ct_1 - 0} = \dfrac{1}{t_1^2}$. Similarly, $\tan\beta = \dfrac{1}{t_2^2}$, $\tan\gamma = \dfrac{1}{t_3^2}$ and $\tan\delta = \dfrac{1}{t_4^2}$.

$$\therefore\ \tan\alpha\ \tan\beta\ \tan\gamma\ \tan\delta = \frac{1}{t_1^2t_2^2t_3^2t_4^2} = 1 \qquad \text{[From (1)]}$$

If $\overleftrightarrow{OA}$, $\overleftrightarrow{OB}$, $\overleftrightarrow{OC}$ and $\overleftrightarrow{OD}$ make angles α, β, γ and δ with the other asymptote the Y-axis then $\cot\alpha$, $\cot\beta$, $\cot\gamma$ and $\cot\delta$ are their respective inclinations so that $\cot\alpha\cot\beta\cot\gamma\cot\delta$

$= \tan\alpha\tan\beta\tan\gamma\tan\delta$

$= 1$.

Exercise 5(a)

I. 1. One focus of a hyperbola is located at the point $(1, -3)$ and the corresponding directrix is the line $y = 2$. Find the equation of the hyperbola if its eccentricity is $\frac{3}{2}$.

2. Find the equations of the hyperbola whose foci are $(\pm 5, 0)$, the transverse axis is of length 8.
3. Find the equation of the hyperbola, whose asymptotes are the straight lines $(x + 2y + 3) = 0, (3x + 4y + 5) = 0$ and which passes through the point $(1, -1)$.
4. If $3x - 4y + k = 0$ is a tangent to $x^2 - 4y^2 = 5$ find the value of k.
5. Find the product of lengths of the perpendiculars from any point on the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$ to its asymptotes.
6. If the eccentricity of a hyperbola is $\frac{5}{4}$, then find the eccentricity of its conjugate hyperbola.
7. Find the equation of the hyperbola whose asymptotes are $3x = \pm 5y$ and the vertices are $(\pm 5, 0)$.
8. Find the equation of the normal at $\theta = \pi/3$ to the hyperbola $3x^2 - 4y^2 = 12$.
9. If the angle between the asymptotes is 30° then find its eccentricity.

II. 1. Find the centre, foci, eccentricity, equation of the directrices, length of the latus rectum of the following hyperbolas

(i) $16y^2 - 9x^2 = 144$ (ii) $x^2 - 4y^2 = 4$

(iii) $5x^2 - 4y^2 + 20x + 8y = 4$ (iv) $9x^2 - 16y^2 + 72x - 32y - 16 = 0$

2. Find the equation to the hyperbola whose foci are $(4, 2)$ and $(8, 2)$ and eccentricity is 2.
3. Find the equation of the hyperbola of given length of transverse axis 6 whose vertex bisects the distance between the centre and the focus.
4. Find the equations of the tangents to the hyperbola $x^2 - 4y^2 = 4$ which are (i) parallel (ii) perpendicular to the line $x + 2y = 0$.
5. Find the equations of tangents drawn to the hyperbola $2x^2 - 3y^2 = 6$ through $(-2, 1)$.
6. Prove that the product of the perpendicular distances from any point on a hyperbola to its asymptotes is constant.

III. 1. Tangents to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ make angles θ_1, θ_2 with transverse axis of a hyperbola. Show that the point of intersection of these tangents lies on the curve $2xy = k(x^2 - a^2)$ when $\tan\theta_1 + \tan\theta_2 = k$.

2. Show that the locus of feet of the perpendiculars drawn from foci to any tangent of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is the auxiliary circle of the hyperbola.

3. Show that the equation $\dfrac{x^2}{9-c} + \dfrac{y^2}{5-c} = 1$ represents
   (i) an ellipse if 'c' is a real constant less than 5.
   (ii) a hyperbola if 'c' is any real constant between 5 and 9.
   (iii) show that each ellipse in (i) and each hyperbola (ii) has foci at the two points $(\pm 2, 0)$, independent of the value 'c'.
4. Show that angle between the two asymptotes of a hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ is $2\ \text{Tan}^{-1}\left(\dfrac{b}{a}\right)$ or $2\ \text{Sec}^{-1}(e)$.

Key Concepts

- Equation of hyperbola in standard form is $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, centered C(0, 0) and foci $(\pm ae, 0)$, directrices $x = \pm\dfrac{a}{e}$ and eccentricity $e = \sqrt{\dfrac{a^2+b^2}{a^2}}$.
- If P is any point on the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ and foci are S and S′ then $S'P - SP = 2a$.
- $x^2 - y^2 = a^2$ is the equation of the rectangular hyperbola whose eccentricity is $\sqrt{2}$.
- The equation of the 'auxiliary circle' of the hyperbola S = 0 is $x^2 + y^2 = a^2$.
- $x = a\sec\theta, y = b\tan\theta$ are called the parametric equations of the hyperbola and θ is called the parameter.
- The condition for straight line $y = mx + c$ to be a tangent to the hyperbola S = 0 is $c^2 = a^2m^2 - b^2$.
- $y = mx \pm \sqrt{a^2m^2 - b^2}$ is always a tangent to the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ at $\left(-\dfrac{a^2m}{c}, -\dfrac{b^2}{c}\right)$ and $\left(\dfrac{a^2m}{c}, \dfrac{b^2}{c}\right)$ respectively $(c \neq 0, c^2 = a^2m^2 - b^2)$.
- The equation of the tangent at $P(x_1, y_1)$ is $S_1 = 0$.
- The equation of the normal at $P(x_1, y_1)$ is $\dfrac{a^2x}{x_1} + \dfrac{b^2y}{y_1} = a^2 + b^2$.
- Equation of the tangent at $P(\theta)$ is $\dfrac{x}{a}\sec\theta - \dfrac{y}{b}\tan\theta = 1$.
- Equation of the normal at $P(\theta)$ is $\dfrac{ax}{\sec\theta} + \dfrac{by}{\tan\theta} = a^2 + b^2$.
- The equation of the asymptotes of a hyperbola S = 0 are $y = \pm\dfrac{b}{a}x$ and the combined equation of the asymptotes is $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 0$.

Historical Note

A special case of Hyperbola was first studied by *Menaechmus* (B.C. 350). The special case was $xy = ab$, the Rectangular Hyperbola. *Euclid* and *Aristaeus* wrote about the general hyperbola, but studied only one branch of it, while the hyperbola was given its present name by *Apollonius*, who was the first to study the two branches of the curve. The focus - directrix property was considered by *Pappus*.

Answers

Exercise 5(a)

I. 1. $4x^2 - 5y^2 - 8x + 60y + 4 = 0$

2. $9x^2 - 16y^2 = 144$

3. $3x^2 + 10xy + 8y^2 + 14x + 22y + 7 = 0$

4. $k = \pm 5$

5. $\frac{144}{25}$

6. $\frac{5}{3}$

7. $9x^2 - 25y^2 = 225$

8. $x + y = 7$

9. $\sqrt{6} - \sqrt{2}$

II. 1.

Q. No	Centre	Foci	Eccentricity	Directrices	Length of the latus rectums.
(i)	$(0, 0)$	$(0, \pm 5)$	$5/3$	$5y \pm 9 = 0$	$32/3$
(ii)	$(0, 0)$	$(\pm\sqrt{5}, 0)$	$\sqrt{5}/2$	$\sqrt{5}x \pm 4 = 0$	1
(iii)	$(-2, 1)$	$(1, 1), (-5, 1)$	$3/2$	$3x + 2 = 0, 3x + 10 = 0$	5
(iv)	$(-4, -1)$	$(1, -1), (-9, -1)$	$5/4$	$5x+4 = 0, 5x+36 = 0$	$9/2$

2. $3x^2 - y^2 - 36x + 4y + 101 = 0$

3. $3x^2 - y^2 = 27$

4. (i) No parallel tangents to $x + 2y = 0$.

   (ii) Perpendicular tangents are $2x - y \pm \sqrt{15} = 0$.

5. $x + y + 1 = 0,\ 3x + y + 5 = 0$

Calculus

Chapter 6

Integration

"The art of doing mathematics consists in finding that special case which contains all the gems of generality"

- David Hilbert

Introduction

We have learnt the concept of differentiation in the first year of the Intermediate course.

If a function f is differentiable in an interval I, i.e., the derivative of f, namely f' exists at each point of I, then the following question arises naturally : "given f' on I, can we determine f?". In this chapter, we answer this question by introducing the concept of integration as the inverse process of differentiation. Also, we discuss standard forms and properties of integrals.

Throughout this chapter, **R** denotes the set of all real numbers and I, an interval in **R**. Unless otherwise stated, all the functions considered here are real valued functions defined over subsets of **R**.

John Wallis
(1616-1703)

John Wallis was born in 1616. He was one of the ablest and most original mathematicians of his days. He wrote extensively on number of areas. In 1649 he was appointed Savilian professor of geometry at Oxford, a position he held for 54 years until his death in 1703. His work in analysis did much to prepare the way for his great contemporary, Isaac Newton. Wallis was one of the first to discuss conics as curves of second degree equation rather than sections of a cone.

6.1 Integration as the inverse process of differentiation, standard forms and properties of integrals

We begin with the definition of an indefinite integral of a function and then state the standard forms of integrals for certain functions.

6.1.1 Definition

Let E be a subset of $\mathbf{R}$ *such that E contains a right or a left neighbourhood of each of its points and let* $f : E \to \mathbf{R}$ *be a function. If there is a function F on E such that* $F'(x) = f(x)$ *for all* $x \in E$, *then we call F an* ***antiderivative of f*** *or a* ***primitive of f***.

For example, we know that

$$\frac{d}{dx}(\sin x) = \cos x, \quad x \in \mathbf{R}.$$

Hence, if f is the function given by $f(x) = \cos x$, $x \in \mathbf{R}$, then the function F given by $F(x) = \sin x, x \in \mathbf{R}$ is an antiderivative or a primitive of f.

If F is an antiderivative of f on E, then for any real number k, we have

$$(F + k)'(x) = f(x) \text{ for all } x \in E.$$

Hence $F + k$ is also an antiderivative of f.

Thus, in the above example, if c is any real constant then the function G given by $G(x) = \sin x + c, x \in \mathbf{R}$ is also an antiderivative of $\cos x$.

6.1.2 Remark : If F and G are antiderivatives of a function f then $F - G$ need not be a constant, in general. Consider the following :

6.1.3 Example : Let $E = (-\infty, 0) \cup (0, \infty)$ and $f : E \to \mathbf{R}$ be defined as

$$f(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}.$$

Let us consider the following two functions :

$$F(x) = \begin{cases} x & \text{if } x > 0 \\ -x & \text{if } x < 0 \end{cases} \quad \text{and} \quad G(x) = \begin{cases} x+5 & \text{if } x > 0 \\ -x+10 & \text{if } x < 0 \end{cases}.$$

$$\text{Then} \quad F'(x) = f(x) = G'(x) \quad \text{for all } x \in E.$$

Therefore, both F and G are antiderivatives of f. But $F - G$ is not a constant on E, since $(F - G)(2) = -5$ and $(F - G)(-2) = -10$.

In this example, note that the domain E of f is not an interval. However, if a function is defined on an interval, then we prove that the difference of any two antiderivatives is a constant. First, we prove the following :

6.1.4 Theorem : *Let φ be a function defined on an interval I. Then φ is a constant function if and only if φ is an antiderivative of the zero function on I.*

Proof : Assume that $\varphi(x) = c$ for all x in I for some constant c. Then $\varphi'(x) = 0$ for all x in I. Therefore, φ is an antiderivative of the zero function on I.

Conversely, assume that φ is an antiderivative of the zero function on I. Then

$$\varphi'(x) = 0 \quad \text{...(1)}$$

for all x in I.

Let $a, b \in$ I be such that $a < b$. Then $[a, b] \subseteq$ I, φ is continuous on $[a, b]$ and differentiable in (a, b). Hence by Lagrange's mean value theorem (10.7.5 of Mathematics -IB Text Book) there exists a point d in (a, b) such that

$$\frac{\varphi(b) - \varphi(a)}{b - a} = \varphi'(d). \quad \text{...(2)}$$

Since $[a, b] \subseteq$ I we have $d \in$ I; Now from (1), $\varphi'(d) = 0$. Hence, from (2), $\varphi(b) = \varphi(a)$.

Since $a, b \in$ I are arbitrary, φ must be a constant function.

6.1.5 Corollary : *Let f be a function defined on an interval I and F be an antiderivative of f. Then a function G on I is an antiderivative of f if and only if $G = F + c$ for some constant function c on I.*

Proof : Let $G = F + c$. Then $G' = F' = f$. Hence G is an antiderivative of f.

Conversely, suppose that G is an antiderivative of f. Then $G' = f = F'$. Hence $(G - F)' = G' - F' = f - f = 0$ on I. Hence $G - F$ is an antiderivative of the zero function on I. Now, by Theorem 6.1.4, $G - F$ is a constant function on I, and hence $G = F + c$ for some constant c.

6.1.6 Note

(i) The above corollary is not true if the domain of f is not an interval. (see Example 6.1.3)

(ii) In the subsequent discussion, we restrict our attention to functions defined on intervals.

(iii) If F is an antiderivative of f then $\{F + c : c \in \mathbf{R}\}$ is the set of all antiderivatives of f. Hence the general form of an antiderivative of f is $F + c$, c is a constant.

6.1.7 Definition (Indefinite integral)

Let $f : I \to \mathbf{R}$. Suppose that f has an antiderivative F on I. Then we say that f has an integral on I and for any real constant c, we call F + c an indefinite integral of f over I, denote it by $\int f(x)\, dx$ and read it as 'integral $f(x)\, dx$'. We also denote $\int f(x)\, dx$ as $\int f$. Thus we have

$$\int f = \int f(x)dx = F(x) + c.$$

Here c is called a 'constant of integration'.

In the indefinite integral $\int f(x)dx$, f is called the 'integrand' and x is called the 'variable of integration'.

6.1.8 Remarks

(i) In the Definition 6.1.7, we can regard any member of $\{F + c : c \in \mathbf{R}\}$ as $\int f(x)dx$.

(ii) From the definition of indefinite integral, $\frac{d}{dx}\left(\int f(x)dx\right) = f(x)$.

(iii) If $f : I \rightarrow \mathbf{R}$ is differentiable on I, then $\int f'(x)\,dx = f(x) + c$, where c is a constant of integration.

6.1.9 Standard forms : In the I year Intermediate course, we have studied the derivatives of some functions. With this background, let us obtain indefinite integrals of some functions.

1. We know that $\frac{d}{dx}\left(\frac{x^{n+1}}{n+1}\right) = x^n$ for $n \neq -1$.

   Hence, if $n \neq -1$, we have $\int x^n\,dx = \frac{x^{n+1}}{n+1} + c$, where c is a constant.

   In particular, when $n = 0$, we have

   $$\int dx = \int 1 \,.\, dx = x + c.$$

2. We know that $\frac{d}{dx}\left(\log_e x\right) = \frac{1}{x}$ if $x > 0$

   and $\frac{d}{dx}\left[\log_e(-x)\right] = \frac{1}{x}$ if $x < 0$.

   Hence $\int \frac{1}{x} dx = \begin{cases} \log_e x & \text{on any interval } I \subseteq (0, \infty) \\ \log_e(-x) & \text{on any interval } I \subseteq (-\infty, 0), \end{cases}$

   so that $\int \frac{1}{x} dx = \log_e |x| + c$ on any interval $I \subseteq \mathbf{R} \setminus \{0\}$,

   where c is a constant.

Note : Throughout this chapter where ever $\log x$ appears, it is to be understood as $\log_e x$.

In a similar way, the following indefinite integrals can be easily obtained wherein c is a constant. The intervals on which the integrals are valid are often specified by stipulating conditions on x.

3. If $a > 0$ and $a \neq 1$, then

   $\int a^x dx = \frac{a^x}{\log_e a} + c, \quad x \in \mathbf{R}.$

4. $\int e^x dx = e^x + c, \quad x \in \mathbf{R}.$

5. $\int \sin x\,dx = -\cos x + c, \quad x \in \mathbf{R}.$

6. $\int \cos x\,dx = \sin x + c, \quad x \in \mathbf{R}.$

7. $\int \sec^2 x \, dx = \tan x + c, \quad x \in I \subset \mathbf{R} \setminus \left\{ \frac{n\pi}{2} : n \text{ is an odd integer} \right\}.$

8. $\int \operatorname{cosec}^2 x \, dx = -\cot x + c, \quad x \in I \subset \mathbf{R} \setminus \{ n\pi : n \in \mathbf{Z} \}.$

9. $\int \sec x \tan x \, dx = \sec x + c, \quad x \in I \subset \mathbf{R} \setminus \left\{ \frac{n\pi}{2} : n \text{ is an odd integer} \right\}.$

10. $\int \operatorname{cosec} x \cot x \, dx = -\operatorname{cosec} x + c, \quad x \in I \subset \mathbf{R} \setminus \{ n\pi : n \in \mathbf{Z} \}.$

11. $\int \frac{1}{\sqrt{1-x^2}} dx = \operatorname{Sin}^{-1} x + c \quad \text{for } |x| < 1$

$= -\operatorname{Cos}^{-1} x + c \quad \left(\text{since } \operatorname{Cos}^{-1} x = \frac{\pi}{2} - \operatorname{Sin}^{-1} x\right).$

12. $\int \frac{1}{1+x^2} dx = \operatorname{Tan}^{-1} x + c \quad \text{for } x \in \mathbf{R}.$

$= -\operatorname{Cot}^{-1} x + c \quad (\text{since } \operatorname{Cot}^{-1} x = \frac{\pi}{2} - \operatorname{Tan}^{-1} x).$

13. $\int \frac{1}{|x| \sqrt{x^2 - 1}} dx = \operatorname{Sec}^{-1} x + c \text{ on } I \subset \mathbf{R} \setminus [-1, 1]$

$= -\operatorname{Cosec}^{-1} x + c \text{ on } I \subset \mathbf{R} \setminus [-1, 1].$

(since $\operatorname{Cosec}^{-1} x = \frac{\pi}{2} - \operatorname{Sec}^{-1} x$ on any interval $I \subset (-\infty, -1) \cup (1, \infty)$).

14. $\int \sinh x \, dx = \cosh x + c, \quad x \in \mathbf{R}.$

15. $\int \cosh x \, dx = \sinh x + c, \quad x \in \mathbf{R}.$

16. $\int \operatorname{sech}^2 x \, dx = \tanh x + c, \quad x \in \mathbf{R}.$

17. $\int \operatorname{cosech}^2 x \, dx = -\coth x + c, \quad x \in I \subset \mathbf{R} \setminus \{0\}.$

18. $\int \operatorname{sech} x \tanh x \, dx = -\operatorname{sech} x + c, \quad x \in \mathbf{R}.$

19. $\int \operatorname{cosech} x \coth x \, dx = -\operatorname{cosech} x + c, \quad x \in I \subset \mathbf{R} \setminus \{0\}.$

20. $\int \frac{1}{\sqrt{1+x^2}} dx = \sinh^{-1} x + c, \quad x \in \mathbf{R}.$

$= \log_e (x + \sqrt{x^2 + 1}), \quad x \in \mathbf{R}.$

21. $$\int \frac{1}{\sqrt{x^2-1}}dx = \begin{cases} \cosh^{-1}x + c \text{ on } (1, \infty) \\ -\cosh^{-1}(-x) + c \text{ on } (-\infty, -1). \end{cases}$$

$$= \begin{cases} \log_e(x+\sqrt{x^2-1}) + c \text{ on } (1, \infty) \\ -\log_e(-x+\sqrt{x^2-1}) + c \text{ on } (-\infty, -1). \end{cases}$$

$$= \log_e|x+\sqrt{x^2-1}| + c \text{ on } I \subset \mathbf{R} \setminus [-1, 1].$$

6.1.10 Properties of integrals : We shall now prove the following algebraic properties of indefinite integrals.

6.1.10 (a) Theorem : *If the functions f and g have integrals on I, then f + g has an integral on I and*

$$\int (f+g)(x)\, dx = \int f(x)\, dx + \int g(x)\, dx + c,$$

where c is a constant.

Proof : Since f and g have integrals on I, there exist functions F and G on I such that $F' = f$ and $G' = g$ on I. Therefore

$$\int f(x)dx = F(x) + c_1 \text{ and } \int g(x)dx = G(x) + c_2,$$

where c_1, c_2 are constants. Now

$$(F+G)' = F' + G' = f + g \text{ on I.}$$

Hence F + G is an antiderivative of $f + g$. Therefore, $f + g$ has integral on I, and

$$\int (f+g)(x)\, dx = F(x) + G(x) + c_3$$
$$= \int f(x)dx - c_1 + \int g(x)dx - c_2 + c_3$$

where c_3 is a constant.

Let $c = c_3 - c_1 - c_2$. Then

$$\int (f+g)(x)dx = \int f(x)dx + \int g(x)dx + c.$$

6.1.10(b) Theorem : *If f has an integral on I and a is a real number, then af also has an integral on I and*

$$\int (af)(x)\, dx = a\int f(x)\, dx + c,$$

where c is a constant.

Proof : Since f has an integral, there is a differentiable function F on I such that $F' = f$ on I. We have $(aF)' = aF' = af$.

Hence af has an integral on I and by definition,

$$\int (af)(x)\, dx = (aF)(x) + c = aF(x) + c = a\int f(x)\, dx + c,$$

where c is a constant.

6.1.11 Remarks

From Theorems 6.1.10(a) and 6.1.10(b), we can easily prove the following statements in which c is a constant.

(i) If f and g have integrals on I then so does $f-g$ and

$$\int (f-g)(x)\,dx = \int f(x)\,dx - \int g(x)\,dx + c.$$

(ii) If $f_1, f_2, \ldots, f_n$ have integrals on I then so does $f_1 + f_2 + \cdots + f_n$ and

$$\int (f_1 + f_2 + \cdots + f_n)(x)\,dx = \int f_1(x)dx + \int f_2(x)dx + \cdots + \int f_n(x)dx + c.$$

(iii) If $f_1, f_2, \ldots, f_n$ have integrals on I and $k_1, k_2, \ldots, k_n$ are constants, then so does $\sum_{i=1}^{n} k_i f_i$ and

$$\int \left(\sum_{i=1}^{n} k_i f_i(x)\right) dx = \sum_{i=1}^{n} k_i \int f_i(x)\,dx + c.$$

Note that the finite sums in (iii) above can not be replaced by infinite sums.

(iv) When we consider logarithms to the base e, we do not make specific mention of the base. Thus, for example, we write $\log t$ for $\log_e t$.

6.1.12 Solved Problems

1. Problem : *Find* $\int 2x^7 dx$ *on* **R**.

Solution : $\int 2x^7 dx = 2\int x^7 dx + c$ (by Theorem 6.1.10(b))

$$= 2 \cdot \frac{x^{7+1}}{7+1} + c \quad \text{(by 6.1.9(1))}$$

$$= \frac{x^8}{4} + c.$$

2. Problem : *Evaluate* $\int \cot^2 x\,dx$ *on* $\mathrm{I} \subset \mathbf{R} \setminus \{n\pi : n \in \mathbf{Z}\}$.

Solution : $\int \cot^2 x\,dx = \int (\operatorname{cosec}^2 x - 1)dx$

$$= \int \operatorname{cosec}^2 x\,dx - \int dx + c \quad \text{(by Remark 6.1.11(i))}$$

$$= -\cot x - x + c. \quad \text{(by 6.1.9(8))}$$

3. Problem : *Evaluate* $\int \left(\frac{x^6 - 1}{1 + x^2}\right) dx$ for $x \in \mathbf{R}$.

Solution : $\int \left(\frac{x^6 - 1}{1 + x^2}\right) dx = \int [(x^4 - x^2 + 1) - \frac{2}{1+x^2}]\,dx$

$$= \int x^4 dx - \int x^2 dx + \int dx - \int \frac{2}{1+x^2} dx + c$$

$$= \frac{x^5}{5} - \frac{x^3}{3} + x - 2\operatorname{Tan}^{-1} x + c.$$

4. Problem : *Find* $\int (1-x)(4-3x)(3+2x)dx,\ x \in \mathbf{R}$.

Solution: $(1 - x)(4 - 3x)(3 + 2x) = 6x^3 - 5x^2 - 13x + 12$.

Hence

$$\int (1-x)(4-3x)(3+2x)dx = \int (6x^3 - 5x^2 - 13x + 12)dx$$

$$= 6\int x^3 dx - 5\int x^2 dx - 13\int x\,dx + 12\int dx + c$$

$$= \frac{3}{2}x^4 - \frac{5}{3}x^3 - \frac{13}{2}x^2 + 12x + c\,.$$

5. Problem : *Evaluate* $\int \left(x + \frac{1}{x}\right)^3 dx,\ x > 0.$

Solution : $\left(x + \frac{1}{x}\right)^3 = x^3 + 3x + \frac{3}{x} + \frac{1}{x^3}.$

Hence $\int \left(x + \frac{1}{x}\right)^3 dx = \int \left(x^3 + 3x + \frac{3}{x} + \frac{1}{x^3}\right) dx$

$$= \int x^3\,dx + 3\int x\,dx + 3\int \frac{dx}{x} + \int \frac{dx}{x^3} + c$$

$$= \frac{x^4}{4} + \frac{3}{2}x^2 + 3\log x - \frac{1}{2x^2} + c.$$

6. Problem : *Find* $\int \sqrt{1 + \sin 2x}\ dx$ *on* $\mathbf{R}$.

Solution: We know that $1 + \sin 2x = (\sin x + \cos x)^2$.

Therefore

$$\sqrt{1+\sin 2x} = \begin{cases} \sin x + \cos x, & \text{if } 2n\pi - \frac{\pi}{4} \le x \le 2n\pi + \frac{3\pi}{4} \text{ for some } n \in \mathbf{Z} \\ -(\sin x + \cos x), & \text{otherwise.} \end{cases}$$

Hence, if $2n\pi - \frac{\pi}{4} \le x \le 2n\pi + \frac{3\pi}{4}$, then

$$\int \sqrt{1+\sin 2x}\ dx = \int (\sin x + \cos x)dx$$

$$= \int \sin x\ dx + \int \cos x\,dx + c$$

$$= -\cos x + \sin x + c.$$

If $2n\pi + \frac{3\pi}{2} \le x \le 2n\pi + \frac{7\pi}{4}$,

then $\int \sqrt{1+\sin 2x}\ dx = \int [-(\sin x + \cos x)]\ dx$

$$= -\left(\int \sin x\,dx + \int \cos x\,dx\right) + c$$

$$= -\left(-\cos x + \sin x\right) + c$$

$$= \cos x - \sin x + c.$$

7. Problem : *Evaluate* $\int \frac{2x^3 - 3x + 5}{2x^2}\,dx$ *for* $x > 0$ *and verify the result by differentiation.*

Solution : $\int \frac{2x^3 - 3x + 5}{2x^2}\,dx = \int x\,dx - \frac{3}{2}\int \frac{dx}{x} + \frac{5}{2}\int \frac{1}{x^2}\,dx + c$

$$= \frac{x^2}{2} - \frac{3}{2}\log x - \frac{5}{2x} + c.$$

Verification : $\frac{d}{dx}\left(\frac{x^2}{2} - \frac{3}{2}\log x - \frac{5}{2x} + c\right) = x - \frac{3}{2x} + \frac{5}{2x^2} = \frac{2x^3 - 3x + 5}{2x^2}.$

This is the given expression and hence the result is correct.

Exercise 6(a)

I. Evaluate the following integrals.

1. $\int (x^3 - 2x^2 + 3)dx$ on **R**.
2. $\int 2x\sqrt{x}\,dx$ on $(0, \infty)$
3. $\int \sqrt[3]{2x^2}\,dx$ on $(0, \infty)$.
4. $\int \frac{x^2 + 3x - 1}{2x}\,dx,\ x \in I \subset \mathbf{R} \setminus \{0\}$.
5. $\int \frac{1 - \sqrt{x}}{x}\,dx$ on $(0, \infty)$.
6. $\int \left(1 + \frac{2}{x} - \frac{3}{x^2}\right)dx$ on $I \subset \mathbf{R} \setminus \{0\}$.
7. $\int \left(x + \frac{4}{1 + x^2}\right)dx$ on **R**.
8. $\int \left(e^x - \frac{1}{x} + \frac{2}{\sqrt{x^2 - 1}}\right)dx$ on $I \subset \mathbf{R} \setminus [-1, 1]$.
9. $\int \left(\frac{1}{1 - x^2} + \frac{1}{1 + x^2}\right)dx$ on $(-1, 1)$.
10. $\int \left(\frac{1}{\sqrt{1 - x^2}} + \frac{2}{\sqrt{1 + x^2}}\right)dx$ on $(-1, 1)$.
11. $\int e^{\log(1 + \tan^2 x)}\,dx$ on $I \subset \mathbf{R} \setminus \left\{\frac{(2n+1)\pi}{2} : n \in \mathbf{Z}\right\}$.
12. $\int \frac{\sin^2 x}{1 + \cos 2x}\,dx$ on $I \subset \mathbf{R} \setminus \{(2n \pm 1)\pi : n \in \mathbf{Z}\}$.

II. Evaluate the following integrals.

1. $\int (1-x^2)^3\, dx$ on $(-1, 1)$.

2. $\int \left(\frac{3}{\sqrt{x}} - \frac{2}{x} + \frac{1}{3x^2}\right) dx$ on $(0, \infty)$.

3. $\int \left(\frac{\sqrt{x}+1}{x}\right)^2 dx$ on $(0, \infty)$.

4. $\int \frac{(3x+1)^2}{2x}\, dx,\ x \in \mathrm{I} \subset \mathbf{R} \setminus \{0\}$.

5. $\int \left(\frac{2x-1}{3\sqrt{x}}\right)^2 dx$ on $(0, \infty)$.

6. $\int \left(\frac{1}{\sqrt{x}} + \frac{2}{\sqrt{x^2-1}} - \frac{3}{2x^2}\right) dx$ on $(1, \infty)$.

7. $\int (\sec^2 x - \cos x + x^2)\, dx,\ x \in \mathrm{I} \subset \mathbf{R} \setminus \left\{\frac{n\pi}{2} : n \text{ is an odd integer}\right\}$.

8. $\int \left(\sec x \tan x + \frac{3}{x} - 4\right) dx,\ x \in \mathrm{I} \subset \mathbf{R} \setminus \left(\left\{\frac{n\pi}{2} : n \text{ is an odd integer}\right\} \cup \{0\}\right)$.

9. $\int \left(\sqrt{x} - \frac{2}{1-x^2}\right) dx$ on $(0, 1)$.

10. $\int \left(x^3 - \cos x + \frac{4}{\sqrt{x^2+1}}\right) dx,\quad x \in \mathbf{R}$.

11. $\int \left(\cosh x + \frac{1}{\sqrt{x^2+1}}\right) dx,\quad x \in \mathbf{R}$.

12. $\int \left(\sinh x + \frac{1}{(x^2-1)^{\frac{1}{2}}}\right) dx,\quad x \in \mathrm{I} \subset (-\infty, -1) \cup (1, \infty)$.

13. $\int \frac{(a^x - b^x)^2}{a^x\, b^x}\, dx,\ (a > 0,\ a \neq 1 \text{ and } b > 0,\ b \neq 1)$ on $\mathbf{R}$.

14. $\int \sec^2 x \operatorname{cosec}^2 x\, dx$ on $\mathrm{I} \subset \mathbf{R} \setminus \left(\{n\pi : n \in \mathbf{Z}\} \cup \left\{(2n+1)\frac{\pi}{2} : n \in \mathbf{Z}\right\}\right)$.

15. $\int \frac{1+\cos^2 x}{1-\cos 2x}\, dx$ on $\mathrm{I} \subset \mathbf{R} \setminus \{n\pi : n \in \mathbf{Z}\}$.

16. $\int \sqrt{1-\cos 2x}\, dx$ on $\mathrm{I} \subset [2n\pi, (2n+1)\pi],\ n \in \mathbf{Z}$.

17. $\int \frac{1}{\cosh x + \sinh x}\, dx$ on $\mathbf{R}$.

18. $\int \frac{1}{1+\cos x}\, dx$ on $\mathrm{I} \subset \mathbf{R} \setminus \{(2n+1)\pi : n \in \mathbf{Z}\}$.

6.2 Method of substitution - Integration of algebraic, exponential, logarithmic, trigonometric and inverse trigonometric functions - Integration by parts

This section is divided into the following two subsections, namely

6.2(A) Integration by the method of substitution - Integration of algebraic and trigonometric functions

and

6.2(B) Integration by parts - Integration of exponential, logarithmic and inverse trigonometric functions

In section 6.2(A), we discuss the integration by the method of substitution. In section 6.2(B), we describe one more useful method of integration called 'integration by parts' for integrating a product of functions. In the sequel, we evaluate the integrals of the form

$$\int \frac{1}{ax^2+bx+c}dx,\ \int \frac{dx}{\sqrt{ax^2+bx+c}},\ \int \sqrt{ax^2+bx+c}\ dx,$$

$$\int \frac{px+q}{ax^2+bx+c}dx,\ \int (px+q)\sqrt{ax^2+bx+c}\ dx,$$

$$\int \frac{px+q}{\sqrt{ax^2+bx+c}}dx,\ \int \frac{dx}{(ax+b)\sqrt{px+q}},\ \int \frac{1}{a+b\cos x}dx,$$

$$\int \frac{1}{a+b\sin x}dx,\ \int \frac{a\cos x+b\sin x+c}{d\cos x+e\sin x+f}dx.$$

6.2(A) Integration by the method of substitution - Integration of algebraic and trigonometric functions

In this section, we reduce certain integrals to some standard forms by using a suitable substitution. Here we discuss mainly the integration of algebraic, trigonometric functions and simple forms of exponential functions and some functions which are combinations of these forms.

6.2.1 Theorem : *Let $f: I \to \mathbf{R}$ have an integral on I and F be a primitive of f on I. Let J be an interval in* $\mathbf{R}$ *and $g : J \to I$ be a differentiable function. Then (fog) g' has an integral on J, and*

$$\int f(g(x))\ g'(x)\ dx = \text{F}(g(x)) + c.$$

i.e.,
$$\int f(g(x))\ g'(x)\ dx = \left[\int f(t)\,dt\right]_{t=g(x)}.$$

Notation : If $\int f(t)\,dt = \text{F}(t) + c$, then $\left[\int f(t)\,dt\right]_{t=g(x)}$ denotes the value of F(t) + c evaluated at $g(x)$.

i.e.,
$$\left[\int f(t)\,dt\right]_{t=g(x)} = \left[\text{F}(t)+c\right]_{t=g(x)} = \text{F}(g(x))+c.$$

Proof of the theorem : Since F is a primitive of f on I, we have $F'(t) = f(t)$ for all t in I. We have

$$(Fog)' = (F'og)\ g' = (fog)\ g'.$$

Hence $(fog)\, g'$ has an integral on J and

$$\int f(g(x))\ g'(x)\,dx = \int [(fog)g'](x)\ dx = (Fog)\ (x) + c$$
$$= F(g(x)) + c.$$

6.2.2 Remark

Sometimes when we are given a function h and asked to find $\int h(x)\,dx$, it might be possible to find easily a differentiable function g and a function f such that $\int f(t)\,dt$ can be easily found and $f(g(x))\, g'(x) = h(x)$ for all x. In such a case, we evaluate $\int f(t)\,dt$. The substitution $t = g(x)$ so obtained yields $\int h(x)\,dx$. Evaluation of integrals in this way is known as '*the method of substitution*'.

6.2.3 Example : Let us now evaluate $\int 2x\cos(1+x^2)dx$. Let $I = [1, \infty)$. We define $f : I \to \mathbf{R}$ by $f(x) = \cos x$. Let $J = \mathbf{R}$. Define g on J by $g(x) = 1 + x^2$. Then, g is differentiable on J, $g(J) \subset I$ and $f(g(x))\ g'(x) = 2x\cos(1 + x^2)$ for all x in J. Clearly f has an integral on I and

$$\int f(t)\,dt = \int \cos t\ dt = \sin t + c.$$

Hence by Theorem 6.2.1, we have

$$\int 2x\ \cos\ (1+x^2)\ dx = \left[\int f(t)\,dt\right]_{t=g(x)}$$
$$= (\sin t + c)_{t=1+x^2}$$
$$= \sin(1 + x^2) + c.$$

6.2.4 Note

In Example 6.2.3 we specifically mentioned the intervals I, J and the functions f, g to illustrate Theorem 6.2.1. Generally, the intervals I and J are not explicitly specified and can be taken suitably. The general practice is to guess an appropriate function g to make the substitution $t = g(x)$ and $dt = g'(x)\,dx$ in the given integral so that it reduces to $\left[\int f(t)\,dt\right]_{t=g(x)}$. It is customary to write $\int f(g(x))\,g'(x)\,dx = \int f(t)\ dt$ with the understanding that in the later integral t is to be replaced by $g(x)$ after evaluation.

6.2.5 Example

Let us now evaluate $\int \frac{e^x}{e^x+1}dx$. Here we suppose that $f(t) = \frac{1}{t}, t \in (1, \infty)$; we define $g : \mathbf{R} \to \mathbf{R}$ by $g(x) = e^x + 1$. Then

$$f(g(x))\,g'(x) = \frac{e^x}{e^x+1}.$$

Put $t = g(x) = e^x + 1$. Then $dt = e^x\, dx$.

Therefore $$\int \frac{e^x}{e^x+1}\,dx = \left[\int \frac{1}{t}\,dt\right]_{t=e^x+1} = [\log t + c]_{t=e^x+1}$$

$$= \log(e^x + 1) + c.$$

6.2.6 Corollary : *Let $f : I \to \boldsymbol{R}$ have an integral on I and F be a primitive of f. Let $a, b \in \boldsymbol{R}$ with $a \neq 0$. Then*

$$\int f(ax+b)\,dx = \frac{1}{a}\mathrm{F}(ax+b)+c$$

for all $x \in J$, where $J = \{x \in \boldsymbol{R} : ax + b \in I\}$ and c is an integration constant.

Proof : Follows from Theorem 6.2.1 by substituting $g(x) = ax + b$.

6.2.7 Example

Let us evaluate $\int \frac{1}{ax+b}\,dx$ on an interval $\mathrm{J} \subset \mathbf{R} \setminus \left\{-\frac{b}{a}\right\}$ with $a, b \in \mathbf{R}, a \neq 0$.

Let $\mathrm{I} = \{ax + b : x \in \mathrm{J}\}$ so that $0 \notin \mathrm{I}$ and I is an interval. We define $f : \mathrm{I} \to \mathbf{R}$ by $f(t) = \frac{1}{t}$ so that $\mathrm{F}(t) = \log|t| + c$ on I. Hence by Corollary 6.2.6, we have

$$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\log|ax+b| + c \text{ on J.}$$

6.2.8 Some Important Formulae

The following formulae can be obtained by using some of the standard integrals given in 6.1.9 and Corollary 6.2.6.

1. $\int e^{ax}\,dx = \frac{1}{a}e^{ax} + c,\ a \neq 0,$ on $\mathbf{R}$.

2. $\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b)+c,\ a \neq 0,$ on $\mathbf{R}$.

3. $\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b)+c,\ a \neq 0,$ on $\mathbf{R}$.

4. $\int \sec^2(ax+b)\,dx = \frac{1}{a}\tan(ax+b)+c,\ a \neq 0,$

$$\text{on } I \subset \mathbf{R} \setminus \left\{ \left[\frac{(2n+1)\pi}{2} - b\right]\frac{1}{a} : n \in \mathbf{Z} \right\}.$$

5. $\int \operatorname{cosec}^2(ax+b)\,dx = -\frac{1}{a}\cot(ax+b)+c,\ a \neq 0,$

$$\text{on } I \subset \mathbf{R} \setminus \left\{ \frac{1}{a}(n\pi - b) : n \in \mathbf{Z} \right\}.$$

6. $\int \operatorname{cosec}(ax+b)\cot(ax+b)\,dx = -\frac{1}{a}\operatorname{cosec}(ax+b)+c,\ a \neq 0$

$$\text{on } I \subset \mathbf{R} \setminus \left\{ \frac{1}{a}(n\pi - b) : n \in \mathbf{Z} \right\}.$$

7. $\int \sec(ax+b)\tan(ax+b)\,dx = \frac{1}{a}\sec(ax+b)+c,\ a \neq 0$

$$\text{on } I \subset \mathbf{R} \setminus \left\{ \frac{1}{a}\left[\frac{(2n+1)\pi}{2} - b\right] : n \in \mathbf{Z} \right\}.$$

Let us now write integrals of functions of particular form by using the method of substitution.

6.2.9 Theorem : *Let $f : I \to \mathbf{R}$ be a differentiable function. Then the following statements are true.*

(i) *If f is never zero on I then $\frac{f'}{f}$ has an integral on I and* $\int \frac{f'(x)}{f(x)}dx = \log|f(x)|+c$ *on I.*

(ii) *If α is a positive integer or if $\alpha \in \mathbf{R} \setminus \{-1\}$ and $f(x) > 0\ \forall\ x \in I$, then $f^{\alpha} f'$ has an integral on* I

and $\int (f(x))^{\alpha} f'(x)dx = \frac{(f(x))^{\alpha+1}}{(\alpha+1)} + c,\ \alpha \neq -1.$

In particular, when $\alpha = -\frac{1}{2}$, *we have* $\int \frac{f'(x)}{\sqrt{f(x)}}dx = 2\sqrt{f(x)} + c$ on I.

(iii) $\int f'(ax+b)\,dx = \frac{1}{a}f(ax+b)+c,\ a \neq 0$ on $J = \{x \in \mathbf{R} : ax+b \in I\}$.

Proof

(i) Since the function φ given by $\varphi(t) = \frac{1}{t}$ has $\log|t|$ as primitive on any interval not containing the origin, by Theorem 6.2.1 it follows that $\frac{f'}{f}$ has an integral on I and that

$$\int \frac{f'(x)}{f(x)}dx = \left[\int \frac{1}{t}dt\right]_{t=f(x)}$$

$$= [\log|t| + c]_{t=f(x)}$$

$$= \log|f(x)| + c.$$

Thus (i) follows.

(ii) Proof of (ii) follows like that of (i).

(iii) Since f is a primitive of f', by Corollary 6.2.6, clearly (iii) follows.

Note : While working out the problems we do not mention the intervals I and J but make the substitution with the tacit assumption that the substitutions are carried on relevant intervals.

6.2.10 Examples

1. Example : Let us find $\int \frac{6x}{3x^2-2} dx$ on any interval $I \subset \mathbf{R} \setminus \left\{ -\sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}} \right\}$.

We define $f: I \to \mathbf{R}$ by $f(x) = 3x^2 - 2$. Then $f'(x) = 6x$.

Hence $$\int \frac{6x}{3x^2-2} dx = \int \frac{f'(x)}{f(x)} dx = \log|f(x)| + c \quad \text{(by Theorem 6.2.9(i))}$$

$$= \log|3x^2 - 2| + c.$$

2. Example : Let us evaluate $\int \frac{(\operatorname{Sin}^{-1} x)^2}{\sqrt{1-x^2}} dx$ on $I = (-1, 1)$.

We define $f: I \to \mathbf{R}$ by $f(x) = \operatorname{Sin}^{-1} x$. Then $f'(x) = \frac{1}{\sqrt{1-x^2}}$.

Hence $$\int \frac{(\operatorname{Sin}^{-1} x)^2}{\sqrt{1-x^2}} dx = \int (f(x))^2 f'(x)\, dx$$

$$= \frac{(f(x))^3}{3} + c \quad \text{(by Theorem 6.2.9(ii))}$$

$$= \frac{(\operatorname{Sin}^{-1} x)^3}{3} + c.$$

3. Example : Let us evaluate $\int \frac{1}{1+(2x+1)^2} dx$ on $\mathbf{R}$.

We define $f: \mathbf{R} \to \mathbf{R}$ by $f(x) = \operatorname{Tan}^{-1} x$. Then $f'(x) = \frac{1}{1+x^2}$.

Now $$\int \frac{1}{1+(2x+1)^2} dx = \int f'(2x+1)\, dx = \frac{1}{2} f(2x+1) + c \quad \text{(by Theorem 6.2.9(iii))}$$

$$= \frac{1}{2} \operatorname{Tan}^{-1}(2x+1) + c.$$

6.2.11 Evaluation of integrals of trigonometric functions; $\tan x$, $\cot x$, $\sec x$ and $\operatorname{cosec} x$

1. Let us show that $\int \tan x \, dx = \log|\sec x| + c,\ x \in I \subset \mathbf{R} \setminus \left\{\frac{(2n+1)\pi}{2} : n \in \mathbf{Z}\right\}$.

$$\int \tan x \, dx = \int \frac{\sin x}{\cos x} dx.$$

Let $f(x) = \cos x$. Then $f'(x) = -\sin x$, and therefore $-\tan x = \frac{f'(x)}{f(x)}$.

Hence $\int \frac{\sin x}{\cos x} dx = \int \left(-\frac{f'(x)}{f(x)}\right) dx = -\int \frac{f'(x)}{f(x)} dx + c$ (by Theorem 6.1.10(b))

$= -\log|f(x)| + c$ (by Theorem 6.2.9(ii))

$= -\log|\cos x| + c = \log|\sec x| + c.$

$\int \tan x \, dx$ can also be evaluated in the following way.

$$\int \tan x \, dx = \int \frac{\sec x \tan x}{\sec x} dx$$

$= \log|\sec x| + c$ (by Theorem 6.2.9(i)).

2. Let us show that $\int \cot x \, dx = \log|\sin x| + c,\ x \in I \subset \mathbf{R} \setminus \{n\pi : n \in \mathbf{Z}\}$.

$\int \cot x \, dx = \int \frac{\cos x}{\sin x} dx = \log|\sin x| + c$. (by Theorem 6.2.9(i))

3. Let us show that

$$\int \sec x \, dx = \log|\sec x + \tan x| + c,\ x \in I \subset \mathbf{R} \setminus \left\{\frac{(2n+1)\pi}{2} : n \in \mathbf{Z}\right\}.$$

$$= \log\left|\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right| + c.$$

$$\int \sec x \, dx = \int \frac{\sec x(\sec x + \tan x)}{(\sec x + \tan x)} dx.$$

Let $f(x) = \sec x + \tan x$. Then $f'(x) = \sec x(\sec x + \tan x)$.

Hence $\int \sec x \, dx = \int \frac{f'(x)}{f(x)} dx = \log|f(x)| + c$ (by Theorem 6.2.9(i))

$= \log|\sec x + \tan x| + c$

$= \log\left|\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right| + c.$

4. Let us show that

$$\int \text{cosec } x \, dx = \log|\text{cosec } x - \cot x| + c$$

$$= \log|\tan \frac{x}{2}| + c \quad \text{for } x \in \text{I} \subset \mathbf{R} \setminus \{n\pi : n \in \mathbf{Z}\}.$$

$$\int \text{cosec } x \, dx = \int \frac{\text{cosec } x(\text{cosec } x - \cot x)}{(\text{cosec } x - \cot x)} dx$$

$$= \log|\text{cosec } x - \cot x| + c.$$

$$= \log|\tan \frac{x}{2}| + c.$$

Observation : By substituting $\frac{\pi}{2} - x$ for x in (1) and (3) and using Corollary 6.2.6, we get the integrals (2) and (4) respectively.

6.2.12 Solved Problems

1. Problem : *Evaluate* $\int \frac{x^5}{1+x^{12}} dx$ *on* **R**.

Solution : We define $f : \mathbf{R} \to \mathbf{R}$ by $f(t) = \frac{1}{1+t^2}$ and $g : \mathbf{R} \to \mathbf{R}$ by $g(x) = x^6$. Then $g'(x) = 6x^5$. Define $\text{F} : \mathbf{R} \to \mathbf{R}$ by $\text{F}(t) = \text{Tan}^{-1} t$. Now F is a primitive of f.

Hence
$$\int \frac{x^5}{1+x^{12}} dx = \frac{1}{6} \int f(g(x)) \, g'(x) \, dx$$

$$= \frac{1}{6} (\text{F}(t) + c)_{t=g(x)}$$

$$= \frac{1}{6} (\text{Tan}^{-1} t + c)_{t=x^6}$$

$$= \frac{1}{6} \text{Tan}^{-1} x^6 + c.$$

2. Problem : *Evaluate* $\int \cos^3 x \sin x \, dx$ *on* **R**.

Solution : We define $f : \mathbf{R} \to \mathbf{R}$ by $f(x) = \cos x$. Then $f'(x) = -\sin x$.

Hence
$$\int \cos^3 x \sin x \, dx = \int (f(x))^3 (-f'(x)) \, dx$$

$$= -\int (f(x))^3 f'(x) \, dx$$

$$= -\frac{(f(x))^4}{4} + c \quad \text{(by Theorem 6.2.9(ii))}$$

$$= -\frac{\cos^4 x}{4} + c.$$

3. Problem : *Find* $\int\left(1-\frac{1}{x^2}\right) e^{\left(x+\frac{1}{x}\right)} dx$ *on* I *where* I $=(0, \infty)$.

Solution : Let $J = I = (0, \infty)$. Define $f: I \to \mathbf{R}$ by $f(t) = e^t$; and $g: J \to \mathbf{R}$ by $g(x) = x + \frac{1}{x}$. Then $g(J) \subset I$, $g'(x) = 1 - \frac{1}{x^2}$. Now by Theorem 6.2.1, it follows that

$$\int\left(1-\frac{1}{x^2}\right) e^{(x+\frac{1}{x})} dx = \int f(g(x))\, g'(x)\, dx = \left[\int f(t)\, dt\right]_{t=g(x)} = \left[\int e^t\, dt\right]_{t=g(x)}$$

$$= \left[e^t + c\right]_{t=x+\frac{1}{x}} = e^{x+\frac{1}{x}} + c.$$

4. Problem : *Evaluate* $\int \frac{1}{\sqrt{\mathrm{Sin}^{-1}x}\,\sqrt{1-x^2}} dx$ *on* I $= (0, 1)$.

Solution : We define $f: I \to \mathbf{R}$ by $f(x) = \mathrm{Sin}^{-1} x$. Then $f'(x) = \frac{1}{\sqrt{1-x^2}}$.

Now $$\int \frac{1}{\sqrt{\mathrm{Sin}^{-1}x}\,\sqrt{1-x^2}} dx = \int \frac{f'(x)}{\sqrt{f(x)}} dx$$

$$= 2\sqrt{f(x)} + c \quad \text{(by Theorem 6.2.9(ii))}$$

$$= 2\sqrt{\mathrm{Sin}^{-1} x} + c.$$

5. Problem : *Evaluate* $\int \frac{\sin^4 x}{\cos^6 x} dx$, $x \in I \subset \mathbf{R} \setminus \left\{\frac{(2n+1)\pi}{2} : n \in \mathbf{Z}\right\}$.

Solution : $$\int \frac{\sin^4 x}{\cos^6 x} dx = \int \tan^4 x \sec^2 x\, dx.$$

We define $f: I \to \mathbf{R}$ by $f(x) = \tan x$. Then $f'(x) = \sec^2 x$.

Therefore $$\int \frac{\sin^4 x}{\cos^6 x} dx = \int [f(x)]^4 f'(x)\, dx = \frac{[f(x)]^5}{5} + c \quad \text{(by Theorem 6.2.9(ii))}$$

$$= \frac{1}{5}\tan^5 x + c.$$

6. Problem: *Evaluate* $\int \sin^2 x\, dx$ *on* **R.**

Solution : $$\int \sin^2 x\, dx = \int\left(\frac{1-\cos 2x}{2}\right) dx$$

$$= \frac{1}{2}\int dx - \frac{1}{2}\int \cos 2x \ dx + c$$

$$= \frac{1}{2}x - \frac{1}{4}\sin 2x + c$$

(since $\int \cos 2x \ dx = \frac{1}{2}\sin 2x + c$ by Theorem 6.2.9 (iii)).

7. Problem : *Evaluate* $\int \frac{1}{a\sin x + b\cos x}\, dx$ *where* $a, b \in$ **R** *and* $a^2 + b^2 \neq 0$ *on* **R**.

Solution : We can find real numbers r and θ such that $r > 0$, $a = r\cos\theta$ and $b = r\sin\theta$. Then $r = \sqrt{a^2+b^2}$; $\cos\theta = \frac{a}{r}$ and $\sin\theta = \frac{b}{r}$.

We have $a\sin x + b\cos x = r\cos\theta\sin x + r\sin\theta\cos x = r\sin(x+\theta)$.

Hence
$$\int \frac{dx}{a\sin x + b\cos x} = \frac{1}{r}\int \frac{1}{\sin(x+\theta)}\, dx$$

$$= \frac{1}{r}\int \operatorname{cosec}(x+\theta)\, dx$$

$$= \frac{1}{r}\log\left|\tan\left(\frac{1}{2}(x+\theta)\right)\right| + c \quad \text{(by 6.2.11(4))}$$

$$= \frac{1}{\sqrt{a^2+b^2}}\log\left|\tan\frac{1}{2}(x+\theta)\right| + c,$$

for all $x \in$ I, where I is an interval disjoint with $\{n\pi - \theta : n \in \mathbf{Z}\}$.

8. Problem : *Find* $\int \frac{x^2}{\sqrt{x+5}}\, dx$ *on* $(-5, \infty)$.

Solution : Put $t = x + 5$ so that $t > 0$ on $(-5, \infty)$. Then $dt = dx$ and $x = t - 5$.

Now
$$\int \frac{x^2}{\sqrt{x+5}}\, dx = \int \frac{(t-5)^2}{\sqrt{t}}\, dt = \int \frac{t^2 - 10t + 25}{\sqrt{t}}\, dt$$

$$= \int t^{\frac{3}{2}}\, dt - 10\int \sqrt{t}\, dt + 25\int t^{-\frac{1}{2}}\, dt + c$$

$$= \frac{2}{5}t^{\frac{5}{2}} - \frac{20}{3}t^{\frac{3}{2}} + 50\sqrt{t} + c$$

$$= \frac{2}{5}(x+5)^{\frac{5}{2}} - \frac{20}{3}(x+5)^{\frac{3}{2}} + 50\sqrt{x+5} + c.$$

6.2.13 Theorem (Integration by the method of substitution - continued) : *Let J be an interval and $\varphi : J \to I\ (\subset \boldsymbol{R})$ be a bijective differentiable function such that φ^{-1} is differentiable on I. Let $f : I \to \boldsymbol{R}$ be such that $(f \circ \varphi)\varphi'$ has a primitive F on I. Then f has an integral on I and*

$$\int f(x)dx = F(\varphi^{-1}(x)) + c = \left[\int f(\varphi(t))\,\varphi'(t)\, dt\right]_{t=\varphi^{-1}(x)}.$$

Proof : We have $(\varphi o \varphi^{-1})(x) = x$ for all x in I.

Since φ and φ^{-1} are differentiable in their domains, from the above equation, we have

$$\varphi'(\varphi^{-1}(x))\,(\varphi^{-1})'(x) = 1 \text{ for all } x \text{ in I.}$$

Hence φ^{-1} is never zero on J and

$$(\varphi^{-1})'(x) = \frac{1}{\varphi'(\varphi^{-1}(x))} \qquad \text{...(1)}$$

for all x in I. We have $(F o \varphi^{-1})' = (F' o \varphi^{-1})\,(\varphi^{-1})'$.

Hence
$$(F o \varphi^{-1})'(x) = F'(\varphi^{-1}(x))\,(\varphi^{-1})'(x)$$
$$= \frac{F'(\varphi^{-1}(x))}{\varphi'(\varphi^{-1}(x))}. \quad \text{(by (1))} \qquad \text{...(2)}$$

Since F is a primitive of $(f o \varphi)\varphi'$, we have $F' = (f \text{ o } \varphi)\varphi'$.

Hence $F'(t) = f(\varphi(t))\,\varphi'(t)$ for $t \in J$.

Therefore $F'(\varphi^{-1}(x)) = f(\varphi(\varphi^{-1}(x)))\,\varphi'(\varphi^{-1}(x))$

$$= f(x)\,\varphi'(\varphi^{-1}(x)).$$

Hence $\dfrac{F'(\varphi^{-1}(x))}{\varphi'(\varphi^{-1}(x))} = f(x)$.

Therefore, by (2), $(F o \varphi^{-1})'(x) = f(x)$.

Hence $\int f(x)\,dx = F(\varphi^{-1}(x)) + c$.

6.2.14 Remark

If J is a closed bounded interval, $\varphi : J \to I\ (\subset \mathbf{R})$ is a bijective differentiable function and φ' is never zero on J then it can be shown that φ^{-1} is differentiable on I.

6.2.15 Example

Let us evaluate $\int \frac{x^2}{\sqrt{1-x^2}}\,dx$ on $(-1, 1)$. Let $I = (-1, 1)$ and $f(x) = \frac{x^2}{\sqrt{1-x^2}}$ for all x in I. Let $J = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Define $\varphi : J \to I$ by $\varphi(\theta) = \sin\theta$. Then φ is a bijective function from J to I. Further, φ and φ^{-1} are differentiable on their respective domains. We have

$$f(\varphi(\theta))\,\varphi'(\theta) = \frac{\sin^2\theta}{\sqrt{1-\sin^2\theta}}\cdot\cos\theta = \frac{\sin^2\theta}{\cos\theta}\cdot\cos\theta = \sin^2\theta.$$

We have $\int \sin^2\theta \, d\theta = \frac{1}{2}\left[\theta - \frac{\sin 2\theta}{2}\right] + c$

$$= \frac{1}{2}[\theta - \sin\theta \ \cos\theta] + c, \text{ where } c \text{ is a constant.}$$

We have $\varphi^{-1}(x) = \text{Sin}^{-1} x$ for all x in $(-1, 1)$. When $\theta = \text{Sin}^{-1} x$, we have $\cos\theta = \sqrt{1-x^2}$ for all $x \in (-1, 1)$.

Hence $\int f(x)\, dx = \frac{1}{2}[\theta - \sin\theta \cos\theta]_{\theta=\text{Sin}^{-1}x} + c$

$$= \frac{1}{2}\left[\text{Sin}^{-1}x - x\sqrt{1-x^2}\right] + c.$$

6.2.16 Remark

Sometimes when we are given a function f and asked to find $\int f(x)\,dx$, it might be possible to find a one-to-one differentiable function φ on an interval J such that the range of φ is the domain of f, φ^{-1} is differentiable and $\int f(\varphi(t))\, \varphi'(t)\, dt$ can be easily evaluated. In such a case, the later integral is evaluated and in the value so obtained, on replacing t with $\varphi^{-1}(x)$, we obtain $\int f(x)\,dx$.

To evaluate $\int f(x)\,dx$ using Theorem 6.2.13, it is customary to make the substitution $x = \varphi(t)$ and $dx = \varphi'(t)\, dt$ and write

$$\int f(x)\,dx = \int f(\varphi(t))\, \varphi'(t)\, dt$$

with the understanding that t is to be replaced by $\varphi^{-1}(x)$ after evaluating the later integral.

In order to be able to use Theorem 6.2.13, one should be in a position to guess an appropriate substitution $\varphi(t)$ for x and have a clear idea about the domain of the given integrand and also the domain of the function φ, so that all the conditions of Theorem 6.2.13 are fulfilled.

For example, when $\sqrt{1-x^2}$ occurs in the denominator of the integrand, the substitution $x = \sin\theta$ must be useful. Since $1 - x^2$ is greater than zero for $|x| < 1$ and $-x^2 \leq 0$ for $|x| \geq 1$, the domain of the integrand must be a subset of $(-1, 1)$. If it is $(-1, 1)$, we have to restrict θ to the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ so that the sine function is a bijective map to $(-1, 1)$.

6.2.17 Solved Problems

1. Problem : *Find* $\int \frac{x}{\sqrt{1-x}}\, dx,\ x \in \text{I} = (0, 1)$.

Solution: We define $f : \text{I} \to \mathbf{R}$ by $f(x) = \frac{x}{\sqrt{1-x}}$. Let $\text{J} = \left(0, \frac{\pi}{2}\right)$. Define $\varphi : \text{J} \to \text{I}$ by $\varphi(\theta) = \sin^2\theta$. Then φ is a bijective mapping from J to I. Further, φ and φ^{-1} are differentiable on their respective domains. Put $x = \varphi(\theta) = \sin^2\theta$. Then $dx = 2\sin\theta\cos\theta\, d\theta$.

Therefore $\int \frac{x}{\sqrt{1-x}}dx = \int \frac{\sin^2\theta}{\sqrt{1-\sin^2\theta}} \cdot 2\sin\theta\cos\theta\, d\theta$

$$= \int \frac{\sin^2\theta}{\cos\theta} \cdot 2\sin\theta\cos\theta\, d\theta = \int 2\sin^3\theta\, d\theta$$

$$= 2\int \frac{1}{4}(3\sin\theta - \sin 3\theta)\, d\theta \qquad (\because \sin 3\theta = 3\sin\theta - 4\sin^3\theta)$$

$$= 2\left[\frac{1}{12}(\cos 3\theta - 9\cos\theta)\right] + c \qquad (\because \cos 3\theta = 4\cos^3\theta - 3\cos\theta)$$

$$= \frac{1}{6}\left[4\cos^3\theta - 3\cos\theta - 9\cos\theta\right] + c$$

$$= \frac{1}{6}\left[4\cos^3\theta - 12\cos\theta\right] + c$$

$$= \frac{2}{3}\cos^3\theta - 2\cos\theta + c = \frac{2}{3}(1-x)^{\frac{3}{2}} - 2\sqrt{1-x} + c.$$

2. Problem : *Evaluate* $\int \frac{dx}{(x+5)\sqrt{x+4}}$ *on* $(-4, \infty)$.

Solution: Let $I = (-4, \infty)$. Define f on I as $f(x) = \frac{dx}{(x+5)\sqrt{x+4}}$.

Let $J = (0, \infty)$. We define $\varphi : J \to I$ by $\varphi(t) = t^2 - 4$. Then φ is differentiable and is a bijection. Now put $x = \varphi(t) = t^2 - 4$. Then $t = \sqrt{x+4}$. Hence $dx = 2t\, dt$.

Thus $\int \frac{dx}{(x+5)\sqrt{x+4}} = \int \frac{2t}{(t^2+1)t} dt = \int \frac{2}{t^2+1} dt = 2\,\mathrm{Tan}^{-1} t + c.$

$$= 2\,\mathrm{Tan}^{-1}(\sqrt{x+4}) + c.$$

6.2.18 Evaluation of integrals of algebraic functions of special forms

In the following integrals, a is a positive real number.

1. Let us show that $\int \frac{1}{x^2+a^2} dx = \frac{1}{a}\mathrm{Tan}^{-1}\frac{x}{a} + c$ on **R**.

$$\int \frac{1}{x^2+a^2} dx = \frac{1}{a^2}\int \frac{1}{1+(\frac{x}{a})^2} dx + c$$

$$= \frac{1}{a^2} \cdot a\,\mathrm{Tan}^{-1}\left(\frac{x}{a}\right) + c \quad \text{(by Corollary 6.2.6)}$$

$$= \frac{1}{a}\,\mathrm{Tan}^{-1}\left(\frac{x}{a}\right) + c.$$

We can also evaluate the same integral by putting $x = a\tan\theta$.

2. Let us show that $\int \frac{1}{x^2-a^2}\,dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c$ on any interval containing neither $-a$ nor a.

$$\frac{1}{x^2-a^2} = \frac{1}{2a}\left[\frac{1}{x-a} - \frac{1}{x+a}\right].$$

Hence $$\int \frac{1}{x^2-a^2}\,dx = \frac{1}{2a}\left[\int \frac{1}{x-a}dx - \int \frac{1}{x+a}dx\right] + c$$

$$= \frac{1}{2a}\left[\log|x-a| - \log|x+a|\right] + c = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + c.$$

3. Let us show that $\int \frac{1}{\sqrt{a^2+x^2}}\,dx = \sinh^{-1}\left(\frac{x}{a}\right) + c$ on **R**.

$$\int \frac{1}{\sqrt{a^2+x^2}}\,dx = \frac{1}{a}\int \frac{1}{\sqrt{1+\left(\frac{x}{a}\right)^2}}\,dx$$

$$= \frac{1}{a}\cdot a\sinh^{-1}\left(\frac{x}{a}\right) + c = \sinh^{-1}\left(\frac{x}{a}\right) + c.$$

Also $$\int \frac{1}{\sqrt{a^2+x^2}}\,dx = \log\left(\frac{x+\sqrt{x^2+a^2}}{a}\right) + c.$$

(since $a > 0$ and $x+\sqrt{x^2+a^2}$ is positive for all x in **R**, we need not write modulus for the expression $\frac{x+\sqrt{x^2+a^2}}{a}$).

We can also evaluate the same integral by using the method of substitution.

For example, to evaluate $\int \frac{dx}{\sqrt{a^2+x^2}}$ on **R**, we substitute $x = \varphi(\theta) = a\sinh\theta$, $\theta \in$ **R**.

Observe that in this example I = **R** and J = **R**, $dx = a\cosh\theta\, d\theta$ and

$$\int \frac{dx}{\sqrt{x^2+a^2}} = \int \frac{a\cosh\theta}{a\cosh\theta}\,d\theta = \int d\theta = \theta + c$$

$$= \sinh^{-1}\left(\frac{x}{a}\right) + c = \log\left(\frac{x+\sqrt{x^2+a^2}}{a}\right) + c.$$

(OR)

We substitute $x = \varphi(\theta) = a \tan\theta$ for $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. In this case, $J = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $\theta : J \to \mathbf{R}$ is a bijection. φ and φ^{-1} are differentiable on their respective domains.

$$\sqrt{x^2+a^2} = \sqrt{a^2\tan^2\theta + a^2} = a\sec\theta;$$

$$dx = a\sec^2\theta \, d\theta.$$

Therefore $$\int \frac{dx}{\sqrt{x^2+a^2}} = \int \frac{a\sec^2\theta}{a\sec\theta} d\theta = \int \sec\theta \, d\theta$$

$$= \log|\sec\theta + \tan\theta| + c$$

$$= \log\left|\sqrt{1+\frac{x^2}{a^2}} + \frac{x}{a}\right| + c$$

$$= \log\left(\frac{\sqrt{a^2+x^2}+x}{a}\right) + c.$$

4. Let us show that $\int \frac{dx}{\sqrt{a^2-x^2}} = \text{Sin}^{-1}\frac{x}{a} + c$ for $x \in (-a, a)$.

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \frac{1}{a}\int \frac{dx}{\sqrt{1-\left(\frac{x}{a}\right)^2}} + c = \text{Sin}^{-1}\left(\frac{x}{a}\right) + c.$$

Here we note that $\int \frac{dx}{\sqrt{a^2-x^2}}$ can also be evaluated by substituting $x = a\sin\theta$, $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

5. Let us evaluate $\int \frac{dx}{\sqrt{x^2-a^2}}$ on I, where $I = (a, \infty)$ or $(-\infty, -a)$.

$$\int \frac{dx}{\sqrt{x^2-a^2}} = \frac{1}{a}\int \frac{dx}{\sqrt{\left(\frac{x}{a}\right)^2 - 1}} + c$$

$$= \begin{cases} \cosh^{-1}\left(\frac{x}{a}\right) + c \text{ on } (a, \infty) \\ -\cosh^{-1}\left(-\frac{x}{a}\right) + c \text{ on } (-\infty, -a) \end{cases}$$

$$= \begin{cases} \log\left(\frac{x+\sqrt{x^2-a^2}}{a}\right) + c \text{ on } (a, \infty) \\ -\log\left(\frac{-x+\sqrt{x^2-a^2}}{a}\right) + c \text{ on } (-\infty, -a) \end{cases} \quad \text{(from 6.1.9 (21))}$$

Alternative method : The function $\dfrac{1}{\sqrt{x^2-a^2}}$ is defined on $(-\infty, -a) \cup (a, \infty)$, $a > 0$. We can evaluate the integral on an interval I only when $I \subset (-\infty, -a) \cup (a, \infty)$.

Let $I \subset (a, \infty)$, put $x = \varphi(\theta) = a\cosh\theta,\ \theta \in (0, \infty)$.

Then $\varphi : (0, \infty) \to (a, \infty)$ is a bijective function, φ and φ^{-1} are differentiable,

$dx = a\sinh\theta\, d\theta$ and

$$\sqrt{x^2-a^2} = \sqrt{a^2\cosh^2\theta - a^2} = a\sqrt{\cosh^2\theta - 1} = a\sinh\theta.$$

Hence $$\int \frac{dx}{\sqrt{x^2-a^2}} = \int \frac{a\sinh\theta}{a\sinh\theta}\,d\theta = \int d\theta = \theta + c = \cosh^{-1}\left(\frac{x}{a}\right) + c \text{ on } (a, \infty).$$

Now let $I \subset (-\infty, -a)$.

On substituting $x = -y,\ y \in (a, \infty)$, we observe that

$$\int \frac{dx}{\sqrt{x^2-a^2}} = -\int \frac{dy}{\sqrt{y^2-a^2}} = -\cosh^{-1}\left(-\frac{x}{a}\right) + c \text{ on } (-\infty, -a).$$

We know from hyperbolic functions (Intermediate Mathematics - I(A) Text Book) that

$$\cosh^{-1}x = \log\left(x + \sqrt{x^2-1}\right) \text{ if } x > 1.$$

Hence for $x > a$ we have

$$\cosh^{-1}\left(\frac{x}{a}\right) = \log\left(\frac{x}{a} + \sqrt{\frac{x^2}{a^2} - 1}\right) = \log\left(\frac{x + \sqrt{x^2-a^2}}{a}\right).$$

If $x < -a$ then $-\dfrac{x}{a} > 1$.

Hence

$$\cosh^{-1}\left(-\frac{x}{a}\right) = \log\left(-\frac{x}{a} + \sqrt{\frac{x^2}{a^2} - 1}\right)$$

$$= \log\left(\frac{-x + \sqrt{x^2-a^2}}{a}\right) = -\log\left(\frac{-x - \sqrt{x^2-a^2}}{a}\right).$$

Thus it follows that

$$\int \frac{dx}{\sqrt{x^2-a^2}} = \begin{cases} \log\left(\dfrac{x+\sqrt{x^2-a^2}}{a}\right) + c & \text{if } I \subset (a, \infty) \\[2ex] \log\left(\dfrac{-x-\sqrt{x^2-a^2}}{a}\right) + c & \text{if } I \subset (-\infty, -a) \end{cases}.$$

Hence , $\int \frac{dx}{\sqrt{x^2-a^2}} = \ln \frac{|x+\sqrt{x^2-a^2}|}{a} + c$ on $I \subset \mathbf{R} \setminus [-a, a]$.

6. Let us show that $\int \sqrt{a^2-x^2}\, dx = \frac{a^2}{2} \text{Sin}^{-1} \frac{x}{a} + \frac{x}{2}\sqrt{a^2-x^2} + c$ on $(-a, a)$.

Put $x = a \sin\theta$ for $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ then $dx = a\cos\theta\, d\theta$.

Hence $\int \sqrt{a^2-x^2}\, dx = \int \sqrt{a^2-a^2\sin^2\theta}\,.\, a\cos\theta\, d\theta$

$$= a^2 \int \cos^2\theta\, d\theta = a^2 \int \frac{1+\cos 2\theta}{2} d\theta$$

$$= \frac{a^2}{2}\left[\int d\theta + \int \cos 2\theta\, d\theta\right] + c = \frac{a^2}{2}\left[\theta + \frac{\sin 2\theta}{2}\right] + c$$

$$= \frac{a^2}{2}[\theta + \sin\theta\cos\theta] + c$$

$$= \frac{a^2}{2}\left[\text{Sin}^{-1}\frac{x}{a} + \frac{x}{a}\sqrt{1-\frac{x^2}{a^2}}\right] + c$$

$$= \frac{a^2}{2}\text{Sin}^{-1}\frac{x}{a} + \frac{x}{2}\sqrt{a^2-x^2} + c.$$

Note : This integral $\int \sqrt{a^2-x^2}\, dx$ can also be evaluated by using the formula for integration by parts (see 6.2.26(1)).

7. Let us show that

$$\int \sqrt{x^2-a^2}\, dx = \frac{x\sqrt{x^2-a^2}}{2} - \frac{a^2}{2}\cosh^{-1}\frac{x}{a} + c \text{ on } [a, \infty).$$

put $x = a\cosh\theta$ for $\theta \in [0, \infty)$. Then $dx = a\sinh\theta\, d\theta$

and $\sqrt{x^2-a^2} = \sqrt{a^2\cosh^2\theta - a^2} = a\sinh\theta$.

$$\int \sqrt{x^2-a^2}\, dx = \int a\sinh\theta\,.\,a\sinh\theta\, d\theta = a^2\int \sinh^2\theta\,.\,d\theta$$

$$= a^2\int\left(\frac{\cosh 2\theta - 1}{2}\right)d\theta = \frac{a^2}{2}\left[\frac{\sinh 2\theta}{2} - \theta\right] + c$$

$$= \frac{a^2}{2}[\sinh\theta\cosh\theta - \theta] + c$$

$$= \frac{a^2}{2}\left[\sqrt{\cosh^2\theta - 1}\,.\,\cosh\theta - \theta\right] + c$$

$$= \frac{a^2}{2}\left[\sqrt{\frac{x^2}{a^2}-1}\,.\frac{x}{a} - \cosh^{-1}\frac{x}{a}\right]+c$$

$$= \frac{x\sqrt{x^2-a^2}}{2} - \frac{a^2}{2}\cosh^{-1}\frac{x}{a} + c.$$

(Also see 6.2.26(2)).

Similarly, it can be shown that

$$\int\sqrt{x^2-a^2}\,dx = \frac{x\sqrt{x^2-a^2}}{2} + \frac{a^2}{2}\cosh^{-1}\left(-\frac{x}{a}\right)+c \text{ on } (-\infty, -a) \text{ by substituting}$$

$x = -a\cosh\theta,\ \theta\in[0, \infty)$.

8. Let us show that

$$\int\sqrt{a^2+x^2}\,dx = \frac{x\sqrt{a^2+x^2}}{2} + \frac{a^2}{2}\sinh^{-1}\frac{x}{a} + c \text{ on } \mathbf{R}.$$

The given integral can be evaluated by substituting $x = a\sinh\theta,\ \theta\in\mathbf{R}$ or by substituting

$x = a\tan\theta,\ \theta\in\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. (Also, see 6.2.26(3)).

6.2.19 Solved Problems

1. Problem : *Evaluate* $\int\frac{dx}{\sqrt{4-9x^2}}$ *on* $\mathrm{I} = \left(-\frac{2}{3}, \frac{2}{3}\right)$.

Solution : $\int\frac{dx}{\sqrt{4-9x^2}} = \int\frac{dx}{\sqrt{2^2-(3x)^2}}$.

Put $x = \varphi(\theta) = \frac{2}{3}\sin\theta$ for $\theta\in\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Then $dx = \frac{2}{3}\cos\theta\, d\theta$.

Hence $$\int\frac{dx}{\sqrt{4-9x^2}} = \int\frac{\frac{2}{3}\cos\theta\, d\theta}{\sqrt{4-9.\frac{4}{9}\sin^2\theta}} = \int\frac{\frac{2}{3}\cos\theta}{2\cos\theta}\,d\theta$$

$$= \frac{1}{3}\int d\theta = \frac{1}{3}\theta + c = \frac{1}{3}\mathrm{Sin}^{-1}\left(\frac{3x}{2}\right)+c.$$

2. Problem : *Evaluate* $\int\frac{1}{a^2-x^2}\,dx$ *for* $x\in\mathrm{I} = (-a, a)$.

Solution : We have $\frac{1}{a^2-x^2} = \frac{1}{(a-x)(a+x)} = \frac{1}{2a}\left(\frac{1}{a-x}+\frac{1}{a+x}\right)$.

Hence $$\int \frac{1}{a^2 - x^2} dx = \frac{1}{2a}\left[\int \frac{1}{a-x} dx + \int \frac{1}{a+x} dx\right] + c$$

$$= \frac{1}{2a}[-\log|a-x| + \log|a+x|\,] + c$$

$$= \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + c.$$

3. Problem : *Evaluate* $\int \frac{1}{1+4x^2} dx$ *on* **R**

Solution : $$\int \frac{1}{1+4x^2} dx = \int \frac{dx}{4\left[\left(\frac{1}{2}\right)^2 + x^2\right]} = \frac{1}{4}\int \frac{dx}{\left(\frac{1}{2}\right)^2 + x^2}$$

$$= \frac{1}{4}\cdot\left[2\ \mathrm{Tan}^{-1}\ 2x\right] + c \quad \text{(by 6.2.18(1))}$$

$$= \frac{1}{2}\ \mathrm{Tan}^{-1}(2x) + c.$$

4. Problem : *Find* $\int \frac{1}{\sqrt{4-x^2}} dx$ *on* $(-2, 2)$.

Solution : $$\int \frac{1}{\sqrt{4-x^2}} dx = \int \frac{1}{\sqrt{2^2 - x^2}} dx = \mathrm{Sin}^{-1}\left(\frac{x}{2}\right) + c.$$

5. Problem : *Evaluate* $\int \sqrt{4x^2 + 9}\ dx$ *on* **R**.

Solution : $$\int \sqrt{4x^2+9}\ dx = 2\int \sqrt{x^2 + \left(\frac{3}{2}\right)^2}\ dx$$

$$= 2\left[\frac{x\sqrt{(\frac{3}{2})^2 + x^2}}{2} + \frac{(\frac{3}{2})^2}{2}\sinh^{-1}\left(\frac{x}{(\frac{3}{2})}\right)\right] + c \quad \text{(by 6.2.18(8))}$$

$$= \frac{1}{2}x\sqrt{4x^2+9} + \frac{9}{4}\sinh^{-1}\left(\frac{2x}{3}\right) + c.$$

6. Problem : *Evaluate* $\int \sqrt{9x^2 - 25}\ dx$ *on* $\left[\frac{5}{3}, \infty\right)$.

Solution: $$\int \sqrt{9x^2 - 25}\ dx = 3\int \sqrt{x^2 - \left(\frac{5}{3}\right)^2}\ dx$$

$$= 3\left[\frac{x\sqrt{x^2-(\frac{5}{3})^2}}{2} - \frac{(\frac{5}{3})^2}{2}\cosh^{-1}\left(\frac{x}{(\frac{5}{3})}\right)\right] + c \quad \text{(by 6.2.18(7))}$$

$$= \frac{1}{2}x\sqrt{9x^2-25} - \frac{25}{6}\cosh^{-1}\left(\frac{3x}{5}\right) + c.$$

7. Problem : *Evaluate* $\int \sqrt{16-25x^2}\, dx$ *on* $\left(-\frac{4}{5}, \frac{4}{5}\right)$.

Solution : $\int \sqrt{16-25x^2}\, dx = 5\int \sqrt{\left(\frac{4}{5}\right)^2 - x^2}\, dx$

$$= 5\left[\frac{x}{2}\sqrt{\left(\frac{4}{5}\right)^2 - x^2} + \frac{(\frac{4}{5})^2}{2}\operatorname{Sin}^{-1}\frac{x}{(\frac{4}{5})}\right] + c$$

$$= \frac{x}{2}\sqrt{16-25x^2} + \frac{16}{10}\operatorname{Sin}^{-1}\left(\frac{5x}{4}\right) + c$$

$$= \frac{x}{2}\sqrt{16-25x^2} + \frac{8}{5}\operatorname{Sin}^{-1}\left(\frac{5x}{4}\right) + c.$$

Exercise 6(b)

I. Evaluate the following integrals.

1. $\int e^{2x}\, dx,\ x \in \mathbf{R}$.
2. $\int \sin 7x\, dx,\ x \in \mathbf{R}$.
3. $\int \frac{x}{1+x^2}\, dx,\ x \in \mathbf{R}$.
4. $\int 2x\sin(x^2+1)\, dx,\ x \in \mathbf{R}$.
5. $\int \frac{(\log x)^2}{x}\, dx$ on $(0, \infty)$.
6. $\int \frac{e^{\operatorname{Tan}^{-1}x}}{1+x^2}\, dx$ on $(0, \infty)$
7. $\int \frac{\sin(\operatorname{Tan}^{-1}x)}{1+x^2}\, dx,\ x \in \mathbf{R}$.
8. $\int \frac{1}{8+2x^2}\, dx$ on $\mathbf{R}$.
9. $\int \frac{3x^2}{1+x^6}\, dx$ on $\mathbf{R}$.
10. $\int \frac{2}{\sqrt{25+9x^2}}\, dx$ on $\mathbf{R}$.
11. $\int \frac{3}{\sqrt{9x^2-1}}\, dx$ on $\left(\frac{1}{3}, \infty\right)$.

12. $\int \sin mx \cos nx\, dx$ on $\mathbf{R}$, $m \neq n$, m and n are positive integers.

13. $\int \sin mx \sin nx\, dx$ on $\mathbf{R}$, $m \neq n$, m and n are positive integers.

14. $\int \cos mx \cos nx\, dx$ on $\mathbf{R}$, $m \neq n$, m and n are positive integers.

15. $\int \sin x \sin 2x \sin 3x\, dx$ on $\mathbf{R}$.

16. $\int \frac{\sin x}{\sin(a+x)}\, dx$ on $I \subset \mathbf{R} \setminus \{n\pi - a : n \in \mathbf{Z}\}$.

II. Evaluate the following integrals.

1. $\int (3x-2)^{\frac{1}{2}}$ on $\left(\frac{2}{3}, \infty\right)$.

2. $\int \frac{1}{7x+3}\, dx$ on $I \subset \mathbf{R} \setminus \left\{-\frac{3}{7}\right\}$.

3. $\int \frac{\log(1+x)}{1+x}\, dx$ on $(-1, \infty)$

4. $\int (3x^2-4)x\, dx$ on $\mathbf{R}$.

5. $\int \frac{dx}{\sqrt{1+5x}}\, dx$ on $\left(-\frac{1}{5}, \infty\right)$

6. $\int (1-2x^3)x^2\, dx$ on $\mathbf{R}$.

7. $\int \frac{\sec^2 x}{(1+\tan x)^3} dx$ on $I \subset \mathbf{R} \setminus \left\{n\pi - \frac{\pi}{4} : n \in \mathbf{Z}\right\}$.

8. $\int x^3 \sin x^4\, dx$ on $\mathbf{R}$.

9. $\int \frac{\cos x}{(1+\sin x)^2}\, dx$ on $I \subset \mathbf{R} \setminus \left\{2n\pi + \frac{3\pi}{2} : n \in \mathbf{Z}\right\}$.

10. $\int \sqrt[3]{\sin x} \cos x\, dx$ on $\mathbf{R}$.

11. $\int 2x\, e^{x^2}\, dx$ on $\mathbf{R}$.

12. $\int \frac{e^{\log x}}{x} dx$ on $(0, \infty)$.

13. $\int \frac{x^2}{\sqrt{1-x^6}} dx$ on $(-1, 1)$.

14. $\int \frac{2x^3}{1+x^8}\, dx$ on $\mathbf{R}$.

15. $\int \frac{x^8}{1+x^{18}} dx$ on $\mathbf{R}$.

16. $\int \frac{e^x(1+x)}{\cos^2(x e^x)} dx$ on $I \subset \mathbf{R} \setminus \{x \in \mathbf{R} : \cos(xe^x) = 0\}$.

17. $\int \frac{\operatorname{cosec}^2 x}{(a+b\cot x)^5} dx$ on $I \subset \mathbf{R} \setminus \{x \in \mathbf{R} : a + b\cot x = 0\}$, where $a, b \in \mathbf{R}$, $b \neq 0$.

18. $\int e^x \sin e^x \, dx$ on $\mathbf{R}$.

19. $\int \frac{\sin (\log x)}{x} dx$ on $(0, \infty)$.

20. $\int \frac{1}{x \log x} dx$ on $(1, \infty)$.

21. $\int \frac{(1+\log x)^n}{x} dx$ on (e^{-1}, ∞), $n \neq -1$.

22. $\int \frac{\cos (\log x)}{x} dx$ on $(0, \infty)$.

23. $\int \frac{\cos \sqrt{x}}{\sqrt{x}} dx$ on $(0, \infty)$.

24. $\int \frac{2x+1}{x^2+x+1} dx$ on $\mathbf{R}$.

25. $\int \frac{ax^{n-1}}{bx^n+c} dx$, where $n \in \mathbf{N}$, a, b, c are real numbers, $b \neq 0$ and $x \in \mathrm{I} \subset \{x \in \mathbf{R} : x^n \neq -\frac{c}{b}\}$.

26. $\int \frac{1}{x \log x [\log(\log x)]} dx$ on $\mathrm{I} \subset (1, \infty) - \{e\}$.

27. $\int \coth x \, dx$ on $\mathbf{R}$.

28. $\int \frac{1}{\sqrt{1-4x^2}} dx$ on $\left(-\frac{1}{2}, \frac{1}{2}\right)$.

29. $\int \frac{dx}{\sqrt{25+x^2}}$ on $\mathbf{R}$.

30. $\int \frac{1}{(x+3)\sqrt{x+2}} dx$ on $(-2, \infty)$.

31. $\int \frac{1}{1+\sin 2x} dx$ on $\mathrm{I} \subset \mathbf{R} \setminus \{\frac{n\pi}{2} + (-1)^n \frac{\pi}{4} : n \in \mathbf{Z}\}$.

32. $\int \frac{x^2+1}{x^4+1} dx$ on $\mathbf{R}$.

33. $\int \frac{dx}{\cos^2 x + \sin 2x}$ on $\mathrm{I} \subset \mathbf{R} \setminus \left(\left\{ (2n+1)\frac{\pi}{2} : n \in \mathbf{Z} \right\} \cup \left\{ n\pi - \mathrm{Tan}^{-1}\left(\frac{1}{2}\right) : n \in \mathbf{Z} \right\} \right)$.

34. $\int \sqrt{1-\sin 2x} \, dx$ on $\mathrm{I} \subset \left[2n\pi - \frac{3\pi}{4}, 2n\pi + \frac{\pi}{4}\right]$, $n \in \mathbf{Z}$

35. $\int \sqrt{1+\cos 2x} \, dx$ on $\mathrm{I} \subset \left[2n\pi - \frac{\pi}{2}, 2n\pi + \frac{\pi}{2}\right]$, $n \in \mathbf{Z}$

36. $\int \frac{\cos x + \sin x}{\sqrt{1+\sin 2x}} dx$ on $\mathrm{I} \subset \left(2n\pi - \frac{\pi}{4}, 2n\pi + \frac{3\pi}{4}\right)$, $n \in \mathbf{Z}$.

37. $\int \frac{\sin 2x}{(a+b\cos x)^2}\,dx$ on $\begin{cases} \mathbf{R}, \text{ if } |a|>|b| \\ I\subset\{x\in\mathbf{R}: a+b\cos x\neq 0\}, \text{ if } |a|<|b| \end{cases}$.

38. $\int \frac{\sec x}{(\sec x+\tan x)^2}\,dx$ on $I\subset \mathbf{R} \setminus \left\{(2n+1)\frac{\pi}{2} : n\in \mathbf{Z}\right\}$.

39. $\int \frac{dx}{a^2\sin^2 x+b^2\cos^2 x}$ on $\mathbf{R}$, $a\neq 0, b\neq 0$.

40. $\int \frac{dx}{\sin(x-a)\sin(x-b)}$ on $I\subset\mathbf{R}\setminus(\{a+n\pi: n\in\mathbf{Z}\}\cup\{b+n\pi: n\in\mathbf{Z}\})$.

41. $\int \frac{1}{\cos(x-a)\cos(x-b)}\,dx$ on $I\subset\mathbf{R}\setminus(\{a+\frac{(2n+1)\pi}{2}: n\in\mathbf{Z}\}\cup\{b+\frac{(2n+1)\pi}{2}: n\in\mathbf{Z}\}$.

III. Evaluate the following integrals.

1. $\int \frac{\sin 2x}{a\cos^2 x+b\sin^2 x}\,dx$ on $I\subset\mathbf{R}\setminus\{x\in\mathbf{R} \mid a\cos^2 x+b\sin^2 x=0\}$.

2. $\int \frac{1-\tan x}{1+\tan x}\,dx$ for $x\in I\subset\mathbf{R}\setminus\{n\pi-\frac{\pi}{4}: n\in\mathbf{Z}\}$.

3. $\int \frac{\cot(\log x)}{x}\,dx$, $x\in I\subset(0,\infty)\setminus\{e^{n\pi}: n\in\mathbf{Z}\}$.

4. $\int e^x\cot e^x\,dx$, $x\in I\subset\mathbf{R}\setminus\{\log n\pi: n\in\mathbf{N}\}$.

5. $\int \sec(\tan x)\sec^2 x\,dx$, on $I\subset\{x\in E: \tan x\neq\frac{(2k+1)\pi}{2}$ for any $k\in\mathbf{Z}\}$,

where $E=\mathbf{R}\setminus\left\{\frac{(2n+1)\pi}{2}: n\in\mathbf{Z}\right\}$.

6. $\int \sqrt{\sin x}\cos x\,dx$ on $[2n\pi, (2n+1)\pi]$, $(n\in\mathbf{Z})$.

7. $\int \tan^4 x\sec^2 x\,dx$, $x\in I\subset\mathbf{R}\setminus\{\frac{(2n+1)\pi}{2}: n\in\mathbf{Z}\}$.

8. $\int \frac{2x+3}{\sqrt{x^2+3x-4}}\,dx$, $x\in I\subset\mathbf{R}\setminus[-4, 1]$.

9. $\int \mathrm{cosec}^2 x\sqrt{\cot x}\,dx$ on $\left(0, \frac{\pi}{2}\right]$

10. $\int \sec x \log(\sec x + \tan x)\, dx$ on $\left(0, \frac{\pi}{2}\right)$.

11. $\int \sin^3 x \; dx$ on $\mathbf{R}$.

12. $\int \cos^3 x \; dx$ on $\mathbf{R}$.

13. $\int \cos x \cos 2x \; dx$ on $\mathbf{R}$.

14. $\int \cos x \cos 3x \; dx$ on $\mathbf{R}$.

15. $\int \cos^4 x \; dx$ on $\mathbf{R}$.

16. $\int x\sqrt{4x+3}\; dx$ on $\left(-\frac{3}{4}, \infty\right)$.

17. $\int \frac{dx}{\sqrt{a^2-(b+cx)^2}}$ on $\{x \in \mathbf{R} : |b+cx| < a\}$, where a, b, c are real numbers $c \neq 0$ and $a > 0$.

18. $\int \frac{dx}{a^2+(b+cx)^2}$ on $\mathbf{R}$, where a, b, c are real numbers, $c \neq 0$ and $a > 0$.

19. $\int \frac{dx}{1+e^x}$, $x \in \mathbf{R}$.

20. $\int \frac{x^2}{(a+bx)^2}\; dx$, $x \in \mathrm{I} \subset \mathbf{R} \setminus \left\{-\frac{a}{b}\right\}$, where a, b are real numbers, $b \neq 0$.

21. $\int \frac{x^2}{\sqrt{1-x}}\; dx$, $x \in (-\infty, 1)$.

6.2(B) Integration by parts - Integration of exponential, logarithmic and inverse trigonometric functions

In section 6.2(A), we discussed mainly the integration of algebraic and trigonometric functions by the method of substitution. In this section, while continuing the discussion of the integration of algebraic and trigonometric functions, we discuss integration of exponential, logarithmic and inverse trigonometric functions and some functions obtained as combinations of these.

6.2.20 Theorem : *Let u, v be real valued differentiable functions on I. Suppose that $u'v$ has an integral on I. Then uv' has an integral on I and*

$$\int (uv')(x)\, dx = (uv)(x) - \int (u'v)(x)\, dx + c \qquad \text{... (1)}$$

where c is a constant.

Proof : From the product rule for differentiation of two functions, we know that uv is differentiable in I and that $(uv)' = u'v + uv'$.

Then $\quad uv' = (uv)' - u'v. \qquad$ ... (2)

Since $(uv)'$ has an integral, namely, uv on I and by hypothesis, $u'v$ has an integral on I, from equation (2) it follows that uv' has an integral on I, and

$$\int (uv')\,(x)\;dx = \int (uv)'\,(x)\;dx - \int (u'v)(x)\;dx + c$$
$$= (uv)\,(x) - \int (u'v)(x)\;dx + c$$

for some constant c.

6.2.21 Note

1. Formula (1) is known as the 'formula for integration by parts'. It is customary to write it in the form

$$\int u\;dv = uv - \int v\;du$$

by absorbing the constant c in the integral on the R.H.S. Here $\int u\;dv$ stands for

$\int u(x)\left(\frac{dv}{dx}\right)(x)\;dx$, $\int v\,du$ for $\int v(x)\left(\frac{du}{dx}\right)(x)\;dx$ and uv for $u(x)\,v(x)$.

2. Sometimes for a given function f, it might be possible to find differentiable functions u and v such that $f = uv'$ and $u'v$ has an integral and $\int u'v$ can easily be evaluated. In such a case, integration by parts might be convenient. Sometimes it may be necessary to use the formula more than once for evaluating a given integral.

6.2.22 Integration of exponential functions

We know that $\int e^x\;dx = e^x + c$, c a constant (see 6.1.9(4)).

1. Let us evaluate $\int xe^x\;dx$ on **R**.

We take $u(x) = x$ and $v(x) = e^x$. We have $u'(x) = 1$.

Now, by the formula for integration by parts, we have

$$\int xe^x\;dx = \int u(x)\;v'(x)\,dx = u(x)\;v(x) - \int u'(x)\;v(x)\,dx$$
$$= xe^x - \int e^x dx = xe^x - e^x + c = (x-1)\,e^x + c.$$

Observation: In the evaluation of $\int xe^x dx$, suppose we choose $u(x) = e^x$ and $v(x) = \frac{x^2}{2}$ so that $v'(x) = x$ and $u(x)\,v'(x) = x\,e^x$. Now, with this choice of u and v, the formula for integration by parts leads us to

$$\int xe^x\;dx = \frac{x^2e^x}{2} - \int \frac{x^2}{2}e^x\;dx.$$

The evaluation of $\int \frac{x^2}{2} e^x \, dx$ is lengthy. Thus the present selection of u and v has led us to a more complicated integral. Hence a judicious choice of u and v is essential to use the formula for integration by parts, for the evaluation of a given integral.

2. Let us show that, for a given differentiable function f on I,

$$\int e^x \, [f(x)+f'(x)] \, dx = e^x f(x)+c. \qquad \text{... (A)}$$

For this purpose, since

$$[e^x f(x)]' = e^x f'(x) + e^x f(x).$$
$$= e^x[f(x)+f'(x)],$$

we have by the definition of the indefinite integral, it follows that

$$\int e^x [f(x)+f'(x)] \, dx = e^x f(x)+c.$$

Formula(A) is useful in evaluating integrals of the form $\int e^x g(x) \, dx$ when g is of the form $f + f'$ for some differentiable function f.

Example : Let us find $\int e^x \frac{(1+x)}{(2+x)^2} \, dx$ on $\mathrm{I} \subset \mathbf{R} \setminus \{-2\}$.

We have $$\frac{1+x}{(2+x)^2} = \frac{(2+x)-1}{(2+x)^2} = \frac{1}{2+x} - \frac{1}{(2+x)^2}.$$

Define $f(x) = \frac{1}{2+x}$ so that $f'(x) = -\frac{1}{(2+x)^2}$.

Hence $\frac{1+x}{(2+x)^2} = \frac{1}{2+x} - \frac{1}{(2+x)^2} = f(x)+f'(x)$ so that by formula (A), we have

$$\int e^x \frac{1+x}{(2+x)^2} \, dx = \int e^x (f(x)+f'(x)) \, dx$$

$$= e^x f(x)+c = \frac{e^x}{2+x}+c.$$

6.2.23 Integration of logarithmic functions

Here we evaluate $\int \log x \, dx$ by using the formula for integration by parts.

We take $u(x) = \log x$ and $v(x) = x$. Then $v'(x) = 1$.

Hence $$\int \log x \, dx = \int u(x) \, v'(x) \, dx = u(x) \, v(x) - \int u'(x) \, v(x) \, dx$$

$$= x \log x - \int x \cdot \frac{d}{dx}(\log x)\, dx$$

$$= x \log x - \int x \cdot \frac{1}{x}\, dx$$

$$= x \log x - x + c.$$

Example : Let us evaluate $\int x \log x \, dx$ on $(0, \infty)$.

We take $u(x) = \log x$ and $v(x) = \frac{x^2}{2}$. Then $v'(x) = x$.

Now on using the formula for integration by parts,

$$\int x \log x \, dx = \frac{x^2}{2}\log x - \int \frac{1}{x} \cdot \frac{x^2}{2}\, dx$$

$$= \frac{x^2}{2}\log x - \frac{1}{2} \cdot \frac{x^2}{2} + c.$$

$$= \frac{x^2}{2}\log x - \frac{1}{4}x^2 + c.$$

6.2.24 Integration of inverse trigonometric functions

1. Let us show that

$$\int \mathrm{Sin}^{-1} x \, dx = x\, \mathrm{Sin}^{-1} x + \sqrt{1-x^2} + c \quad \text{on } (-1, 1).$$

We take $u(x) = \mathrm{Sin}^{-1} x$ and $v(x) = x$. Then $v'(x) = 1$. Hence by the formula for integration by parts, we have

$$\int \mathrm{Sin}^{-1} x \, dx = \int u(x)\, v'(x)\, dx = u(x)\, v(x) - \int u'(x)\, v(x)\, dx$$

$$= x\, \mathrm{Sin}^{-1} x - \int x \frac{d}{dx}(\mathrm{Sin}^{-1} x)\, dx$$

$$= x\, \mathrm{Sin}^{-1} x - \int \frac{x}{\sqrt{1-x^2}}\, dx$$

$$= x\, \mathrm{Sin}^{-1} x + \sqrt{1-x^2} + c. \qquad \text{...(1)}$$

2. We now evaluate $\int \mathrm{Cos}^{-1} x \, dx$ on $(-1, 1)$ by using (1).

Since $\mathrm{Cos}^{-1} x = \frac{\pi}{2} - \mathrm{Sin}^{-1} x$, we have

$$\int \mathrm{Cos}^{-1} x \, dx = \int \left(\frac{\pi}{2} - \mathrm{Sin}^{-1} x\right) dx$$

$$= \frac{\pi}{2}x - \int \text{Sin}^{-1} x \ dx$$

$$= \frac{\pi}{2}x - x\text{Sin}^{-1} x - \sqrt{1-x^2} + c \quad \text{(by(1))}$$

$$= x\ \text{Cos}^{-1} x - \sqrt{1-x^2} + c. \qquad ...(2)$$

3. We show that

$$\int \text{Tan}^{-1} x \ dx = x\ \text{Tan}^{-1} x - \log\sqrt{1+x^2} + c \ \text{ on } \mathbf{R}.$$

We take $u(x) = \text{Tan}^{-1}x$ and $v(x) = x$. Then $v'(x) = 1$.

Hence by the formula for integration by parts, we have

$$\int \text{Tan}^{-1} x\, dx = x\ \text{Tan}^{-1} x - \int \frac{x}{1+x^2}\, dx.$$

$$= x\ \text{Tan}^{-1} x - \frac{1}{2}\log(1+x^2) + c$$

$$= x\ \text{Tan}^{-1} x - \log\sqrt{1+x^2} + c. \qquad ...(3)$$

4. We now evaluate $\int \text{Cot}^{-1} x\ dx$ on **R**.

On using $\text{Cot}^{-1} x = \frac{\pi}{2} - \text{Tan}^{-1}x$, and integrating both sides and using (3), we get

$$\int \text{Cot}^{-1}x\, dx = x\frac{\pi}{2} - x\text{Tan}^{-1}x + \log\sqrt{1+x^2} + c.$$

$$= x\ \text{Cot}^{-1}x + \log\sqrt{1+x^2} + c \qquad ...(4)$$

5. Let us evaluate $\int \text{Sec}^{-1} x\ dx$ on $\text{I} \subset (-\infty,-1) \cup (1,\infty)$.

Let $x \in \text{I} \subset (1, \infty)$. Then

$$\int \text{Sec}^{-1} x\ dx = x\ \text{Sec}^{-1}x - \int \frac{1}{\sqrt{x^2-1}}$$

$$= x\ \text{Sec}^{-1}x - \cosh^{-1}x + c$$

or

$$= x\ \text{Sec}^{-1}x - \log(x + \sqrt{x^2-1}) + c. \quad \text{(by 6.2.18(5))}$$

Let $x \in \text{I} \subset (-\infty, -1)$. Then

$$\int \text{Sec}^{-1} x\ dx = x\ \text{Sec}^{-1}x + \int \frac{1}{\sqrt{x^2-1}}\, dx$$

$$= x\ \text{Sec}^{-1}x - \cosh^{-1}(-x) + c$$

or

$$= x\,\mathrm{Sec}^{-1}x - \log(-x+\sqrt{x^2-1})+c \qquad \text{(by 6.2.18(5))}$$

Hence $\int \mathrm{Sec}^{-1}x\,dx = x\,\mathrm{Sec}^{-1}x - \log(|x|+\sqrt{x^2-1})+c$ on $I \subset \mathbf{R} \setminus [-1, 1]$.

6. Similar to (5), it can be easily verified that, for $x \in I \subset (1, \infty)$,

$$\int \mathrm{Cosec}^{-1}x\,dx = x\,\mathrm{Cosec}^{-1}x + \cosh^{-1}x + c$$

or

$$= x\,\mathrm{Cosec}^{-1}x + \log(-x+\sqrt{x^2-1}) + c.$$

For $x \in I \subset (-\infty, -1)$,

$$\int \mathrm{Cosec}^{-1}x\,dx = x\,\mathrm{Cosec}^{-1}x + \cosh^{-1}(-x) + c$$

or

$$= x\,\mathrm{Cosec}^{-1}x + \log(-x+\sqrt{x^2-1}) + c.$$

Hence $\int \mathrm{Cosec}^{-1}x\,dx = x\,\mathrm{Cosec}^{-1}x + \log(|x|+\sqrt{x^2-1})+c$ on $I \subset \mathbf{R} \setminus [-1, 1]$.

6.2.25 Solved Problems

1. Problem : *Evaluate* $\int x\,\mathrm{Sin}^{-1}x\,dx$ on $(-1, 1)$.

Solution : Let $u(x) = \mathrm{Sin}^{-1}x$ and $v(x) = \dfrac{x^2}{2}$ so that $v'(x) = x$. Then $u(x)v'(x) = x\,\mathrm{Sin}^{-1}x$. Eventhough the domain of u is $[-1, 1]$, the function u is differentiable only in $(-1, 1)$. Hence the formula for integration by parts can be used here in $(-1, 1)$ only. From the same formula, we have

$$\int x\,\mathrm{Sin}^{-1}x\,dx = \mathrm{Sin}^{-1}x \,.\, \frac{x^2}{2} - \int \frac{x^2}{2}\frac{d}{dx}(\mathrm{Sin}^{-1}x)\,dx$$

$$= \frac{x^2}{2}\mathrm{Sin}^{-1}x - \frac{1}{2}\int \frac{x^2}{\sqrt{1-x^2}}\,dx$$

$$= \frac{x^2}{2}\mathrm{Sin}^{-1}x - \frac{1}{4}[\mathrm{Sin}^{-1}x - x\sqrt{1-x^2}] + c. \qquad \text{(from 6.2.15)}$$

2. Problem : *Evaluate* $\int x^2 \cos x\,dx$.

Solution : Let us take $u(x) = x^2$, $v(x) = \sin x$ so that $v'(x) = \cos x$ and $u(x)\,v'(x) = x^2\cos x$. By using the formula for integration by parts, we have

$$\int x^2 \cos x\,dx = x^2 \sin x - \int \sin x\,(x^2)'\,dx$$

$$= x^2 \sin x - 2\int x \sin x\,dx + c_1.$$

Again, by applying the formula for integration by parts to $\int x \sin x \, dx$, we get

$$\int x \sin x \, dx = -x \cos x - \int (-\cos x) \, dx$$

$$= -x \cos x + \sin x + c_2.$$

Hence $\int x^2 \cos x \, dx = x^2 \sin x - 2(\sin x - x \cos x) + c$

$$= (x^2 - 2)\sin x + 2x \cos x + c.$$

In evaluating certain integrals by using the formula for integration by parts, more than once, we come across the given integral with change of sign. This enables us to evaluate the given integral.

3. Problem : *Evaluate* $\int e^x \sin x \, dx$ *on* **R**.

Solution : Let $A = \int e^x \sin x \, dx$. Then

$$A = \int e^x (-\cos x)' \, dx$$

$$= e^x (-\cos x) - \int (-\cos x)(e^x)' \, dx$$

$$= -e^x \cos x + \int e^x \cos x \, dx + c_1 . \qquad \text{... (1)}$$

Now $\int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx + c_2$

$$= e^x \sin x - A + c_2 \qquad \text{... (2)}$$

From (1) and (2),

$$A = -e^x \cos x + e^x \sin x - A + c_1 + c_2.$$

Hence $2A = e^x(\sin x - \cos x) + c_1 + c_2.$

Therefore $A = \frac{1}{2} e^x (\sin x - \cos x) + c,$ where $c = \frac{c_1 + c_2}{2}.$

i.e., $\int e^x \sin x \, dx = \frac{1}{2} e^x (\sin x - \cos x) + c.$

4. Problem : *Find* $\int e^{ax} \cos (bx + c) \, dx$ *on* **R**, *where a, b, c are real numbers, and* $b \neq 0$.

Solution : Let $A = \int e^{ax} \cos (bx + c) \, dx$. Then from the formula for integration by parts,

$$A = e^{ax} \left[\frac{\sin (bx + c)}{b} \right] - \int a \, e^{ax} \left\{ \frac{\sin \, (bx + c)}{b} \right\} dx$$

$$= \frac{1}{b} e^{ax} \sin (bx + c) - \frac{a}{b} \int e^{ax} \sin (bx + c) dx$$

$$= \frac{1}{b} e^{ax} \sin(bx+c) - \frac{a}{b}\left[e^{ax}\left(\frac{-\cos(bx+c)}{b}\right) - \int a\, e^{ax}\left(-\frac{\cos(bx+c)}{b}\right) dx\right] + c_1$$

$$= \frac{1}{b} e^{ax} \sin(bx+c) + \frac{a}{b^2} e^{ax} \cos(bx+c) - \frac{a^2}{b^2} \mathrm{A} + c_2.$$

Hence $\left(1+\frac{a^2}{b^2}\right)\mathrm{A} = \frac{a}{b^2} e^{ax} \cos(bx+c) + \frac{1}{b} e^{ax} \sin(bx+c) + c_2.$

Therefore $(a^2+b^2)\mathrm{A} = a\, e^{ax} \cos(bx+c) + b\, e^{ax} \sin(bx+c) + c_3.$

Hence $\mathrm{A} = \frac{e^{ax}}{a^2+b^2}[a \cos(bx+c) + b \sin(bx+c)] + k$ where $k = \frac{c_3}{a^2+b^2}$, a constant.

Note : In the problem (4) above, by taking $c = 0$, we get

$$\int e^{ax} \cos bx\, dx = \frac{e^{ax}}{a^2+b^2}[a \cos bx + b \sin bx] + k.$$

5. Problem : *Evaluate* $\int \mathrm{Tan}^{-1}\sqrt{\frac{1-x}{1+x}}\, dx$, *on* $(-1, 1)$.

Solution: Put $x = \cos\theta$, $(\theta \in (0, \pi))$. Then $dx = -\sin\theta\, d\theta$, and

$$\frac{1-x}{1+x} = \frac{1-\cos\theta}{1+\cos\theta} = \frac{2\sin^2\frac{\theta}{2}}{2\cos^2\frac{\theta}{2}} = \tan^2\frac{\theta}{2}.$$ Hence

$$\int \mathrm{Tan}^{-1}\sqrt{\frac{1-x}{1+x}}\, dx = \int \mathrm{Tan}^{-1}\sqrt{\tan^2\frac{\theta}{2}}\,(-\sin\theta)\, d\theta$$

$$= -\int \mathrm{Tan}^{-1}\left(\tan\frac{\theta}{2}\right)(\sin\theta)\, d\theta$$

$$= -\frac{1}{2}\int \theta \sin\theta\, d\theta$$

$$= -\frac{1}{2}\left[\theta(-\cos\theta) - \int(-\cos\theta)\, d\theta\right] + c$$

(on using the formula for integration by parts)

$$= \frac{1}{2}[\theta\cos\theta - \sin\theta] + c$$

$$= \frac{1}{2}\left[x\, \mathrm{Cos}^{-1}x - \sqrt{1-x^2}\right] + c.$$

6. Problem : *Evaluate* $\int e^x \left(\frac{1-\sin x}{1-\cos x}\right) dx$ *on* $I \subset \mathbf{R} \setminus \{2n\pi : n \in \mathbf{Z}\}$.

Solution : $\frac{1-\sin x}{1-\cos x} = \frac{1-\sin x}{2\sin^2 \frac{x}{2}} = \frac{1}{2}\operatorname{cosec}^2 \frac{x}{2} - \cot\frac{x}{2}$.

Let $f(x) = -\cot\frac{x}{2}$. Then $f'(x) = \frac{1}{2}\operatorname{cosec}^2\frac{x}{2}$. Hence

$$\int e^x \left(\frac{1-\sin x}{1-\cos x}\right) dx = \int e^x \left(\frac{1}{2}\operatorname{cosec}^2\frac{x}{2} - \cot\frac{x}{2}\right) dx$$

$$= \int e^x \,(f'(x) + f(x))\, dx$$

$$= e^x f(x) + c \qquad \text{(by 6.2.22, formula (A))}$$

$$= -e^x \cot\frac{x}{2} + c.$$

To evaluate some integrals, we use the method of substitution as well as the formula for integration by parts.

7. Problem : *Evaluate* $\int \operatorname{Tan}^{-1}\left(\frac{2x}{1-x^2}\right) dx$ *on* $I \subset \mathbf{R} \setminus \{-1, 1\}$.

Solution : Let $\operatorname{Tan}^{-1} x = \theta$. Then $x = \tan\theta$ and $dx = \sec^2\theta\, d\theta$.

$$\frac{2x}{1-x^2} = \frac{2\tan\theta}{1-\tan^2\theta} = \tan 2\theta.$$

Hence $\operatorname{Tan}^{-1}\frac{2x}{1-x^2} = 2\theta + n\pi,$

where $n = \begin{cases} 0 & \text{if } |x| < 1 \\ -1 & \text{if } x > 1 \\ 1 & \text{if } x < -1. \end{cases}$

We have $d\theta = \frac{1}{1+x^2}\, dx$ and $1 + x^2 = 1 + \tan^2\theta = \sec^2\theta$.

Therefore $\int \operatorname{Tan}^{-1}\left(\frac{2x}{1-x^2}\right) dx = \int \left[\operatorname{Tan}^{-1}\left(\frac{2x}{1-x^2}\right)\right](1+x^2)\frac{1}{1+x^2}\, dx$

$$= \int (2\theta + n\pi)\sec^2\theta\, d\theta$$

$$= 2\int \theta \sec^2\theta\, d\theta + n\pi \int \sec^2\theta\, d\theta + c$$

$$= 2\left[\theta\tan\theta - \int \tan\theta\, d\theta\right] + n\pi\tan\theta + c \quad \text{(on integrating by parts)}$$

$$= 2\left[\theta\tan\theta + \log|\cos\theta|\right] + n\pi\tan\theta + c$$

$$= (2\theta + n\pi)\tan\theta + \log\cos^2\theta + c$$

$$= x\,\mathrm{Tan}^{-1}\left(\frac{2x}{1-x^2}\right) - \log(\sec^2\theta) + c$$

$$= x\,\mathrm{Tan}^{-1}\left(\frac{2x}{1-x^2}\right) - \log(1+x^2) + c.$$

8. Problem : *Find* $\int \frac{x^2 \exp\{m\,\mathrm{Sin}^{-1}x\}}{\sqrt{1-x^2}}\,dx$ *on* $(-1, 1)$, *where m is a real number.*

(*Here, for* $y \in \mathbf{R}$, $\exp\{y\}$ *stands for* e^y).

Solution : Let $t = \mathrm{Sin}^{-1}x$. Then $x = \sin t$ and $\frac{dx}{\sqrt{1-x^2}} = dt$ for $x \in (-1, 1)$.

Hence $\int \frac{x^2 \exp\{m\,\mathrm{Sin}^{-1}x\}}{\sqrt{1-x^2}}\,dx = \int e^{mt}\sin^2 t\,dt$

$$= \int\left(\frac{1-\cos 2t}{2}\right)e^{mt}\,dt$$

$$= \frac{1}{2}\int e^{mt}\,dt - \frac{1}{2}\int e^{mt}\cos 2t\,dt + c. \qquad \text{...(1)}$$

Case (i) : $m = 0$

From (1), $\int \frac{x^2 \exp\{m\,\mathrm{Sin}^{-1}x\}}{\sqrt{1-x^2}}\,dx = \frac{1}{2}\int dt - \frac{1}{2}\int \cos 2t\,dt + c$

$$= \frac{t}{2} - \frac{1}{2}\,\frac{\sin 2t}{2} + c$$

$$= \frac{1}{2}\mathrm{Sin}^{-1}x - \frac{1}{4}\sin 2(\mathrm{Sin}^{-1}x) + c.$$

Case (ii) : $m \neq 0$.

$$\int \frac{x^2 \exp\{m\,\mathrm{Sin}^{-1}x\}}{\sqrt{1-x^2}}\,dx = \frac{1}{2}\frac{e^{mt}}{m} - \frac{1}{2}\frac{e^{mt}}{m^2+4}[m\cos 2t + 2\sin 2t] + c_1$$

(from (1) and Problem 6.2.25(4))

$$= \frac{1}{2}e^{mt}\left[\frac{1}{m} - \frac{1}{m^2+4}(m\cos 2t + 2\sin 2t)\right] + c_1$$

$$= \frac{1}{2}e^{m\,\mathrm{Sin}^{-1}x}\left[\frac{1}{m} - \frac{1}{m^2+4}(m\cos 2(\mathrm{Sin}^{-1}x) + 2\sin 2(\mathrm{Sin}^{-1}x))\right] + c_1.$$

6.2.26 Evaluation of integrals of some special type of algebraic functions (using the method of integration by parts)

1. *Evaluation of* $\int\sqrt{a^2-x^2}\,dx$ *on* $(-a,\ a)$, *where* $a>0$.

Let $A=\int\sqrt{a^2-x^2}\,.1\ dx$.

Then by the formula for integration by parts, we have

$$A=\sqrt{a^2-x^2}\cdot x-\int\frac{-2x}{2\sqrt{a^2-x^2}}\cdot x\,dx$$

$$=x\sqrt{a^2-x^2}-\int\frac{a^2-x^2}{\sqrt{a^2-x^2}}dx+a^2\int\frac{1}{\sqrt{a^2-x^2}}dx+c_1$$

$$=x\sqrt{a^2-x^2}-\int\sqrt{a^2-x^2}\,dx+a^2\,\text{Sin}^{-1}\left(\frac{x}{a}\right)+c_1\qquad\text{(by 6.2.18(4))}$$

$$=x\sqrt{a^2-x^2}-A+a^2\,\text{Sin}^{-1}\left(\frac{x}{a}\right)+c_1.$$

Hence $2A=x\sqrt{a^2-x^2}+a^2\,\text{Sin}^{-1}\left(\frac{x}{a}\right)+c_1$.

Thus $A=\frac{x}{2}\sqrt{a^2-x^2}+\frac{1}{2}a^2\,\text{Sin}^{-1}\left(\frac{x}{a}\right)+c$ where $c=\frac{c_1}{2}$.

2. *Evaluation of* $\int\sqrt{x^2-a^2}\,dx$ *on* $(a,\ \infty)$, *where* $a>0$.

Let $A=\int\sqrt{x^2-a^2}\,.1\ dx$.

Then $A=\sqrt{x^2-a^2}\cdot x-\int\frac{2x}{2\sqrt{x^2-a^2}}\cdot x\,dx$

$$=x\sqrt{x^2-a^2}-\int\frac{x^2-a^2}{\sqrt{x^2-a^2}}dx-a^2\int\frac{1}{\sqrt{x^2-a^2}}dx+c_1$$

$$=x\sqrt{x^2-a^2}-\int\sqrt{x^2-a^2}\,dx-a^2\cosh^{-1}\left(\frac{x}{a}\right)+c_1.$$

Hence $A=\frac{1}{2}x\sqrt{x^2-a^2}-\frac{1}{2}a^2\cosh^{-1}\left(\frac{x}{a}\right)+c$ where $c=\frac{c_1}{2}$.

This integral is evaluated earlier by using the substitution $x=a\cosh\theta$ (see 6.2.18(7)).

3. *Evaluation of* $\int \sqrt{a^2+x^2}\, dx$ *on* **R**, *where* $a > 0$.

Let $A = \int \sqrt{a^2+x^2}\, dx$.

$$= \int \sqrt{a^2+x^2} \,.\, 1\, dx$$

$$= \sqrt{a^2+x^2}\,.\, x - \int \frac{2x}{2\sqrt{a^2+x^2}} .x\, dx$$

$$= x\sqrt{a^2+x^2} - \int \frac{x^2+a^2}{\sqrt{a^2+x^2}}\, dx + a^2 \int \frac{1}{\sqrt{a^2+x^2}}\, dx + c_1$$

$$= x\sqrt{a^2+x^2} - \int \sqrt{a^2+x^2}\, dx + a^2 \sinh^{-1}\left(\frac{x}{a}\right) + c_1$$

Hence $A = \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}\sinh^{-1}\left(\frac{x}{a}\right) + c$, where $c = \frac{c_1}{2}$.

Exercise 6(c)

I. Evaluate the following integrals.

1. $\int x \sec^2 x\, dx$ on $I \subset \mathbf{R} \setminus \left\{\frac{(2n+1)\pi}{2} : n \text{ is an integer}\right\}$.

2. $\int e^x \left(\text{Tan}^{-1} x + \frac{1}{1+x^2}\right) dx,\ x \in \mathbf{R}$.

3. $\int \frac{\log x}{x^2}\, dx$ on $(0, \infty)$.

4. $\int (\log x)^2\, dx$ on $(0, \infty)$.

5. $\int e^x(\sec x + \sec x \tan x)\, dx$ on $I \subset \mathbf{R} \setminus \left\{(2n+1)\frac{\pi}{2} : n \in \mathbf{Z}\right\}$.

6. $\int e^x \cos x\, dx$ on **R.**

7. $\int e^x(\sin x + \cos x)\, dx$ on **R**.

8. $\int (\tan x + \log \sec x)\, e^x\, dx$ on $((2n-\frac{1}{2})\pi, (2n+\frac{1}{2})\pi),\ n \in \mathbf{Z}$.

II. Evaluate the following integrals.

1. $\int x^n \log x\, dx$ on $(0, \infty)$, n is a real number and $n \neq -1$.

2. $\int \log(1+x^2)\, dx$ on **R**

3. $\int \sqrt{x} \log x\, dx$ on $(0, \infty)$.

4. $\int e^{\sqrt{x}}\, dx$ on $(0, \infty)$.

5. $\int x^2 \cos x\, dx$ on **R**.

6. $\int x \sin^2 x\,dx$ on $\mathbf{R}$.

7. $\int x \cos^2 x\,dx$ on $\mathbf{R}$.

8. $\int \cos\sqrt{x}\;dx$ on $\mathbf{R}$.

9. $\int x \sec^2 2x\;dx$ on $I \subset \mathbf{R} \setminus \left\{(2n\pi+1)\frac{\pi}{4} : n \in \mathbf{Z}\right\}$.

10. $\int x \cot^2 x\;dx$ on $I \subset \mathbf{R} \setminus \{n\pi \;:\; n \in \mathbf{Z}\}$.

11. $\int e^x (\tan x + \sec^2 x)\;dx$ on $I \subset \mathbf{R} \setminus \{(2n+1)\frac{\pi}{2} \;:\; n \in \mathbf{Z}\}$.

12. $\int e^x \left(\frac{1+x\log x}{x}\right) dx$ on $(0, \infty)$.

13. $\int e^{ax} \sin bx\;dx$ on $\mathbf{R}$, $a, b \in \mathbf{R}$.

14. $\int \frac{x\,e^x}{(x+1)^2} dx$ on $I \subset \mathbf{R} \setminus \{-1\}$

15. $\int \frac{dx}{(x^2+a^2)^2}$, $(a>0)$ on $\mathbf{R}$.

16. $\int e^x \log(e^{2x}+5e^x+6)\;dx$ on $\mathbf{R}$.

17. $\int e^x \frac{(x+2)}{(x+3)^2}\;dx$ on $I \subset \mathbf{R} \setminus \{-3\}$.

18. $\int \cos(\log x)\;dx$ on $(0, \infty)$.

III. Evaluate the following integrals.

1. $\int x\,\mathrm{Tan}^{-1} x\;dx,\quad x \in \mathbf{R}$.

2. $\int x^2\,\mathrm{Tan}^{-1} x\;dx,\quad x \in \mathbf{R}$.

3. $\int \frac{\mathrm{Tan}^{-1} x}{x^2}\;dx,\quad x \in I \subset \mathbf{R} \setminus \{0\}$.

4. $\int x\,\mathrm{Cos}^{-1} x\;dx,\quad x \in (-1, 1)$.

5. $\int x^2\,\mathrm{Sin}^{-1} x\;dx,\quad x \in (-1, 1)$.

6. $\int x \log(1+x)\;dx,\quad x \in (-1, \infty)$.

7. $\int \sin\sqrt{x}\;dx,$ on $(0, \infty)$.

8. $\int e^{ax} \sin(bx+c)\;dx,\quad (a, b, c \in \mathbf{R},\; b \neq 0)$ on $\mathbf{R}$.

9. $\int a^x \cos 2x\;dx$ on $\mathbf{R}$ $(a>0$ and $a \neq 1)$.

10. $\int \mathrm{Tan}^{-1}\left(\frac{3x-x^3}{1-3x^2}\right) dx$ on $I \subset \mathbf{R} \setminus \left\{-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right\}$.

11. $\int \sinh^{-1} x\;dx$ on $\mathbf{R}$.

12. $\int \cosh^{-1} x\;dx$ on $[1, \infty)$.

13. $\int \tanh^{-1} x\;dx$ on $(-1, 1)$.

Note : Hereafter we do not mention the intervals I and J.

6.2.27 Evaluation of integrals of the form $\int \frac{1}{ax^2+bx+c}\,dx$ where a, b, c are real numbers, $a \neq 0$.

Working rule : Reduce ax^2+bx+c to the form $a[(x+\alpha)^2+\beta]$ and then integrate using the substitution $t = x+\alpha$.

6.2.28 Solved Problems

1. Problem : *Evaluate* $\int \frac{dx}{4x^2-4x-7}$.

Solution : $4x^2-4x-7 = 4\left(x^2-x-\frac{7}{4}\right)$

$$= \left[4\left(x-\frac{1}{2}\right)^2-2\right].$$

Thus $$\int \frac{dx}{4x^2-4x-7} = \frac{1}{4}\int \frac{1}{\left(x-\frac{1}{2}\right)^2-(\sqrt{2})^2}\,dx+c$$

$$= \frac{1}{4}\int \frac{1}{t^2-(\sqrt{2})^2}\,dt+c \quad \text{(on substituting } t = x-\frac{1}{2})$$

$$= \frac{1}{4}\frac{1}{2\sqrt{2}}\log\left|\frac{t-\sqrt{2}}{t+\sqrt{2}}\right|+c \quad \text{(by 6.2.18(2))}$$

$$= \frac{1}{8\sqrt{2}}\log\left|\frac{t-\sqrt{2}}{t+\sqrt{2}}\right|+c$$

$$= \frac{1}{8\sqrt{2}}\log\left|\frac{2x-1-2\sqrt{2}}{2x-1+2\sqrt{2}}\right|+c.$$

2. Problem : *Find* $\int \frac{dx}{5-2x^2+4x}$.

Solution : $5-2x^2+4x = -2\left(x^2-2x-\frac{5}{2}\right) = -2\left[(x-1)^2-\frac{5}{2}-1\right]$

$$= -2\left[(x-1)^2-\left(\sqrt{\frac{7}{2}}\right)^2\right].$$

Thus $$\int \frac{dx}{5-2x^2+4x} = -\frac{1}{2}\int \frac{1}{(x-1)^2-\left(\sqrt{\frac{7}{2}}\right)^2}\,dx+c$$

$$= -\frac{1}{2}\int \frac{1}{t^2 - \left(\sqrt{\frac{7}{2}}\right)^2}\,dx + c \quad \text{(on substituting } t = x-1\text{)}$$

$$= -\frac{1}{2}\,\frac{1}{2\sqrt{\frac{7}{2}}}\log\left|\frac{t-\sqrt{\frac{7}{2}}}{t+\sqrt{\frac{7}{2}}}\right| + c \quad \text{(by 6.2.18(2))}$$

$$= -\frac{1}{2\sqrt{14}}\log\left|\frac{t-\sqrt{\frac{7}{2}}}{t+\sqrt{\frac{7}{2}}}\right| + c$$

$$= \frac{1}{2\sqrt{14}}\log\left|\frac{t+\sqrt{\frac{7}{2}}}{t-\sqrt{\frac{7}{2}}}\right| + c = \frac{1}{2\sqrt{14}}\log\left|\frac{x-1+\sqrt{\frac{7}{2}}}{x-1-\sqrt{\frac{7}{2}}}\right| + c.$$

3. Problem : *Evaluate* $\int \frac{dx}{x^2+x+1}$.

Solution : $x^2+x+1 = \left(x+\frac{1}{2}\right)^2 - \frac{1}{4} + 1 = \left(x+\frac{1}{2}\right)^2 + \frac{3}{4}$.

Hence $\int \frac{dx}{x^2+x+1} = \int \frac{dx}{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$

$$= \int \frac{dt}{t^2 + \left(\frac{\sqrt{3}}{2}\right)^2} \quad \text{(on substituting } t = x + \tfrac{1}{2}\text{)}$$

$$= \frac{2}{\sqrt{3}}\,\mathrm{Tan}^{-1}\left(\frac{t}{\left(\frac{\sqrt{3}}{2}\right)}\right) + c = \frac{2}{\sqrt{3}}\,\mathrm{Tan}^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + c. \quad \text{(by 6.2.18(1))}$$

6.2.29 Evaluation of integrals of the form

(i) $\int \frac{dx}{\sqrt{ax^2+bx+c}}$ **and (ii)** $\int \sqrt{ax^2+bx+c}\; dx$

where a, b, c are real numbers and $a \neq 0$

Working rule

Case (i) : If $a > 0$ and $b^2 - 4ac < 0$, then reduce $ax^2 + bx + c$ to the form $a[(x+\alpha)^2 + \beta]$ and then integrate.

Case (ii) : If $a < 0$ and $b^2 - 4ac > 0$, then write $ax^2 + bx + c$ as $(-a)\,[\beta - (x+\alpha)^2]$ and then integrate.

6.2.30 Solved Problems

1. Problem : *Evaluate* $\int \frac{dx}{\sqrt{x^2+2x+10}}$.

Solution : $\sqrt{x^2+2x+10} = \sqrt{(x+1)^2+9}$.

Thus
$$\int \frac{dx}{\sqrt{x^2+2x+10}} = \int \frac{dx}{\sqrt{(x+1)^2+3^2}}$$
$$= \int \frac{dt}{\sqrt{t^2+3^2}} \quad \text{(on substituting } t = x+1\text{)}$$
$$= \sinh^{-1}\left(\frac{t}{3}\right)+c = \sinh^{-1}\left(\frac{x+1}{3}\right)+c.$$

2. Problem : *Evaluate* $\int \frac{dx}{\sqrt{1+x-x^2}}$.

Solution :
$$\int \frac{dx}{\sqrt{1+x-x^2}} = \int \frac{dx}{\sqrt{-(x^2-x-1)}}$$
$$= \int \frac{dx}{\sqrt{-\left[(x-\frac{1}{2})^2-\frac{5}{4}\right]}} = \int \frac{dx}{\sqrt{(\frac{\sqrt{5}}{2})^2-(x-\frac{1}{2})^2}}$$
$$= \int \frac{dt}{\sqrt{(\frac{\sqrt{5}}{2})^2-t^2}} \quad \text{(on substituting } t = x-\tfrac{1}{2}\text{)}$$
$$= \operatorname{Sin}^{-1}\left(\frac{t}{(\frac{\sqrt{5}}{2})}\right)+c = \operatorname{Sin}^{-1}\left(\frac{x-\frac{1}{2}}{(\frac{\sqrt{5}}{2})}\right)+c$$
$$= \operatorname{Sin}^{-1}\left(\frac{2x-1}{\sqrt{5}}\right)+c.$$

3. Problem : *Evaluate* $\int \sqrt{3+8x-3x^2}\,dx$.

Solution : $3 + 8x - 3x^2 = (-3)\,(x^2 - \frac{8}{3}x - 1)$
$$= (-3)\left[(x-\tfrac{4}{3})^2-\tfrac{16}{9}-1\right] = (-3)\left[(x-\tfrac{4}{3})^2-\tfrac{25}{9}\right]$$
$$= 3\left[(\tfrac{5}{3})^2-(x-\tfrac{4}{3})^2\right].$$

Therefore

$$\int\sqrt{3+8x-3x^2}\,dx = \sqrt{3}\int\sqrt{\left[\left(\frac{5}{3}\right)^2-\left(x-\frac{4}{3}\right)^2\right]}\,dx + c$$

$$= \sqrt{3}\int\sqrt{\left(\left(\frac{5}{3}\right)^2-t^2\right)}\,dt + c \qquad \left(\text{on substituting } t = x-\frac{4}{3}\right)$$

$$= \sqrt{3}\left[\frac{t}{2}\sqrt{\left(\frac{5}{3}\right)^2-t^2}+\frac{\left(\frac{5}{3}\right)^2}{2}\operatorname{Sin}^{-1}\left(\frac{t}{\left(\frac{5}{3}\right)}\right)\right]+c$$

$$= \sqrt{3}\,\frac{(x-\frac{4}{3})}{2}\sqrt{(\tfrac{5}{3})^2-(x-\tfrac{4}{3})^2}+\frac{25}{18}\operatorname{Sin}^{-1}\left(\frac{(x-\frac{4}{3})}{(\frac{5}{3})}\right)+c$$

$$= \frac{1}{2}\left(x-\frac{4}{3}\right)\sqrt{3+8x-3x^2}+\frac{25}{6\sqrt{3}}\operatorname{Sin}^{-1}\left(\frac{3x-4}{5}\right)+c$$

$$= \frac{(3x-4)\sqrt{3+8x-3x^2}}{6}+\frac{25}{6\sqrt{3}}\operatorname{Sin}^{-1}\left(\frac{3x-4}{5}\right)+c.$$

6.2.31 To evaluate integrals of the form

(i) $\int\frac{px+q}{ax^2+bx+c}\,dx$ **(ii)** $\int(px+q)\sqrt{ax^2+bx+c}\,dx$ **(iii)** $\int\frac{px+q}{\sqrt{ax^2+bx+c}}\,dx$

where a, b, c, p, q are real numbers, $a \neq 0$ and $p \neq 0$.

Working rule : Write $px + q$ in the form $A\frac{d}{dx}(ax^2+bx+c)+B$ and then integrate.

6.2.32 Solved Problems

1. Problem : *Evaluate* $\int\frac{x+1}{x^2+3x+12}\,dx.$

Solution : We write $x+1 = A\frac{d}{dx}(x^2+3x+12)+B$

$$= A(2x+3)+B.$$

On comparing the coefficients of like powers of x on both sides of the above equation, we get

$$A = \frac{1}{2} \text{ and } B = -\frac{1}{2}. \text{ Hence } x+1 = \frac{1}{2}(2x+3)-\frac{1}{2}.$$

Now $\int\frac{x+1}{x^2+3x+12}\,dx = \frac{1}{2}\int\frac{2x+3}{x^2+3x+12}\,dx-\frac{1}{2}\int\frac{dx}{x^2+3x+12}+c$

$$= \frac{1}{2}\log|x^2+3x+12| - \frac{1}{2}\int \frac{dx}{(x+\frac{3}{2})^2+\frac{39}{4}} + c$$

$$= \frac{1}{2}\log|x^2+3x+12| - \frac{1}{2}\int \frac{dx}{(x+\frac{3}{2})^2+(\frac{\sqrt{39}}{2})^2} + c$$

$$= \frac{1}{2}\log|x^2+3x+12| - \frac{1}{2}\,\frac{2}{\sqrt{39}}\,\text{Tan}^{-1}\left(\frac{x+\frac{3}{2}}{(\frac{\sqrt{39}}{2})}\right) + c$$

$$= \frac{1}{2}\log|x^2+3x+12| - \frac{1}{\sqrt{39}}\,\text{Tan}^{-1}\left(\frac{2x+3}{\sqrt{39}}\right) + c.$$

2. Problem : *Evaluate* $\int (3x-2)\sqrt{2x^2-x+1}\,dx.$

Solution : We write $(3x-2) = A\frac{d}{dx}(2x^2-x+1)+B$

$= A(4x-1)+B.$

On comparing the coefficients of like powers of x on both sides of the above equation, we get

$A = \frac{3}{4}$ and $B = -\frac{5}{4}$. Hence $3x-2 = \frac{3}{4}(4x-1)-\frac{5}{4}$.

Therefore $\int (3x-2)\sqrt{2x^2-x+1}\,dx = \int \left[\frac{3}{4}(4x-1)-\frac{5}{4}\right]\sqrt{2x^2-x+1}\,dx$

$$= \frac{3}{4}\int (4x-1)\sqrt{2x^2-x+1}\,dx - \frac{5}{4}\int \sqrt{2x^2-x+1}\,dx + c$$

$$= \frac{3}{4}\cdot\frac{2}{3}(2x^2-x+1)^{\frac{3}{2}} - \frac{5\sqrt{2}}{4}\int\sqrt{\left(x-\frac{1}{4}\right)^2+\frac{7}{16}}\,dx + c$$

$$= \frac{1}{2}(2x^2-x+1)^{\frac{3}{2}} - \frac{5\sqrt{2}}{4}\left[\frac{1}{2}\left(x-\frac{1}{4}\right)\sqrt{\left(x-\frac{1}{4}\right)^2+\frac{7}{16}} + \frac{7}{32}\sinh^{-1}\left(\frac{(x-\frac{1}{4})}{\frac{\sqrt{7}}{4}}\right)\right] + c$$

(from 6.2.26(3))

$$= \frac{1}{2}(2x^2-x+1)^{\frac{3}{2}} - \frac{5}{4\sqrt{2}}\left(x-\frac{1}{4}\right)\sqrt{\left(x-\frac{1}{4}\right)^2+\frac{7}{16}} - \frac{35}{64\sqrt{2}}\sinh^{-1}\left(\frac{4x-1}{\sqrt{7}}\right) + c.$$

3. Problem : *Evaluate* $\int \frac{2x+5}{\sqrt{x^2-2x+10}}\,dx.$

Solution : We write

$$2x+5 = A\frac{d}{dx}(x^2-2x+10)+B = A(2x-2)+B.$$

On comparing the coefficients of the like powers of x on both sides of the above equation, we get A = 1 and B = 7. Thus $2x + 5 = (2x - 2) + 7$.

$$\text{Hence } \int \frac{2x+5}{\sqrt{x^2-2x+10}}\,dx = \int \frac{2x-2}{\sqrt{x^2-2x+10}}\,dx + 7\int \frac{dx}{\sqrt{x^2-2x+10}} + c$$

$$= 2\sqrt{x^2-2x+10} + 7\int \frac{dx}{\sqrt{(x-1)^2+3^2}} + c$$

$$= 2\sqrt{x^2-2x+10} + 7\sinh^{-1}\left(\frac{x-1}{3}\right) + c.$$

6.2.33 To evaluate integrals of the type

$\int \frac{dx}{(ax+b)\sqrt{px+q}}$ **where a, b, p and q are real numbers, $a \neq 0$ and $p \neq 0$**

Working rule : Put $t = \sqrt{px+q}$ and then integrate.

6.2.34 Solved Problem

Evaluate $\int \frac{dx}{(x+5)\sqrt{x+4}}$.

Solution : Put $t = \sqrt{x+4}$. Then $dt = \frac{1}{2\sqrt{x+4}}\,dx$.

We have $t^2 = x + 4$. Hence $x + 5 = t^2 + 1$.

$$\text{Therefore } \int \frac{dx}{(x+5)\sqrt{x+4}} = \int \frac{2}{t^2+1}\,dt = 2\,\mathrm{Tan}^{-1}\,t + c$$

$$= 2\,\mathrm{Tan}^{-1}(\sqrt{x+4}) + c.$$

6.2.35 To evaluate integrals of the type

(i) $\int \frac{1}{a+b\cos x}\,dx$ (ii) $\int \frac{1}{a+b\sin x}\,dx$

where a and b real numbers, $b \neq 0$

Working rule : $\int \frac{1}{a+b\cos x}\,dx = \int \frac{1}{a+b\left(\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)}\,dx \quad \left(\because \cos x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)$

$$= \int \frac{\left(1+\tan^2\frac{x}{2}\right)dx}{a\left(1+\tan^2\frac{x}{2}\right)+b\left(1-\tan^2\frac{x}{2}\right)} = \int \frac{\sec^2\frac{x}{2}\,dx}{(a+b)+(a-b)\tan^2\frac{x}{2}}$$

Put $\tan\frac{x}{2} = t$. Then $\frac{1}{2}\sec^2\frac{x}{2}\,dx = dt$.

Therefore $\int \frac{1}{a+b\cos x}dx = \int \frac{2}{(a+b)+(a-b)t^2}dt$

and we can now integrate it by known methods.

The integral in (ii) can be evaluated in a similar way by replacing sin$x$ by $\dfrac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}$.

6.2.36 Solved Problems

1. Problem : *Evaluate* $\int \frac{dx}{5+4\cos x}$.

Solution : $\int \frac{dx}{5+4\cos x} = \int \frac{dx}{5+4\left(\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)} = \int \frac{\left(1+\tan^2\frac{x}{2}\right)dx}{5\left(1+\tan^2\frac{x}{2}\right)+4\left(1-\tan^2\frac{x}{2}\right)} = \int \frac{\sec^2\frac{x}{2}\,dx}{9+\tan^2\frac{x}{2}}$

Put $\tan\frac{x}{2} = t$. Then $\frac{1}{2}\sec^2\frac{x}{2}\,dx = dt$.

Therefore $\int \frac{dx}{5+4\cos x} = \int \frac{2dt}{9+t^2} = \frac{2}{3}\mathrm{Tan}^{-1}\left(\frac{t}{3}\right)+c = \frac{2}{3}\mathrm{Tan}^{-1}\left(\frac{\tan\frac{x}{2}}{3}\right)+c.$

2. Problem : *Find* $\int \frac{dx}{3\cos x+4\sin x+6}$.

Solution : $\int \frac{dx}{3\cos x+4\sin x+6} = \int \frac{dx}{3\left(\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)+4\left(\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)+6}$

$$= \int \frac{\left(1+\tan^2\frac{x}{2}\right)dx}{3\left(1-\tan^2\frac{x}{2}\right)+8\tan\frac{x}{2}+6\left(1+\tan^2\frac{x}{2}\right)}$$

Put $\tan\frac{x}{2} = t$. Then $\frac{1}{2}\sec^2\frac{x}{2}\,dx = dt$.

Therefore $\displaystyle\int\frac{dx}{3\cos x+4\sin x+6} = \int\frac{2dt}{3(1-t^2)+8t+6(1+t^2)} = \int\frac{2dt}{3t^2+8t+9}$

$$= \frac{2}{3}\int\frac{dt}{(t+\frac{4}{3})^2+\frac{11}{9}} \qquad = \frac{2}{3}\cdot\frac{3}{\sqrt{11}}\,\mathrm{Tan}^{-1}\left(\frac{(t+\frac{4}{3})}{(\frac{\sqrt{11}}{3})}\right)+c$$

$$= \frac{2}{\sqrt{11}}\,\mathrm{Tan}^{-1}\left(\frac{3\tan\frac{x}{2}+4}{\sqrt{11}}\right)+c.$$

6.2.37 Evaluation of integrals of the type

$$\int\frac{a\cos x+b\sin x+c}{d\cos x+e\sin x+f}\,dx \qquad \text{...(A)}$$

where a, b, c, d, e, f are real numbers, $d \neq 0$, $e \neq 0$

Working rule : We find real numbers λ, μ and γ such that

$$(a\cos x + b\sin x + c) = \lambda[d\cos x + e\sin x + f]' + \mu[d\cos x + e\sin x + f] + \gamma$$

and then by substituting this expression in the integrand, we evaluate the given integral.

6.2.38 Solved Problems

1. Problem : *Find* $\displaystyle\int\frac{dx}{d+e\tan x}$.

Solution : We have $\displaystyle\frac{1}{d+e\tan x} = \frac{\cos x}{d\cos x+e\sin x}$.

Let us find λ, μ and γ such that

$$\cos x \equiv \lambda(d\cos x+e\sin x)'+\mu(d\cos x+e\sin x)+\gamma$$

$$\equiv \lambda(-d\sin x+e\cos x)+\mu(d\cos x+e\sin x)+\gamma.$$

On comparing the coefficients of like terms on both sides of the above equation, we have

$$\lambda e+\mu d = 1,\ -\lambda d+\mu e = 0,\ \gamma = 0.$$

On solving these equations, we obtain $\lambda = \frac{e}{d^2+e^2}$, $\mu = \frac{d}{d^2+e^2}$, $\gamma = 0$.

Therefore

$$\int\frac{dx}{d+e\tan x} = \lambda\int\frac{(d\cos x+e\sin x)'}{(d\cos x+e\sin x)}\,dx+\mu\int\frac{d\cos x+e\sin x}{d\cos x+e\sin x}\,dx+c_1$$

$$= \lambda\log|d\cos x+e\sin x|+\mu x+c_1.$$

$$= \frac{1}{d^2+e^2}[dx+e\log|d\cos x+e\sin x|]+c_1.$$

2. Problem : *Evaluate* $\int \frac{\sin x}{d\cos x + e\sin x}\,dx$ *and* $\int \frac{\cos x}{d\cos x + e\sin x}\,dx.$

Solution : Let $A_1 = \int \frac{\sin x}{d\cos x + e\sin x}\,dx$ and $A_2 = \int \frac{\cos x}{d\cos x + e\sin x}\,dx$.

Now $eA_1 + dA_2 = \int \frac{e\sin x + d\cos x}{d\cos x + e\sin x}\,dx = \int dx = x + c_1$ ...(i)

and $-dA_1 + eA_2 = \int \frac{(-d\sin x + e\cos x)}{d\cos x + e\sin x}\,dx$

$= \log|d\cos x + e\sin x| + c_2$ ...(ii)

From (i) and (ii)

$$A_1 = \frac{1}{d^2+c^2}[ex - d\log|d\cos x + e\sin x|] + c_3 \text{ where } c_3 = \frac{ec_1 - dc_2}{d^2+e^2}; \text{ and}$$

$$A_2 = \frac{1}{d^2+c^2}[dx + e\log|d\cos x + e\sin x|] + c_4 \text{ where } c_4 = \frac{dc_1 + ec_2}{d^2+e^2}.$$

3. Problem : *Evaluate* $\int \frac{\cos x + 3\sin x + 7}{\cos x + \sin x + 1}\,dx$.

Solution : Let us find real numbers λ, μ and γ such that

$$\cos x + 3\sin x + 7 = \lambda(\cos x + \sin x + 1)' + \mu(\cos x + \sin x + 1)\,\gamma$$

$$= \lambda(-\sin x + \cos x) + \mu(\cos x + \sin x + 1)\,\gamma$$

$$= (\lambda + \mu)\cos x + (-\lambda + \mu)\sin x + (\gamma + \mu).$$

On comparing the coefficients of like terms on both sides of the above equation, we have

$\lambda + \mu = 1;\quad -\lambda + \mu = 3;\quad \mu + \gamma = 7.$

On solving these equations, we have $\lambda = -1;\ \mu = 2;$ and $\gamma = 5$. Therefore

$$\int \frac{\cos x + 3\sin x + 7}{\cos x + \sin x + 1}\,dx$$

$$= -\int \frac{(\cos x + \sin x + 1)'}{\cos x + \sin x + 1}\,dx + 2\int \frac{\cos x + \sin x + 1}{\cos x + \sin x + 1}\,dx + 5\int \frac{1}{\cos x + \sin x + 1}\,dx + c$$

$$= -\log|\cos x + \sin x + 1| + 2x + 5\int \frac{1}{\cos x + \sin x + 1}\,dx + c. \quad \text{... (A)}$$

We now evaluate $\int \frac{1}{\cos x + \sin x + 1}\,dx.$

$$\int \frac{1}{\cos x+\sin x+1}\,dx = \int \frac{1}{2\cos^2\frac{x}{2}+2\cos\frac{x}{2}\sin\frac{x}{2}}\,dx = \frac{1}{2}\int \frac{\sec^2\frac{x}{2}}{(1+\tan\frac{x}{2})}\,dx$$

$$= \int \frac{dt}{1+t} \quad \text{(on substituting } t=\tan\frac{x}{2}\text{)}$$

$$= \log|1+t| = \log\left|1+\tan\frac{x}{2}\right|.$$

Hence from (A),

$$\int \frac{\cos x+3\sin x+7}{\cos x+\sin x+1}\,dx = -\log|\cos x+\sin x+1| + 2x+5\log\left|1+\tan\frac{x}{2}\right|+c.$$

6.2.39 Remark

If $f(x)=\dfrac{h(x)}{g(x)}$, $g(x)\neq 0$, where h and g are either polynomials in x or trigonometric expressions, then to find $\int f(x)\,dx$, sometimes it is possible to find constants λ, μ and γ such that

$$h(x) = \lambda g'(x)+\mu\, g(x) + \gamma.$$

In this case, $\displaystyle\int f(x)\,dx = \int\left(\frac{\lambda g'(x)+\mu\, g(x)+\gamma}{g(x)}\right)dx$

$$= \lambda \log|g(x)|+\mu x+\int \frac{\gamma}{g(x)}\,dx+c.$$

If $\displaystyle\int \frac{1}{g(x)}\,dx$ can be evaluated by known methods, then $\int f(x)\,dx$ can be evaluated.

Exercise 6(d)

I. Evaluate the following integrals.

1. $\displaystyle\int \frac{1}{\sqrt{2x-3x^2+1}}\,dx$

2. $\displaystyle\int \frac{\sin\theta}{\sqrt{2-\cos^2\theta}}\,d\theta$

3. $\displaystyle\int \frac{\cos x}{\sin^2 x+4\sin x+5}\,dx$

4. $\displaystyle\int \frac{dx}{1+\cos^2 x}$

5. $\displaystyle\int \frac{dx}{2\sin^2 x+3\cos^2 x}$

6. $\displaystyle\int \frac{1}{1+\tan x}\,dx$

7. $\displaystyle\int \frac{1}{1-\cot x}\,dx$

II. Evaluate the following integrals on any interval contained in the domains of the integrands.

1. $\int \sqrt{1+3x-x^2}\, dx$
2. $\int \frac{9\cos x - \sin x}{4\sin x + 5\cos x}\, dx$
3. $\int \frac{2\cos x + 3\sin x}{4\cos x + 5\sin x}\, dx$
4. $\int \frac{1}{1+\sin x + \cos x}\, dx$
5. $\int \frac{1}{3x^2+x+1}\, dx$
6. $\int \frac{dx}{\sqrt{5-2x^2+4x}}$

III. Evaluate the following integrals on any interval contained in the domains of the integrands.

1. $\int \frac{x+1}{\sqrt{x^2-x+1}}\, dx$
2. $\int (6x+5)\sqrt{6-2x^2+x}\, dx$
3. $\int \frac{dx}{4+5\sin x}$
4. $\int \frac{1}{2-3\cos 2x}\, dx$
5. $\int x\sqrt{1+x-x^2}\, dx$
6. $\int \frac{dx}{(1+x)\sqrt{3+2x-x^2}}$
7. $\int \frac{dx}{4\cos x + 3\sin x}$
8. $\int \frac{1}{\sin x + \sqrt{3}\cos x}\, dx$
9. $\int \frac{dx}{5+4\cos 2x}$
10. $\int \frac{2\sin x + 3\cos x + 4}{3\sin x + 4\cos x + 5}\, dx$
11. $\int \sqrt{\frac{5-x}{x-2}}\, dx$
12. $\int \sqrt{\frac{1+x}{1-x}}\, dx$
13. $\int \frac{dx}{(1-x)\sqrt{3-2x-x^2}}$
14. $\int \frac{dx}{(x+2)\sqrt{x+1}}$
15. $\int \frac{dx}{(2x+3)\sqrt{x+2}}$
16. $\int \frac{1}{(1+\sqrt{x})\sqrt{x-x^2}}\, dx$
17. $\int \frac{dx}{(x+1)\sqrt{2x^2+3x+1}}$

18. $\int \sqrt{e^x - 4}\ dx$

19. $\int \sqrt{1+\sec x}\ dx$

20. $\int \frac{dx}{1+x^4}$

6.3 Integration - Partial fractions method

A rational function in x is the quotient of two polynomials in x. If the degree of the numerator is greater than or equal to that of the denominator, the function can be reduced by actual division to the sum of a polynomial and a rational function whose numerator is of degree less than that of denominator. Since the integral of a polynomial can be easily found, the problem of integrating a rational function reduces to that of integrating a rational function whose numerator is of lesser degree than that the denominator.

In this section we confine our attention to integrate rational functions whose numerators and denominators are all polynomials with real coefficients.

Let R(x) be a rational function. We find $\int R(x)\ dx$ in the following way.

Let $R(x) = \frac{f(x)}{g(x)}$, where f and g are polynomials and $g \neq 0$ on I.

If $\deg f(x) \geq \deg g(x)$, then by dividing $f(x)$ by $g(x)$, we get $R(x) = Q(x) + \frac{h(x)}{g(x)}$,

where Q(x) and $h(x)$ are polynomials and either $h(x) \equiv 0$ or $\deg h(x) < \deg g(x)$.

Here, deg f denotes the degree of the polynomial f.

Now $\int R(x)\,dx = \int Q(x)\ dx + \int \frac{h(x)}{g(x)}\ dx$. We write $R_1(x) = \frac{h(x)}{g(x)}$.

When $h \neq 0$, using the theory of partial fractions, the fraction $\frac{h(x)}{g(x)}$ can be resolved into a sum of simpler fractions, which can be easily integrated. In this resolution, we come across four different types of fractions as shown below.

(i) $\frac{A}{bx+c}$, A, b, c are real constants and $b \neq 0$

(ii) $\dfrac{A}{(bx+c)^k}$, A, b, c are real constants, $b \neq 0$ and k is a positive integer > 1.

(iii) $\dfrac{Ax+B}{ax^2+bx+c}$, A, B, a, b, c are real numbers, $a \neq 0$ and $b^2 - 4ac < 0$ (ax^2+bx+c is an irreducible factor).

(iv) $\dfrac{Ax+B}{(ax^2+bx+c)^k}$, A, B, a, b, c are real numbers, $a \neq 0$ and $b^2 - 4ac < 0$ and k is a positive integer > 1.

Let us now consider the problem of integrating $R_1(x) = \dfrac{h(x)}{g(x)}$.

Case (i) : The roots of $g(x) = 0$ are real and distinct.

Let these roots be $x_0, x_1, ..., x_k$. Then $g(x) = a(x - x_0)(x - x_1) ... (x - x_k)$.

Then, from algebra, we know that there exist unique real constants $A_0, A_1, ..., A_k$ such that

$$R_1(x) = \frac{h(x)}{g(x)} = \frac{A_0}{x - x_0} + \frac{A_1}{x - x_1} + ... + \frac{A_k}{x - x_k}.$$

Hence
$$\int R_1(x)\,dx = A_0\int\frac{dx}{x-x_0} + A_1\int\frac{dx}{x-x_1} + ... + A_k\int\frac{dx}{x-x_k} + \log|c|$$
$$= A_0 \log|x - x_0| + A_1 \log|x - x_1| + ... + A_k \log|x - x_k| + \log|c|.$$
$$= \log|c(x-x_0)^{A_0}(x-x_1)^{A_1} ... (x-x_k)^{A_k}|, \text{ for some constant } c.$$

6.3.1 Solved Problem

Find $\displaystyle\int \frac{x^3 - 2x + 3}{x^2 + x - 2}\,dx.$

Solution : We note that the integrand is a rational function in which the degree of the numerator is greater than that of the denominator. Using synthetic division, it can be shown that

$$\frac{x^3 - 2x + 3}{x^2 + x - 2} = (x-1) + \frac{x+1}{x^2 + x - 2}.$$

Hence, we have

$$\int \frac{x^3 - 2x + 3}{x^2 + x - 2}\,dx = \int (x-1)\,dx + \int \frac{x+1}{x^2 + x - 2}\,dx + c$$

$$= \frac{(x-1)^2}{2} + \int \frac{x+1}{x^2+x-2}\,dx + c.$$

In order to evaluate the integral on the R.H.S. we resolve the integrand into partial fractions. From the theory of partial fractions, we know that

$$\frac{x+1}{x^2+x-2} = \frac{x+1}{(x+2)(x-1)} = \frac{A}{x+2} + \frac{B}{x-1},$$

for some unique constants A and B. We have

$$(x+1) = A(x-1) + B(x+2) = (A+B)x + (-A+2B).$$

On comparing the coefficients of like powers of x on both sides of the above equation, we get

$$A + B = 1 \text{ and } -A + 2B = 1.$$

On solving these equations, we get $A = \frac{1}{3}$ and $B = \frac{2}{3}$. Therefore, we have

$$\int \frac{x+1}{x^2+x-2}\,dx = \frac{1}{3}\int \frac{dx}{x+2} + \frac{2}{3}\int \frac{dx}{x-1} + \log|c|$$

$$= \frac{1}{3}\log|x+2| + \frac{2}{3}\log|x-1| + \log|c|$$

$$= \log|c(x+2)^{\frac{1}{3}}(x-1)^{\frac{2}{3}}|.$$

Thus $$\int \frac{x^3-2x+3}{x^2+x-2}\,dx = \frac{(x-1)^2}{2} + \log|c\,(x+2)^{\frac{1}{3}}\,(x-1)^{\frac{2}{3}}|.$$

6.3.2 Solved Problem

Find $\int \frac{dx}{x^2-81}$.

Solution : Using the methods of partial fractions, it can be shown that

$$\frac{1}{x^2-81} = \frac{1}{x^2-9^2} = \frac{1}{18}\left[\frac{1}{x-9} - \frac{1}{x+9}\right].$$

Hence $$\int \frac{dx}{x^2-81} = \frac{1}{18}\left[\int \frac{1}{x-9}\,dx - \int \frac{1}{x+9}\,dx\right] + c$$

$$= \frac{1}{18}[\log|x-9| - \log|x+9|] + c$$

$$= \frac{1}{18}\log\left|\frac{x-9}{x+9}\right| + c, \text{ on any interval } I \subset \mathbf{R} \setminus \{-9, 9\}.$$

Case (ii) : The roots of $g(x) = 0$ are real but some roots are repeated.

When x_0 is a root of $g(x) = 0$ of multiplicity k, the contribution of x_0 arising from resolving $\frac{h(x)}{g(x)}$ into partial fractions gives a sum of the type

$$\frac{A_1}{x-x_0} + \frac{A_2}{(x-x_0)^2} + \frac{A_3}{(x-x_0)^3} + \ldots + \frac{A_k}{(x-x_0)^k}$$

for some unique constants $A_1, A_2, \ldots, A_k$.

Each of the k terms can be easily integrated. We illustrate this case by the following solved problem.

6.3.3 Solved Problem

Find $\int \frac{2x^2-5x+1}{x^2(x^2-1)}\,dx$.

Solution : We have $x^2(x^2-1) = x^2(x-1)(x+1)$.

From the methods of partial fractions, it follows that there exist unique constants A, B, C, D such that

$$\frac{2x^2-5x+1}{x^2(x^2-1)} \equiv \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{D}{x+1}. \qquad \ldots(1)$$

Hence

$2x^2 - 5x + 1 \equiv Ax(x^2-1) + B(x^2-1) + Cx^2(x+1) + Dx^2(x-1)$. In the above equation, if we let $x = 0$, we have $1 = -B$ or $B = -1$. If $x = 1$, we have $-2 = 2C$ or $C = -1$, and if $x = -1$ we have $8 = -2D$ or $D = 4$. To find A, we equate the co-efficients of x^3 on either side of the equation and obtain $0 = A + C + D$. Since $C = -1$ and $D = -4$, we have $A - 1 - 4 = 0$ so that $A = 5$. Hence, by substituting $A = 5$, $B = -1$, $C = -1$ and $D = -4$ in (1) and then by integrating we get

$$\int \frac{2x^2-5x+1}{x^2(x^2-1)}\,dx = \int \frac{5}{x}\,dx + \int -\frac{dx}{x^2} + \int -\frac{dx}{x-1} + \int \frac{-4}{x+1}\,dx + c$$

$$= 5\log|x| + \frac{1}{x} - \log|x-1| - 4\log|x+1| + c$$

$$= \frac{1}{x} + \log\left|\frac{x^5}{(x^2-1)(x+1)^3}\right| + c.$$

Case (iii) : Some roots of $g(x) = 0$ are non-real (complex numbers), but no such root is repeated.

From algebra, we know that the complex roots of a polynomial equation with real coefficients occur in conjugate pairs. Hence if $a + ib$, $b \neq 0$ is a root of $g(x) = 0$ then $a - ib$ is also a root, where $i = \sqrt{-1}$. Hence, $g(x)$ contains a quadratic expression of the form $\alpha x^2 + \beta x + \gamma$ as one of its factors, where α, β, γ are real numbers and $\beta^2 - 4\alpha\gamma < 0$ iff $g(x)$ has two complex conjugate roots.

The contribution to the resolution of $\frac{h(x)}{g(x)}$ into partial fractions for the real roots has been discussed earlier. The contribution of an irreducible quadratic factor $\alpha x^2+\beta x+\gamma$ in the resolution of $\frac{h(x)}{g(x)}$ into partial fractions is a term of the form $\frac{\mathrm{A}x+\mathrm{B}}{\alpha x^2+\beta x+\gamma}$, which can be integrated following the discussion in 6.2.31.

The following solved problem illustrates this case.

6.3.4 Solved Problem

Find $\int \frac{3x-5}{x(x^2+2x+4)}\,dx$.

Solution : The discriminant of x^2+2x+4 is $4-16=-12<0$.

Hence x^2+2x+4 is irreducible.

From the theory of partial fractions, it follows that there exist unique constants A, B, C such that

$$\frac{3x-5}{x(x^2+2x+4)} \equiv \frac{\mathrm{A}}{x} + \frac{\mathrm{B}x+\mathrm{C}}{x^2+2x+4}.$$

Hence
$$3x-5 \equiv \mathrm{A}(x^2+2x+4)+x(\mathrm{B}x+\mathrm{C})$$
$$= (\mathrm{A}+\mathrm{B})\,x^2+(2\mathrm{A}+\mathrm{C})\,x+4\mathrm{A}.$$

On equating the coefficients of like powers of x on both sides of the above equation, we get

$\mathrm{A}+\mathrm{B} = 0$; $2\mathrm{A}+\mathrm{C}=3$ and $4\mathrm{A}=-5$.

On solving these equations, we get $\mathrm{A}=-\frac{5}{4}$; $\mathrm{B}=\frac{5}{4}$; $\mathrm{C}=\frac{11}{2}$.

Thus
$$\int \frac{3x-5}{x(x^2+2x+4)}\,dx = -\frac{5}{4}\int \frac{dx}{x}+\frac{1}{4}\int \frac{5x+22}{(x+1)^2+3}\,dx+c$$
$$= -\frac{5}{4}\log|x|+\frac{1}{4}\int \frac{5x+22}{(x+1)^2+3}\,dx+c.$$

To evaluate the integral on the RHS, put $u=x+1$. Then $x=u-1$; $dx=du$ and

$$\int \frac{5x+22}{(x+1)^2+3}\,dx = \int \frac{5u+17}{u^2+3}\,du.$$
$$= 5\int \frac{u}{u^2+3}\,du + 17\int \frac{1}{u^2+3}\,du.$$

$$= \frac{5}{2}\log|u^2+3| + \frac{17}{\sqrt{3}}\,\mathrm{Tan}^{-1}\left(\frac{u}{\sqrt{3}}\right)$$

$$= \frac{5}{2}\log|x^2+2x+4| + \frac{17}{\sqrt{3}}\,\mathrm{Tan}^{-1}\left(\frac{x+1}{\sqrt{3}}\right).$$

Thereore $\int \frac{3x-5}{x(x^2+2x+4)}\,dx = -\frac{5}{4}\log|x| + \frac{5}{8}\log|x^2+2x+4| + \frac{17}{4\sqrt{3}}\,\mathrm{Tan}^{-1}\left(\frac{x+1}{\sqrt{3}}\right) + c.$

Case (iv) : The equation $g(x) = 0$ has complex roots and some of them are repeated.

We know that if z_0 is a complex root of $g(x) = 0$ of multiplicity k, then so is $\bar{z}_0$. Hence corresponding to two conjugate complex roots $a + ib, a - ib, b \neq 0$ of $g(x) = 0$ of multiplicity k, there corresponds an irreducible quadratic expression $\alpha x^2 + \beta x + \gamma$ (with $a + ib, a - ib$ as zeros) which occurs exactly k times in the factorization of $g(x)$.

When $k > 1$, the contribution to the partial fraction resolution $R_1(x) = \frac{h(x)}{g(x)}$ from the factor $(\alpha x^2 + \beta x + \gamma)^k$ consists of the sum of the partial fractions of the type

$$\frac{A_1x+B_1}{\alpha x^2+\beta x+\gamma}, \frac{A_2x+B_2}{(\alpha x^2+\beta x+\gamma)^2}, \ldots, \frac{A_kx+B_k}{(\alpha x^2+\beta x+\gamma)^k}.$$

The contribution to the partial fraction resolution from non-real roots of multiplicity one and real roots has been discussed earlier.

We illustrate it in the following solved problem.

6.3.5 Solved Problem

Find $\int \frac{2x+1}{x(x^2+4)^2}\,dx.$

Solution : From the theory of partial fractions, it follows that there exist unique constants A, B, C, D and E such that

$$\frac{2x+1}{x(x^2+4)^2} \equiv \frac{A}{x} + \frac{Bx+c}{x^2+4} + \frac{Dx+E}{(x^2+4)^2}.$$

Hence $2x + 1 \equiv A(x^2 + 4)^2 + (Bx + C)x\,(x^2 + 4) + (Dx + E)x.$

On expanding the right hand side of the above equation, and rearranging, we have

$2x + 1 \equiv (A + B)x^4 + Cx^3 + (8A + 4B + D)\,x^2 + (4C + E)x.$

On equating the coefficients of like powers of x on both sides of the above equation, we obtain

$$A + B = 0;\; C = 0;\; 8A + 4B + D = 0;\; 4C + E = 2,\; A = \frac{1}{16}.$$

On solving these equations, we obtain

$$A = \frac{1}{16},\ B = -\frac{1}{16},\ C = 0,\ D = -\frac{1}{4} \text{ and } E = 2.$$

Hence

$$\int \frac{2x+1}{x(x^2+4)^2}dx = \frac{1}{16}\int \frac{dx}{x} - \frac{1}{32}\int \frac{2x}{x^2+4}dx + \int \frac{(-\frac{1}{4}x+2)}{(x^2+4)^2}dx + c_1 .$$

$$= \frac{1}{16}\log|x| - \frac{1}{32}\log(x^2+4) + \frac{1}{8(x^2+4)} + 2\int \frac{dx}{(x^2+4)^2} + c_1 \qquad \text{...(1)}$$

We now evaluate $\int \frac{dx}{(x^2+4)^2}$.

Put $x = 2\tan\theta$ for $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Then $dx = 2\sec^2\theta\, d\theta$.

Hence
$$\int \frac{dx}{(x^2+4)^2} = \int \frac{2\sec^2\theta}{4^2(1+\tan^2\theta)^2}d\theta = \frac{1}{8}\int \frac{d\theta}{\sec^2\theta} = \frac{1}{8}\int \cos^2\theta\, d\theta$$

$$= \frac{1}{16}\int (1+\cos 2\theta)\, d\theta = \frac{1}{16}\left[\theta + \frac{\sin 2\theta}{2}\right] + c_2$$

$$= \frac{1}{16}\left[\theta + \frac{\tan\theta}{1+\tan^2\theta}\right] + c_2 = \frac{1}{16}\left[\operatorname{Tan}^{-1}\left(\frac{x}{2}\right) + \frac{2x}{4+x^2}\right] + c_2 . \qquad \text{...(2)}$$

Thus from (1) and (2) we get

$$\int \frac{2x+1}{x(x^2+4)^2}dx = \frac{1}{16}\log|x| - \frac{1}{32}\log(x^2+4) + \frac{1}{8(x^2+4)} + \frac{1}{8}\operatorname{Tan}^{-1}\left(\frac{x}{2}\right) + \frac{1}{4}\left(\frac{x}{4+x^2}\right) + c$$

where $c = c_1 + c_2$.

Exercise 6(e)

I. Evaluate the following integrals.

1. $\int \frac{x-1}{(x-2)(x-3)}dx$
2. $\int \frac{x^2}{(x+1)(x+2)^2}dx$
3. $\int \frac{x+3}{(x-1)(x^2+1)}dx$
4. $\int \frac{dx}{(x^2+a^2)(x^2+b^2)}$
5. $\int \frac{dx}{e^x+e^{2x}}$
6. $\int \frac{dx}{(x+1)(x+2)}$
7. $\int \frac{1}{e^x-1}dx$
8. $\int \frac{1}{(1-x)(4+x^2)}dx$
9. $\int \frac{2x+3}{x^3+x^2-2x}dx$

II. Evaluate the following integrals.

1. $\int \frac{dx}{6x^2 - 5x + 1}$

2. $\int \frac{dx}{x(x+1)\,(x+2)}$

3. $\int \frac{3x-2}{(x-1)\,(x+2)(x-3)}\,dx$

4. $\int \frac{7x-4}{(x-1)^2\,(x+2)}\,dx$

III. Evaluate the following integrals.

1. $\int \frac{1}{(x-a)\,(x-b)(x-c)}\,dx$

2. $\int \frac{2x+3}{(x+3)\,(x^2+4)}\,dx$

3. $\int \frac{2x^2+x+1}{(x+3)\,(x-2)^2}\,dx$

4. $\int \frac{dx}{x^3+1}$

5. $\int \frac{\sin x \cos x}{\cos^2 x + 3\cos x + 2}\,dx$

6.4 Reduction formulae

There are many functions whose integrals cannot be reduced to one or the other of the well known standard forms of integration. However, in some cases these integrals can be connected algebraically with integrals of other expressions in the form of a recurrance relation which are directly integrable or which may be easier to integrate than the original functions. Such connecting algebraic relations are called 'reduction formulae'. These formulae connect an integral with another one which is of the same type, with a lower integer parameter which is relatively easier to integrate. In this section, we illustrate the method of integration by successive reduction.

6.4.1 Reduction formula for $\int x^n e^{ax}\,dx$, n being a positive integer

Let $I_n = \int x^n e^{ax}\,dx$.

On using formula for integration by parts, we get

$$I_n = \frac{x^n e^{ax}}{a} - \int n\, x^{n-1} \frac{e^{ax}}{a}\,dx$$

$$= \frac{x^n e^{ax}}{a} - \frac{n}{a}\int x^{n-1} e^{ax}\,dx$$

$$= \frac{x^n e^{ax}}{a} - \frac{n}{a} I_{n-1}. \qquad ...(1)$$

This is called a 'reduction formula' for $\int x^n e^{ax}\,dx$. Now I_{n-1} in turn can be connected to I_{n-2}. By successive reduction of n, the original integral I_n finally depends on I_0, where $I_0 = \int e^{ax}\,dx = \frac{e^{ax}}{a}$.

6.4.2 Solved Problem

Evaluate $\int x^3 e^{5x}\, dx$.

Solution : We take $a = 5$ and use the reduction formula 6.4.1(1) for $n = 3, 2, 1$ in that order. Then we have

$$I_3 = \int x^3 e^{5x}\, dx = \frac{x^3 e^{5x}}{5} - \frac{3}{5} I_2.$$

$$I_2 = \frac{x^2 e^{5x}}{5} - \frac{2}{5} I_1$$

$$I_1 = \frac{x e^{5x}}{5} - \frac{1}{5} I_0$$

and $$I_0 = \frac{e^{5x}}{5} + c.$$

Hence $$I_3 = \frac{x^3 e^{5x}}{5} - \frac{3}{5^2} x^2 e^{5x} + \frac{6}{5^3} x e^{5x} - \frac{6}{5^4} e^{5x} + c.$$

6.4.3 Reduction formula for $\int \sin^n x\, dx$ for an integer $n \geq 2$

Let $$I_n = \int \sin^n x\, dx$$

$$= \int \sin^{(n-1)} x \sin x\, dx = \int \sin^{n-1} x \frac{d}{dx}(-\cos x)\, dx$$

$$= \sin^{(n-1)} x\,(-\cos x) - \int (n-1) \sin^{n-2} x \cos x(-\cos x)\, dx$$

$$= -\sin^{n-1} x \cos x + \int (n-1) \sin^{n-2} x\,(1 - \sin^2 x)\, dx$$

$$= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x\, dx - (n-1) \int \sin^n x\, dx$$

$$= -\sin^{n-1} x \cos x + (n-1)\, I_{n-2} - (n-1) I_n.$$

Hence $$I_n = \frac{-\sin^{n-1} x \cos x}{n} + \frac{(n-1)}{n} I_{n-2}. \qquad \text{... (2)}$$

This is called a 'reduction formula' for $\int \sin^n x\, dx$.

If n is even, after successive reduction, we get

$$I_0 = \int (\sin x)^0\, dx = x + c_1.$$

If n is odd, after successive reduction, we get

$$I_1 = \int (\sin x)'\, dx = -\cos x + c_2.$$

6.4.4 Solved Problem

Evaluate $\int \sin^4 x\, dx$.

Solution: On using the reduction formula 6.4.3(2) for $\int \sin^n x\, dx$ with $n = 4$ and 2 in that order we have

$$I_4 = \int \sin^4 x\, dx = -\frac{\sin^3 x \cos x}{4} + \frac{3}{4} I_2$$

$$= \frac{-\sin^3 x \cos x}{4} + \frac{3}{4}\left[-\frac{\sin x \cos x}{2} + \frac{1}{2} I_0\right]$$

$$= \frac{-\sin^3 x \cos x}{4} - \frac{3}{8}\sin x \cos x + \frac{3}{8}x + c.$$

6.4.5 Reduction formula for $\int \sin^m x \cos^n x\, dx$ for a positive integer m and an integer $n \geq 2$

Let $I_{m,n} = \int \sin^m x \cos^n x\, dx$

$$= \int \sin^m x \cos^{n-1} x \,.\, \cos x\, dx$$

$$= \int \sin^m x \cos^{n-1} x \frac{d}{dx}(\sin x) dx$$

$$= \int \cos^{n-1} x \frac{d}{dx}\left(\frac{(\sin x)^{m+1}}{m+1}\right) dx$$

$$= \frac{1}{m+1}\cos^{n-1} x \sin^{m+1} x - \frac{1}{m+1}\int \sin^{m+1} \frac{d}{dx}(\cos^{n-1} x)\, dx$$

$$= \frac{1}{m+1}\cos^{n-1} x \sin^{m+1} x + \frac{n-1}{m+1}\int \sin^{m+2} x \cos^{n-2} x\, dx$$

$$= \frac{1}{m+1}\cos^{n-1} x \sin^{m+1} x + \frac{n-1}{m+1}\int \sin^m x \cos^{n-2} x (1-\cos^2 x)\, dx$$

$$= \frac{1}{m+1}\cos^{n-1} x \sin^{m+1} x + \frac{n-1}{m+1}\int \sin^m x \cos^{n-2} x\, dx - \frac{n-1}{m+1}\int \sin^m x \cos^n x\, dx$$

$$= \frac{1}{m+1}\cos^{n-1} x \sin^{m+1} x + \frac{n-1}{m+1} I_{m,\, n-2} - \frac{n-1}{m+1} I_{m,n}.$$

Hence $$I_{m,n} = \frac{1}{m+n}\cos^{n-1} x \sin^{m+1} x + \frac{n-1}{m+n} I_{m,\, n-2}, \qquad \text{... (3)}$$

which is the required reduction formula.

6.4.6 Reduction formula for $\int \tan^n x\, dx$ for an integer $n \geq 2$

Let
$$\begin{aligned} I_n &= \int \tan^n x\, dx \\ &= \int \tan^{n-2} x \tan^2 x\, dx \\ &= \int \tan^{n-2} x \sec^2 x\, dx - \int \tan^{n-2} x\, dx \qquad (\because\ \tan^2 x = \sec^2 x - 1) \\ &= \frac{\tan^{n-1} x}{n-1} - I_{n-2}, \qquad \ldots (4) \end{aligned}$$

which is the required 'reduction formula'.

When n is even, I_n will finally depend on $I_0 = \int dx = x + c_1$.

When n is odd, I_n will finally depend on $I_1 = \int \tan x\, dx = \log|\sec x| + c_2$.

6.4.7 Solved Problem

Evaluate $\int \tan^6 x\, dx$

Solution : On using 6.4.6(4) with $n = 6, 4, 2$ in that order, we get

$$\begin{aligned} I_6 &= \int \tan^6 x\, dx = \frac{\tan^5 x}{5} - \int \tan^4 x\, dx \\ &= \frac{\tan^5 x}{5} - \frac{\tan^3 x}{3} + \int \tan^2 x\, dx \\ &= \frac{\tan^5 x}{5} - \frac{\tan^3 x}{3} + \tan x - x + c. \end{aligned}$$

6.4.8 Reduction formula for $\int \sec^n x\, dx$ for an integer $n \geq 2$

Let
$$\begin{aligned} I_n &= \int \sec^n x\, dx = \int \sec^{n-2} x \sec^2 x\, dx \\ &= \int \sec^{n-2} x \frac{d}{dx}(\tan x)\, dx \\ &= \sec^{n-2} x \tan x - \int \tan x (n-2) \sec^{n-2} x \tan x\, dx \\ &= \sec^{n-2} x \tan x - (n-2)\left[\int \sec^n x\, dx - \int \sec^{n-2} x\, dx\right] \\ &= \sec^{n-2} x \tan x - (n-2)(I_n - I_{n-2}). \end{aligned}$$

That is, $I_n = \frac{1}{n-1}\sec^{n-2} x \tan x + \frac{n-2}{n-1} I_{n-2}$. ... (5)

That is the required 'reduction formula'.

When n is even, the last integral to which I_n can be reduced is I_0 which is $\int dx = x + c_1$.

When n is odd, the ultimate integral is I_1, which is $\int \sec x\, dx = \log|\sec x + \tan x| + c_2$.

6.4.9 Solved Problem

Evaluate $\int \sec^5 x\, dx$.

Solution : On using the reduction formula 6.4.8 (5) with $n = 5$, we have

$$I_5 = \int \sec^5 x\, dx = \frac{\sec^3 x \tan x}{4} + \frac{3}{4} I_3$$

$$= \frac{\sec^3 x \tan x}{4} + \frac{3}{4}\frac{\sec x \tan x}{2} + \frac{3}{8} I_1$$

$$= \frac{\sec^3 x \tan x}{4} + \frac{3}{8}\sec x \tan x + \frac{3}{8}\log|\sec x + \tan x| + c.$$

Exercise 6(f)

I. Evaluate the following integrals.

1. $\int e^x (1 + x^2)\, dx$
2. $\int x^2 e^{-3x}\, dx$
3. $\int x^3 e^{ax}\, dx$

II. 1. Show that $\int x^n e^{-x}\, dx = -x^n e^{-x} + n \int x^{n-1} e^{-x}\, dx$

2. If $I_n = \int \cos^n x\, dx$, then show that $I_n = \frac{1}{n}\cos^{n-1} x \sin x + \frac{n-1}{n} I_{n-2}$.

III. 1. Obtain reduction formula for $I_n = \int \cot^n x\, dx$, n being a positive integer, $n \geq 2$ and deduce the value of $\int \cot^4 x\, dx$.

2. Obtain the reduction formula for $I_n = \int \text{cosec}^n x\, dx$, n being a positive integer, $n \geq 2$ and deduce the value of $\int \text{cosec}^5 x\, dx$.

3. If $I_{m,n} = \int \sin^m x \cos^n x\, dx$, then show that

$$I_{m,n} = -\frac{\sin^{m-1} x \cos^{n+1} x}{m+n} + \frac{m-1}{m+n} I_{m-2,n},$$ for a positive integer n and an integer $m \geq 2$.

(compare it with the formula obtained in 6.4.5(3)).

4. Evaluate $\int \sin^5 x \cos^4 x\, dx$.

5. If $I_n = \int (\log x)^n\, dx$ then show that $I_n = x(\log x)^n - n I_{n-1}$, and hence find $\int (\log x)^4\, dx$.

Key Concepts

- Let E be a subset of $\mathbf{R}$ and let $f : E \to \mathbf{R}$ be a function. If there is function F on E such that $F'(x) = f(x)$ for all $x \in E$, then we call F an *antiderivative of f* or a *primitive of f.*
- Let I be an interval of $\mathbf{R}$. Let $f : I \to \mathbf{R}$. Suppose that f has an antiderivative F on I. Then we say that f has an *integral* on I and, for any real constant c, we call $F + c$ an *indefinite integral of f over* I and denote it by '$\int f(x)\, dx$'.
  Hence $\int f(x)\, dx = F(x) + c$; c is called a constant of integration.
- $\frac{d}{dx}\left(\int f(x)\, dx\right) = f(x)$.
- If $f : I \to \mathbf{R}$ is differentiable on I, then $\int f'(x)\, dx = f(x) + c$ where c is a consant.
- **Indefinite integrals of certain standard forms**

(a) If $n \in \mathbf{R} \setminus \{-1\}$, then $\int x^n\, dx = \frac{x^{n+1}}{n+1}$ on $\mathbf{R}$.

(b) $\int \frac{dx}{x} = \log|x| + c$ on any interval $I \subset \mathbf{R} \setminus \{0\}$.

(c) If $a > 0$ and $a \neq 1$, then $\int a^x\, dx = \frac{a^x}{\log_e a} + c$ on $\mathbf{R}$.

(d) $\int e^x\, dx = e^x + c,\ x \in \mathbf{R}$.

(e) $\int \sin x\, dx = -\cos x + c,\ x \in \mathbf{R}$.

(f) $\int \cos x\, dx = \sin x + c,\ x \in \mathbf{R}$.

(g) $\int \sec^2 x\, dx = \tan x + c$ on $I \subset \mathbf{R} \setminus \left\{\frac{(2n+1)\pi}{2} : n \in \mathbf{Z}\right\}$.

(h) $\int \operatorname{cosec}^2 x\, dx = -\cot x + c$ on $I \subset \mathbf{R} \setminus \{n\pi : n \in \mathbf{Z}\}$.

(i) $\int \sec x \tan x\, dx = \sec x + c$ on $I \subset \mathbf{R} \setminus \left\{\frac{(2n+1)\pi}{2} : n \in \mathbf{Z}\right\}$.

(j) $\int \operatorname{cosec} x \cot x\, dx = -\operatorname{cosec} x + c$ on $I \subset \mathbf{R} \setminus \{n\pi : n \in \mathbf{Z}\}$.

(k) $\int \frac{1}{\sqrt{1-x^2}}\, dx = \operatorname{Sin}^{-1} x + c$ on $(-1, 1)$.
$= -\operatorname{Cos}^{-1} x + c$ on $(-1, 1)$.

(l) $\int \frac{1}{1+x^2}\, dx = \text{Tan}^{-1} x + c$ on $\mathbf{R}$.

$= -\text{Cot}^{-1} x + c$ on $\mathbf{R}$.

(m) $\int \frac{1}{|x|\sqrt{x^2-1}}\, dx = \text{Sec}^{-1} x + c$.

$= -\text{Cosec}^{-1} x + c$ on any interval $I \subset (-\infty, -1) \cup (1, \infty)$.

(n) $\int \sinh x \; dx = \cosh x + c$ on $\mathbf{R}$.

(o) $\int \cosh x \; dx = \sinh x + c$ on $\mathbf{R}$.

(p) $\int \text{sech}^2 x \; dx = \tanh x + c$ on $\mathbf{R}$.

(q) $\int \text{cosech}^2 x \; dx = -\coth x + c$ on $\mathbf{R} \setminus \{0\}$.

(r) $\int \text{sech}\, x \tanh x \; dx = -\text{sech}\, x + c$ on $\mathbf{R}$.

(s) $\int \text{cosech}\, x \coth x \; dx = -\text{cosech}\, x + c$ on $\mathbf{R} \setminus \{0\}$.

(t) $\int \frac{1}{\sqrt{1+x^2}}\, dx = \sinh^{-1} x + c$

$= \log\,(x + \sqrt{x^2+1})$ on $\mathbf{R}$.

(u) $$\int \frac{1}{\sqrt{x^2-1}}\, dx = \begin{cases} \cosh^{-1} x + c & \text{on } (1, \infty) \\ -\cosh^{-1}(-x) + c & \text{on } (-\infty, -1) \end{cases}$$

$$= \begin{cases} \log\,(x + \sqrt{x^2-1}) + c & \text{on } (1, \infty) \\ -\log_e(-x + \sqrt{x^2-1}) + c & \text{on } (-\infty, -1) \end{cases}$$

$= \log |x + \sqrt{x^2-1}| + c$ on $I \subset \mathbf{R} \setminus [-1, 1]$.

❖ If f and g have integrals on I, then $f+g$ has an integral on I and $\int (f \pm g)(x)\, dx = \int f(x)\, dx \pm \int g(x)\, dx + c$.

❖ If f has an integral on I and a is a real number then af has an integral on I and $\int (af)(x)\, dx = a \int f(x)\, dx + c$.

❖ **Method of substitution**

Let $f: I \to \mathbf{R}$ have an integral on I and F be a primitive of f on I. Let J be an interval of $\mathbf{R}$ and g : J $\to$ I be a differentiable function. Then $(fog)g'$ has an integral on J, and

$$\int f(g(x))\, g'(x)\, dx = F(g(x)) + c.$$

i.e., $$\int f(g(x))\, g'(x)\, dx = \left[\int f(t)\, dt\right]_{t=g(x)}.$$

- Let $f : \mathrm{I} \to \mathbf{R}$ have an integral on I and F be a primitive of f. Let $a, b \in \mathbf{R}$ with $a \neq 0$. Then $\int f(ax+b)\,dx = \frac{1}{a}\mathrm{F}(ax+b)+c$ for all $x \in \mathrm{J}$, where $\mathrm{J} = \{x \in \mathbf{R} : ax+b \in \mathrm{I}\}$.
- Let $f : \mathrm{I} \to \mathbf{R}$ be differentiable. Then the following are true.

  (a) If f is never zero on I, then $\frac{f'}{f}$ has an integral on I and $\int \frac{f'(x)}{f(x)}\,dx = \log|f(x)|+c$ on I.

  (b) If α is a positive integer or if $\alpha \in \mathbf{R} \setminus \{-1\}$ and $f(x) > 0$ for all $x \in \mathrm{I}$ then $f^{\alpha} f'$ has an integral on I and $\int [f(x)]^{\alpha} f'(x)\,dx = \frac{[f(x)]^{\alpha+1}}{\alpha+1}+c.$

  In particular, when $\alpha = -\frac{1}{2}$, we have $\int \frac{f'(x)}{\sqrt{f(x)}}\,dx = 2\sqrt{f(x)}+c$ on I.

  (c) If $a \in \mathbf{R} \setminus \{0\}$, then $\int f'(ax+b)dx = \frac{1}{a} f(ax+b)+c$ on $\mathrm{J} = \{x \in \mathbf{R} : ax+b \in \mathrm{I}\}$.

- **Integration by the method of substitution continued**

  Let J be an interval and $\varphi : \mathrm{J} \to \mathrm{I}$ be a one-to-one differentiable mapping of J onto I such that φ^{-1} is differentiable in I. Let $f : \mathrm{I} \to \mathbf{R}$ be such that $(f o \varphi)\,\varphi'$ has a primitive F on J. Then f has an integral on I and

  $$\int f(x)dx = \mathrm{F}(\varphi^{-1}(x))+c = \left[\int f(\varphi(t))\,\varphi'(t)\,dt\right]_{t=\varphi^{-1}(x)}.$$

- **Evaluation of integrals of special forms**

  Let a be a positive real number. Then we have the following :

  (a) $\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\,\mathrm{Tan}^{-1}\left(\frac{x}{a}\right)+c$ on $\mathbf{R}$.

  (b) $\int \frac{1}{x^2-a^2}\,dx = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right|+c$ on any interval containing neither $-a$ nor a.

  (c) $\int \frac{1}{\sqrt{a^2+x^2}}\,dx = \sinh^{-1}\left(\frac{x}{a}\right)$ on $\mathbf{R}$.

  $= \log\left(\frac{x+\sqrt{x^2+a^2}}{a}\right)+c$ on $\mathbf{R}$.

  (d) $\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \mathrm{Sin}^{-1}\left(\frac{x}{a}\right)+c$ on $(-a, a)$.

(e) $$\int \frac{dx}{\sqrt{x^2-a^2}} = \begin{cases} \cosh^{-1}\left(\frac{x}{a}\right)+c \text{ on } (a,\infty) \\ -\cosh^{-1}\left(-\frac{x}{a}\right)+c \text{ on } (-\infty,-a) \end{cases}$$

$$= \begin{cases} \log\left(\frac{x+\sqrt{x^2-a^2}}{a}\right)+c \text{ on } (a,\infty) \\ -\log\left(\frac{-x+\sqrt{x^2-a^2}}{a}\right)+c \text{ on } (-\infty,-a) \end{cases}$$

$$= \log\frac{|x+\sqrt{x^2-a^2}|}{a}+c \ \ on\ \ I\subset \mathbf{R}\setminus[-a,a].$$

(f) $$\int\sqrt{a^2-x^2}\,dx = \frac{a^2}{2}\,\mathrm{Sin}^{-1}\left(\frac{x}{a}\right)+\frac{x}{2}\sqrt{a^2-x^2}+c \text{ on } (-a,a).$$

(g) $$\int\sqrt{x^2-a^2}\,dx = \begin{cases} \frac{x\sqrt{x^2-a^2}}{2}-\frac{a^2}{2}\cosh^{-1}\left(\frac{x}{a}\right)+c \ on\ [a,\infty) \\ \frac{x\sqrt{x^2-a^2}}{2}+\frac{a^2}{2}\cosh^{-1}\left(-\frac{x}{a}\right)+c \ on\ (-\infty,-a] \end{cases}$$

(h) $$\int\sqrt{a^2+x^2}\,dx = \frac{x\sqrt{x^2+a^2}}{2}+\frac{a^2}{2}\sinh^{-1}\left(\frac{x}{a}\right)+c \text{ on } \mathbf{R}.$$

❖ **Formula for integration by parts**

Let u and v be real valued differentiable functions on I. Suppose that uv' has an integral on I. Then uv' has an integral on I and $\int(uv')(x)\,dx = (uv)(x)-\int(u'v)(x)\,dx$

❖ Given a differentiable function f on I,

$$\int e^x(f(x)+f'(x)\,dx = e^x f(x)+c.$$

❖ To evaluate integrals of the form

$\int\frac{1}{ax^2+bx+c}\,dx$, where a, b, c are real numbers, $a\neq 0$, reduce ax^2+bx+c to the form $a[(x+\alpha)^2+\beta]$ and then integrate using the substitution $t=x+\alpha$.

❖ To evaluate integrals which are of the form (i) $\int\frac{dx}{\sqrt{ax^2+bx+c}}$ or of the form (ii) $\int\sqrt{ax^2+bx+c}\;dx$, where a, b, c are real numbers and $a\neq 0$, we adopt the following working rule :

case (a) : If $a > 0$ and $b^2 - 4ac < 0$ then reduce $ax^2 + bx + c$ to the form $a[(x+\alpha)^2 + \beta]$ and then integrate.

case (b) :

If $a < 0$ and $b^2 - 4ac > 0$ then write $ax^2 + bx + c$ as $(-a)[\beta - (x+\alpha)^2]$ and then integrate.

❖ To evaluate integrals of any of the three forms

(i) $\int \frac{px+q}{ax^2+bx+c}\,dx$ (ii) $\int (px+q)\sqrt{ax^2+bx+c}\,dx$ and (iii) $\int \frac{px+q}{\sqrt{ax^2+bx+c}}\,dx$

where a, b, c, p, q are real numbers, $a \neq 0$ and $p \neq 0$, we write $px + q$ in the form $A\frac{d}{dx}(ax^2+bx+c) + B$ and then integrate.

❖ To evaluate integrals of the type $\int \frac{dx}{(ax+b)\sqrt{px+q}}$, where a, b, p and q are real numbers, $a \neq 0$ and $p \neq 0$, we put $t = \sqrt{px+q}$ and then integrate.

❖ Integrals of the form

(i) $\int \frac{1}{a+b\cos x}\,dx$ and (ii) $\int \frac{1}{a+b\sin x}\,dx$,

where a and b are real numbers and $b \neq 0$, are evaluated by using the substitution $t = \tan\frac{x}{2}$.

❖ Integrals of the form

$$\int \frac{a\cos x + b\sin x + c}{d\cos x + e\sin x + f}\,dx$$

where a, b, c, d, e, f are real numbers $d \neq 0, e \neq 0$ are evaluated by using the following rule :

We find real numbers λ, μ and γ such that

$(a\cos x + b\sin x + c) = \lambda[d\cos x + e\sin x + f]' + \mu[d\cos x + e\sin x + f] + \gamma$

and substitute this expression for the integrand, to evaluate the given integral.

❖ Let $R(x) = \frac{f(x)}{g(x)}$, where f and g are polynomials in x with rational coefficients and $g \neq 0$ on I. If the degree of $f(x)$ is greater than or equal to that of $g(x)$, then by dividing $f(x)$ by $g(x)$, using synthetic division, we find polynomials $Q(x)$ and $h(x)$ such that $f(x) = Q(x)\,g(x) + h(x)$, where h is either the zero polynomial or $h \neq 0$ and degree of $h(x)$ is less than the degree of $g(x)$. In this case, we have

$$R(x) = Q(x) + \frac{h(x)}{g(x)}.$$

When $h \neq 0$, we obtain the resolution of $\frac{h(x)}{g(x)}$ into partial fractions using which we evaluate $\int \frac{h(x)}{g(x)}\,dx$.

We have $\int \frac{f(x)}{g(x)}\,dx = \int Q(x)dx + \int \frac{h(x)}{g(x)}\,dx + c.$

- If $I_n = \int x^n e^{ax} dx$, then $I_n = \frac{x^n e^{ax}}{a} - \frac{n}{a} I_{n-1}$, for a positive integer n.
- If $I_n = \int \sin^n x \, dx$, then $I_n = \frac{-\sin^{n-1} x \cos x}{n} + \frac{(n-1)}{n} I_{n-2}$, for an integer $n \geq 2$.
- If $I_n = \int \cos^n x \, dx$, then $I_n = \frac{\cos^{n-1} x \sin x}{n} + \frac{(n-1)}{n} I_{n-2}$, for an integer $n \geq 2$.
- If $I_n = \int \tan^n x \, dx$, then $I_n = \frac{\tan^{n-1} x}{n-1} - I_{n-2}$, for an integer $n \geq 2$.
- If $I_{m,n} = \int \sin^m x \cos^n x \, dx$, then $I_{m,n} = \frac{1}{m+n} \cos^{n-1} x \sin^{m+1} x + \frac{n-1}{m+n} I_{m,n-2}$, where m is a positive integer and an integer $n \geq 2$.

Historical Note

The beginnings of 'Integral Calculus' can be traced back to antiquity. Ancient mathematicians of Greece developed the method of exhaustion which they have applied to calculate areas of plane surfaces and volumes of solids. Thus this method can be regarded as a primitive procedure for integration. *Eudoxus* (ca.408-355 B.C.) and *Archimedes* (ca. 287-212 B.C.) contributed vastly to the development of the method of exhaustion.

Walli's (1616-1703) chief contribution to the development of calculus in its early period lay in the theory of integration.

With the invention of the calculus, the fundamental concepts of function, continuity, differentiablity and integration got systematized.

Augustin Louis Cauchy (1789 - 1857) gave the modern definition of continuity in his '*Cours d' analyse*' (1821). He defined the definite integral as a limit of a sum. He introduced *Cauchy* sum $\sum_{i=1}^{n} f(x_i)\,(x_i - x_{i-1})$ for a continuous function f on $[a, b]$ and defined the limit of the sum as the definite integral.

Bernhard Riemann (1826-1866) began with the question 'when is a function integrable?', which led him to the investigation of convergence of *Cauchy* sums. Thus he refined and clarified the notion of integral and that is what we call now the '*Riemann Integral*'.

Answers

Exercise 6(a)

I. 1. $\frac{x^4}{4}-\frac{2}{3}x^3+3x+c$

2. $\frac{4}{5}x^{\frac{5}{2}}+c$

3. $\sqrt[3]{2}\,\frac{3}{5}x^{\frac{5}{3}}+c$

4. $\frac{x^2}{4}+\frac{3}{2}x-\frac{1}{2}\log|x|+c$

5. $\log x-2\sqrt{x}+c$

6. $x+2\log|x|+\frac{3}{x}+c$

7. $\frac{x^2}{2}+4\text{Tan}^{-1}x+c$

8. $e^x-\log|x|+2\log|x+\sqrt{x^2-1}|+c$

9. $\tanh^{-1}x+\text{Tan}^{-1}x+c$

10. $\text{Sin}^{-1}x+2\sinh^{-1}x+c$

11. $\tan x+c$

12. $\frac{1}{2}(\tan x-x)+c$

II. 1. $x-x^3+\frac{3}{5}x^5-\frac{1}{7}x^7+c$

2. $6\sqrt{x}-2\log x-\frac{1}{3x}+c$

3. $\log x-\frac{4}{\sqrt{x}}-\frac{1}{x}+c$

4. $\frac{9}{4}x^2+3x+\frac{1}{2}\log|x|+c$

5. $\frac{2}{9}x^2-\frac{4}{9}x+\frac{1}{9}\log x+c$

6. $2\sqrt{x}+2\cosh^{-1}x+\frac{3}{2x}+c$

7. $\tan x-\sin x+\frac{x^3}{3}+c$

8. $\sec x+3\log|x|-4x+c$

9. $\frac{2}{3}x^{\frac{3}{2}}-2\text{Tanh}^{-1}x+c$

10. $\frac{x^4}{4}-\sin x+4\sinh^{-1}x+c$

11. $\sinh x+\sinh^{-1}x+c$

12. $\cosh x+\log|x+\sqrt{x^2-1}|+c$

13. $\frac{1}{(\log a-\log b)}\left[\left(\frac{a}{b}\right)^x-\left(\frac{b}{a}\right)^x\right]-2x+c$

14. $\tan x-\cot x+c$

15. $-\cot x-\frac{1}{2}x+c$

16. $-\sqrt{2}\cos x+c$

17. $-\cosh x+\sinh x+c$

18. $-\cot x+\text{cosec}\,x+c$

Exercise 6(b)

I. 1. $\frac{1}{2}e^{2x}+c$

2. $-\frac{1}{7}\cos 7x+c$

3. $\frac{1}{2}\log(1+x^2)+c$

4. $-\cos(x^2+1)+c$

5. $\frac{(\log x)^3}{3}+c$

6. $e^{\text{Tan}^{-1}x}+c$

7. $-\cos(\text{Tan}^{-1}x)+c$

8. $\frac{1}{4}\tan^{-1}\frac{x}{2}+c$

9. $\text{Tan}^{-1}x^3+c$

10. $\frac{2}{3}\sinh^{-1}\frac{3x}{5}+c$

11. $\cosh^{-1}3x+c$

12. $-\frac{1}{2}\left[\frac{\cos(m+n)x}{(m+n)}+\frac{\cos(m-n)x}{(m-n)}\right]+c$

13. $\frac{1}{2}\left[\frac{\sin(m-n)x}{(m-n)}-\frac{\sin(m+n)x}{(m+n)}\right]+c$

14. $\frac{1}{2}\left[\frac{\sin(m+n)x}{(m+n)}+\frac{\sin(m-n)x}{(m-n)}\right]+c$

15. $\frac{1}{4}\left[\frac{\cos 6x}{6}-\frac{1}{4}\cos 4x-\frac{\cos 2x}{2}\right]+c$

16. $(x+a)\cos a-\sin a\log|\sin(a+x)|+c$

II. 1. $\frac{2}{9}(3x-2)^{\frac{3}{2}}+c$

2. $\frac{1}{7}\log|7x+3|+c$

3. $\frac{(\log(1+x))^2}{2}+c$

4. $\frac{1}{12}(3x^2-4)^2+c$

5. $\frac{2}{5}\sqrt{1+5x}+c$

6. $-\frac{(1-2x^3)^2}{12}+c$

7. $-\frac{1}{2(1+\tan x)^2}+c$

8. $-\frac{1}{4}\cos x^4+c$

9. $-\frac{1}{1+\sin x}+c$

10. $\frac{3}{4}\sin^{\frac{4}{3}}x+c$

11. $e^{x^2}+c$

12. $x+c$

13. $\frac{1}{3}\text{Sin}^{-1}x^3+c$

14. $\frac{1}{2}\text{Tan}^{-1}x^4+c$

15. $\frac{1}{9}\text{Tan}^{-1}x^9+c$

16. $\tan(xe^x)+c$

17. $\frac{1}{4b(a+b\cot x)^4}+c$

18. $-\cos e^x+c$

19. $-\cos(\log x)+c$

20. $\log(|\log x|)+c$

21. $\frac{(1+\log x)^{n+1}}{n+1}+c$

22. $\sin(\log x)+c$

23. $2\sin\sqrt{x}+c$

24. $\log|x^2+x+1|+c$

25. $\frac{a}{bn}\log|bx^n+c|+c_1$

26. $\log|\log(\log x)|+c$

27. $\log|\sinh x|+c$

28. $\frac{1}{2}\text{Sin}^{-1}(2x)+c$

29. $\sinh^{-1}\left(\frac{x}{5}\right)+c$

30. $2\text{Tan}^{-1}(\sqrt{x+2})+c$

31. $-\frac{1}{1+\tan x}+c$

32. $\frac{1}{\sqrt{2}}\text{Tan}^{-1}\left(\frac{x^2-1}{\sqrt{2}x}\right)+c$

33. $\frac{1}{2}\log|1+2\tan x|+c$

34. $(\sin x+\cos x)+c$

35. $\sqrt{2}\sin x+c$

36. $x+c$

37. $-\frac{2}{b^2}\left[\log|a+b\cos x|+\frac{a}{(a+b\cos x)}\right]+c$

38. $-\frac{1}{2(\sec x+\tan x)^2}+c$

39. $\frac{1}{ab}\text{Tan}^{-1}\left(\frac{a}{b}\tan x\right)+c$

40. $\frac{1}{\sin(b-a)}\log\left|\frac{\sin(x-b)}{\sin(x-a)}\right|+c$

41. $\frac{1}{\sin(a-b)}\log\left|\frac{\sec(x-b)}{\sec(x-a)}\right|+c$

III. 1. $\frac{1}{b-a}\log|a\cos^2 x+b\sin^2 x|+c$

2. $\log|\cos x+\sin x|+c$

3. $\log|\sin(\log x)|+c$

4. $\log|\sin e^x|+c$

5. $\log\left|\tan\left(\frac{\pi}{4}+\frac{1}{2}\tan x\right)\right|+c$

6. $\frac{2}{3}(\sin x)^{\frac{3}{2}}+c$

7. $\frac{\tan^5 x}{5}+c$

8. $2\sqrt{x^2+3x-4}+c$

9. $-\frac{2}{3}(\cot x)^{\frac{3}{2}}+c$

10. $\frac{1}{2}(\log(\sec x+\tan x))^2+c$

11. $\frac{1}{12}(\cos 3x-9\cos x)+c$

12. $\frac{1}{12}(\sin 3x+9\sin x)+c$

13. $\frac{1}{6}(\sin 3x+3\sin x)+c$

14. $\frac{\sin 4x}{8}+\frac{\sin 2x}{4}+c$

15. $\frac{1}{32}(12x+\sin 4x+8\sin 2x)+c$

16. $\frac{1}{40}(4x+3)^{\frac{5}{2}}-\frac{1}{8}(4x+3)^{\frac{3}{2}}+c$

17. $\frac{1}{c}\text{Sin}^{-1}\left(\frac{b+cx}{a}\right)+c_1$

18. $\frac{1}{ac}\text{Tan}^{-1}\left(\frac{b+cx}{a}\right)+c_1$

19. $x-\log(1+e^x)+c$

20. $\frac{1}{b^3}\left[(a+bx)-2a\log|a+bx|-\frac{a^2}{a+bx}\right]+c$

21. $\frac{4}{3}(1-x)^{\frac{3}{2}}-\frac{2}{5}(1-x)^{\frac{5}{2}}-2\sqrt{1-x}+c$

Exercise 6(c)

I. 1. $x\tan x - \log|\sec x| + c$

2. $e^x \operatorname{Tan}^{-1} x + c$

3. $-\frac{1}{x}\log x - \frac{1}{x} + c$

4. $x(\log x)^2 - 2x\log x + 2x + c$

5. $e^x \sec x + c$

6. $\frac{e^x}{2}(\sin x + \cos x) + c$

7. $e^x \sin x + c$

8. $e^x \log|\sec x| + c$

II. 1. $\frac{x^{n+1}}{(n+1)^2}[(n+1)\log x - 1] + c$

2. $x\log(1+x^2) - 2x + 2\operatorname{Tan}^{-1} x + c$

3. $\frac{2}{3}x^{\frac{3}{2}}\log x - \frac{4}{9}x^{\frac{3}{2}} + c$

4. $2\sqrt{x}\,e^{\sqrt{x}} - 2e^{\sqrt{x}} + c$

5. $x^2 \sin x + 2x\cos x - 2\sin x + c$

6. $\frac{x^2}{4} - \frac{1}{4}x\sin 2x - \frac{1}{8}\cos 2x + c$

7. $\frac{x^2}{4} + \frac{1}{4}x\sin 2x + \frac{1}{8}\cos 2x + c$

8. $2(\sqrt{x}\sin\sqrt{x} + \cos\sqrt{x}) + c$

9. $x\frac{\tan 2x}{2} - \frac{1}{4}\log|\sec 2x| + c$

10. $-x\cot x + \log|\sin x| - \frac{x^2}{2} + c$

11. $e^x \tan x + c$

12. $e^x \log x + c$

13. $\frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx) + c$

14. $\frac{1}{x+1}e^x + c$

15. $\frac{1}{2a^3}\left[\operatorname{Tan}^{-1}\left(\frac{x}{a}\right) + \frac{1}{2}\sin 2\left(\operatorname{Tan}^{-1}\left(\frac{x}{a}\right)\right)\right] + c$

16. $e^x \log(e^{2x} + 5e^x + 6) + 2\log(e^x + 2) + 3\log(e^x + 3) - 2e^x + c$

17. $\frac{e^x}{x+3} + c$

18. $\frac{x}{2}\cos(\log x) + \sin(\log x)] + c$

III. 1. $\left(\frac{x^2+1}{2}\right)\text{Tan}^{-1}x-\frac{1}{2}x+c$

2. $\frac{x^3}{3}\text{Tan}^{-1}x-\frac{x^2}{6}+\frac{1}{6}\log(1+x^2)+c$

3. $-\frac{\text{Tan}^{-1}x}{x}+\log|x|-\frac{1}{2}\log(1+x^2)+c$

4. $\frac{x^2}{2}\text{Cos}^{-1}x+\frac{1}{4}\text{Sin}^{-1}x-\frac{1}{4}x\sqrt{1-x^2}+c$

5. $\frac{x^3}{3}\text{Sin}^{-1}x-\frac{1}{9}(1-x^2)^{\frac{3}{2}}+\frac{1}{3}\sqrt{1-x^2}+c$

6. $\frac{1}{2}\left[(x^2-1)\log(1+x)-\frac{x^2}{2}+x\right]+c$

7. $2(\sin\sqrt{x}-\sqrt{x}\cos\sqrt{x})+c$

8. $\frac{e^{ax}}{a^2+b^2}[a\sin(bx+c)-b\cos(bx+c)]+c_1$

9. $\frac{2a^x\sin 2x+(\log a)a^x\cos 2x}{(\log a)^2+4}+c$

10. $3x\text{Tan}^{-1}x-\frac{3}{2}\log(1+x^2)+c$

11. $x\sinh^{-1}x-\sqrt{x^2+1}+c$

12. $x\cosh^{-1}x-\sqrt{x^2-1}+c$

13. $x\tanh^{-1}x+\frac{1}{2}\log(1-x^2)+c$

Exercise 6(d)

I. 1. $\frac{1}{\sqrt{3}}\text{Sin}^{-1}\left(\frac{3x-1}{2}\right)+c$

2. $-\text{Sin}^{-1}\left(\frac{\cos\theta}{\sqrt{2}}\right)+c$

3. $\text{Tan}^{-1}(\sin x+2)+c$

4. $\frac{1}{\sqrt{2}}\text{Tan}^{-1}\left(\frac{\tan x}{\sqrt{2}}\right)+c$

5. $\frac{1}{\sqrt{6}}\text{Tan}^{-1}\left(\sqrt{\frac{2}{3}}\tan x\right)+c$

6. $\frac{1}{2}x+\frac{1}{2}\log|\sin x+\cos x|+c$

7. $\frac{1}{2}x+\frac{1}{2}\log|\sin x-\cos x|+c$

II. 1. $\frac{2x-3}{4}\sqrt{1+3x-x^2}+\frac{13}{8}\ \text{Sin}^{-1}\frac{2x-3}{\sqrt{13}}+c$

2. $x+\log|4\sin x+5\cos x|+c$

3. $\frac{23}{41}x-\frac{2}{41}\log|4\cos x+5\sin x|+c$

4. $\log|1+\tan\frac{x}{2}|+c$

5. $\frac{2}{\sqrt{11}}\text{Tan}^{-1}\left(\frac{6x+1}{\sqrt{11}}\right)+c$

6. $\frac{1}{\sqrt{2}}\text{Sin}^{-1}\frac{\sqrt{2}(x-1)}{\sqrt{7}}+c$

III. 1. $\sqrt{x^2-x+1}+\frac{3}{2}\sinh^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)+c$

2. $-(6-2x^2+x)^{\frac{3}{2}}+\frac{637}{32\sqrt{2}}\ \text{Sin}^{-1}\left(\frac{4x-1}{7}\right)+\frac{13}{16}(4x-1)\sqrt{6-2x^2+x}+c$

3. $\frac{1}{3}\log\left|\frac{2\tan\frac{x}{2}+1}{2(\tan\frac{x}{2}+2)}\right|+c$

4. $\frac{1}{2\sqrt{5}}\log\left|\frac{\sqrt{5}\tan x-1}{\sqrt{5}\tan x+1}\right|+c$

5. $-\frac{1}{3}(1+x-x^2)^{\frac{3}{2}}+\frac{5}{16}\text{Sin}^{-1}\left(\frac{2x-1}{\sqrt{5}}\right)+\frac{2x-1}{8}\sqrt{1+x-x^2}+c$

6. $-\frac{1}{2}\sqrt{\frac{3-x}{1+x}}+c$

7. $\frac{1}{5}\log\left|\frac{2\tan\frac{x}{2}+1}{2\tan\frac{x}{2}-4}\right|+c$

8. $\frac{1}{2}\log\left|\frac{\sqrt{3}\tan\frac{x}{2}+1}{\sqrt{3}\tan\frac{x}{2}-3}\right|+c$

9. $\frac{1}{3}\text{Tan}^{-1}\left(\frac{\tan x}{3}\right)+c$

10. $\frac{1}{25}\log|3\sin x+4\cos x+5|+\frac{18}{25}x-\frac{4}{5(\tan\frac{x}{2}+3)}+c$

11. $\sqrt{7x-x^2-10}+\frac{3}{2}\text{Sin}^{-1}\left(\frac{2x-7}{3}\right)+c$

12. $\text{Sin}^{-1}x-\sqrt{1-x^2}+c$

13. $\frac{1}{2}\sqrt{\frac{3+x}{1-x}}+c$

14. $2\text{Tan}^{-1}(\sqrt{x+1})+c$

15. $\frac{1}{\sqrt{2}}\log\left|\frac{\sqrt{2x+4}-1}{\sqrt{2x+4}+1}\right|+c$

16. $\frac{2(\sqrt{x}-1)}{\sqrt{1-x}}+c$

17. $2\sqrt{\frac{2x+1}{x+1}}+c$

18. $2\sqrt{e^x-4}-4\text{Tan}^{-1}\left(\frac{\sqrt{e^x-4}}{2}\right)+c$

19. $2\text{Sin}^{-1}\left(\sqrt{2}\sin\frac{x}{2}\right)+c$

20. $\frac{1}{2\sqrt{2}}\text{Tan}^{-1}\left(\frac{x^2-1}{x\sqrt{2}}\right)+\frac{1}{4\sqrt{2}}\log\left|\frac{x^2+1+\sqrt{2}x}{x^2+1-\sqrt{2}x}\right|+c$

Exercise 6(e)

I. 1. $2\log|x-3|-\log|x-2|+c$

2. $\log|x+1|+\frac{4}{x+2}+c$

3. $2\log|x-1|-\log(x^2+1)-\text{Tan}^{-1}x+c$

4. $\frac{1}{b^2-a^2}\left[\frac{1}{a}\mathrm{Tan}^{-1}\left(\frac{x}{a}\right)-\frac{1}{b}\mathrm{Tan}^{-1}\left(\frac{x}{b}\right)\right]+c$

5. $\log\left(\frac{e^x+1}{e^x}\right)-e^{-x}+c$

6. $\log\left|\frac{x+1}{x+2}\right|+c$

7. $\log\left|\frac{e^x-1}{e^x}\right|+c$

8. $-\frac{1}{5}\log|1-x|+\frac{1}{10}\log(4+x^2)+\frac{1}{10}\mathrm{Tan}^{-1}\frac{x}{2}+c$

9. $-\frac{3}{2}\log|x|-\frac{1}{6}\log|x+2|+\frac{5}{3}\log|x-1|+c$

II. 1. $\log\left|\frac{2x-1}{3x-1}\right|+c$

2. $\frac{1}{2}\log|x|-\log|x+1|+\frac{1}{2}\log|x+2|+c$

3. $-\frac{1}{6}\log|x-1|-\frac{8}{15}\log|x+2|+\frac{7}{10}\log|x-3|+c$

4. $2\log|x-1|-\frac{1}{x-1}-2\log|x+2|+c$

III. 1. $\frac{1}{(a-b)(a-c)}\log|x-a|+\frac{1}{(b-a)(b-c)}\log|x-b|+\frac{1}{(c-a)(c-b)}\log|x-c|+c_1$

2. $-\frac{3}{13}\log|x+3|+\frac{3}{26}\log(x^2+4)+\frac{17}{26}\mathrm{Tan}^{-1}\left(\frac{x}{2}\right)+c$

3. $\frac{16}{25}\log|x+3|+\frac{34}{25}\log|x-2|-\frac{11}{5(x-2)}+c$

4. $\frac{1}{3}\log|x+1|-\frac{1}{6}\log|x^2-x+1|+\frac{1}{\sqrt{3}}\mathrm{Tan}^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)+c$

5. $\log\left|\frac{\cos x+1}{(\cos x+2)^2}\right|+c$

Exercise 6(f)

I. 1. $e^x(x^2-2x+3)+c$

2. $-\frac{e^{-3x}}{27}(9x^2+6x+2)+c$

3. $\frac{e^{ax}}{a^4}(a^3x^3-3a^2x^2+6ax-6)+c$

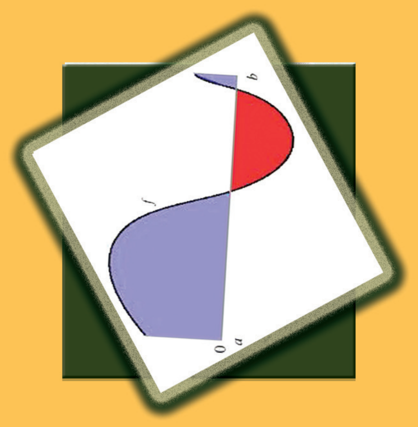
III. 1. $I_n=-\frac{\cot^{n-1}x}{n-1}-I_{n-2};\ -\frac{\cot^3 x}{3}+\cot x+x+c$

2. $-\frac{1}{n-1}\operatorname{cosec}^{n-2}x\cot x+\frac{n-2}{n-1}I_{n-2};$

$-\frac{\operatorname{cosec}^3 x\cot x}{4}-\frac{3}{8}\operatorname{cosec}x\cot x+\frac{3}{8}\log\left|\tan\frac{x}{2}\right|+c$

4. $-\frac{\sin^4 x\cos^5 x}{9}-\frac{4}{63}\sin^2 x\cos^5 x-\frac{8}{315}\cos^5 x+c$

5. $x[(\log|x|)^4-4(\log|x|)^3+12(\log|x|)^2-24(\log|x|)+24]+c.$



Chapter 7

Definite Integrals

"If only I had theorems then I should find the proofs easily enough"

- Riemann

Introduction

Calculus originated to solve mainly two geometric problems : finding the tangent line to a curve and finding the area of a region under a curve. The first was studied by a limit process known as differentiation (which we studied in Intermediate first year) and the second by another limit process - integration - which we study now.

We recall from elementary calculus that to find the area of the region under the graph of a positive and continuous function f defined on $[a, b]$, we subdivide the interval $[a, b]$ into a finite number of subintervals, say n, the k^{th} subinterval having length Δx_k, and we consider sums of the form $\sum_{k=1}^{n} f(t_k)\,\Delta x_k$, where t_k is some point in the k^{th} subinterval. Such a sum is an approximation to the area by means of the sum of the areas of rectangles. Suppose we make subdivisions finer and finer. It so happens that the sequence of the corresponding sums tends to a limit as $n \to \infty$. Thus, roughly speaking, this is Riemann's definition of the definite integeral $\int_a^b f(x)dx$. (A precise definition is given below).

Bernhard Riemann

(1826-1866)

Bernhard Riemann *was a German mathematician who made important contributions to analysis and differential geometry, some of them paving the way for the later development of general relativity. He was a student of C.F. Gauss.*

7.1 Definite integral as the limit of sum

We discussed in the earlier chapter that indefinite integration is an inverse process of differentiation. We recall that, if f is the derivative of F, then $\int f(x)\,dx = F(x) + c$, where c is a real constant. In this case, F is called a primitive of f.

Now to define the definite integral, we need the following :

7.1.1 Definition (Partition)

Let $a, b \in \mathbf{R}$ be such that $a < b$. Then, a finite set $P = \{x_0, x_1, ..., x_{i-1}, x_i, x_{i+1}, ..., x_n\}$ of elements of $[a, b]$ is said to be a ***partition*** *of $[a, b]$ if $a = x_0 < x_1 < ... < x_{i-1} < x_i < x_{i+1} < ... < x_{n-1} < x_n = b$.*

7.1.2 Definition (Norm)

If $P = \{x_0, x_1, ..., x_n\}$ is a partition of $[a, b]$, then the ***norm*** *of the partition P, denoted by $\|P\|$, is defined by $\|P\| = \max\{x_1 - x_0, x_2 - x_1, ..., x_n - x_{n-1}\}$.*
We denote the length of the subinterval $[x_{i-1}, x_i]$ by Δx_i so that $\Delta x_i = x_i - x_{i-1}$. We denote the set of all partitions of $[a, b]$ by $\mathscr{R}([a, b])$.

i.e., $\mathscr{R}([a, b]) = \{P : P \text{ is a partition of } [a, b]\}$.

7.1.3 Definition (Definite integral)

*Let $f : [a, b] \to \mathbf{R}$ be a bounded function (that is, there is a real number M such that $|f(x)| \le M$ for all x in $[a, b]$). Let $P = \{x_0, x_1, ..., x_{i-1}, x_i, x_{i+1}, ..., x_{n-1}, x_n\}$ be a partition of $[a, b]$, and let $t_i \in [x_{i-1}, x_i]$, for $i = 1, 2, ..., n$. A sum of the form $S(P, f) = \sum_{i=1}^{n} f(t_i)\Delta x_i$ is called a '****Riemann sum****' of f relative to P.*

*We say that f is 'Riemann integrable on $[a, b]$', if there exists a real number A such that $S(P, f)$ approaches A as $\|P\|$ approaches zero. In other words, given $\epsilon > 0$, there is a $\delta > 0$ such that $|S(P, f) - A| < \epsilon$ for any partition P of $[a, b]$ with $\|P\| < \delta$ irrespective of the choice of t_i in $[x_{i-1}, x_i]$. Such an A, if exists, is unique and is denote by $\int_a^b f(x)\,dx$. We call '$\int_a^b f(x)\,dx$', the '****definite integral****' of f from a to b, 'a' is called the '****lower limit****' and 'b' is called the '****upper limit****' of the integral.*

The function f in $\int_a^b f(x)\,dx$ is called the '***integrand***'. Here we observe that the numerical value of $\int_a^b f(x)\,dx$ depends on f and does not dependent on the symbol x. The letter "x" is a "dummy symbol" and may be replaced by any other convenient symbol.

Suppose that $\int_a^b f(x)\,dx$ exists on $[a, b]$. Then for every choice of $t_i \in [x_{i-1}, x_i]$

$$\sum_{i=1}^{n} (x_i - x_{i-1})\ f(t_i) = \text{S}(\text{P}, f) \to \int_a^b f(x)\,dx \ \text{ as } \|\text{P}\| \to 0.$$

i.e., the Riemann integral $\int_a^b f(x)\,dx$ is the limit of the sum $\text{S}(\text{P}, f) = \sum_{i=1}^{n} (x_i - x_{i-1})\ f(t_i)$ as $\|\text{P}\| \to 0$.

In this way, we can regard definite integral as the limit of a sum.

Hereafter, we briefly use the word 'integrable' for the phrase 'Riemann integrable'.

We assume the following theorems without proof for our further discussion.

7.1.4 Theorem : *If $f : [a, b] \to \mathbf{R}$ is continuous, then $\int_a^b f(x)\,dx$ exists.*

7.1.5 Theorem : *If $f : [a, b] \to \mathbf{R}$ is continuous, then there exist real numbers p and q in $[a, b]$ such that $f(p) \le f(x) \le f(q)$ for all x in $[a, b]$.*

7.2 Interpretation of definite integral as an area

Let $f : [a, b] \to [0, \infty)$ be continuous. Let $\text{P} = \{x_0, x_1, ..., x_n\}$ be a partition of $[a, b]$. Since f is continuous on $[a, b]$, for each $i \in \{1, 2, ..., n\}$, there exist p_i, q_i in $[x_{i-1}, x_i]$ such that

$f(p_i) \le f(t) \le f(q_i)$ for all $t \in [x_{i-1}, x_i]$. (Theorem 7.1.5).

Let $\text{S}_1(\text{P}, f) = \sum_{i=1}^{n} f(p_i)(x_i - x_{i-1})$

and $\text{S}_2(\text{P}, f) = \sum_{i=1}^{n} f(q_i)(x_i - x_{i-1})$.

Here $f(p_i)\ (x_i - x_{i-1})$ is the area of the rectangle bounded by the lines $x = x_{i-1}$, $x = x_i$, $y = 0$ and $y = f(p_i)$. Similarly, $f(q_i)(x_i - x_{i-1})$ is the area of the rectangle bounded by the lines $x = x_{i-1}$, $x = x_i$, $y = 0$ and $y = f(q_i)$ (See Fig. 7.1).

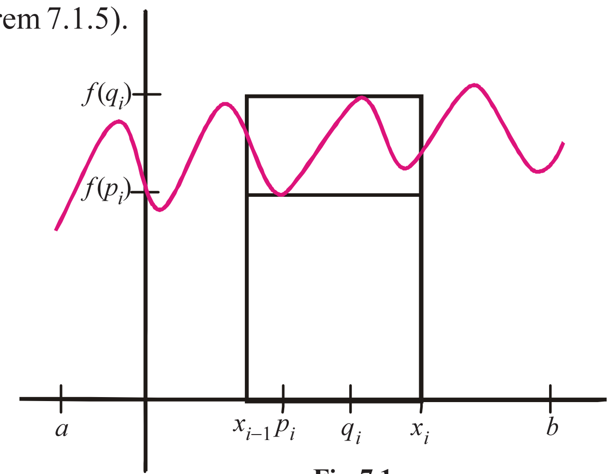


Fig. 7.1

Since $f(p_i) \le f(t) \le f(q_i)$ for all $t \in [x_{i-1}, x_i]$, it follows that the curve $y = f(x)$, $(x_{i-1} \le x \le x_i)$ lies between the lines $y = f(p_i)$ and $y = f(q_i)$. Hence the area below the curve $y = f(x)$, $(x_{i-1} \le x \le x_i)$, (that is, the area bounded by the lines $x = x_{i-1}$, $x = x_i$, $y = 0$ and the curve $y = f(x)$) lies between $f(p_i)(x_i - x_{i-1})$ and $f(q_i)(x_i - x_{i-1})$. Hence

$$\sum_{i=1}^{n} f(p_i)\Delta x_i \le \sum_{i=1}^{n} (\text{area below the curve } y = f(x) \text{ over } [x_{i-1}, x_i]) \le \sum_{i=1}^{n} f(q_i)\Delta x_i .$$

That is,

$S_1(P, f) \le$ area below the curve $y = f(x)$ over the interval $[a, b] \le S_2(P, f)$. ... (1)

Since f is continuous on $[a, b]$, $\int_a^b f(x)\,dx$ exists (Theorem 7.1.4), and under the usual notation, $S(P, f) \to \int_a^b f(x)\,dx$ as $\|P\| \to 0$. Hence, both $S_1(P, f)$ and $S_2(P, f)$ approach $\int_a^b f(x)\,dx$ as $\|P\| \to 0$.

Since the inequality (1) is true for all partitions P of $[a, b]$, and both $S_1(P, f)$ and $S_2(P, f)$ tend to $\int_a^b f(x)\,dx$ as $\|P\| \to 0$, it follows that

$$\int_a^b f(x)\,dx = \text{area below the curve } y = f(x), a \le x \le b, \text{ the ordinates } x = a, x = b \text{ and the X-axis.}$$

Thus definite integral $\int_a^b f(x)\,dx$ of a nonnegative function f on $[a, b]$ can be interpreted as the area of the region bounded by the curve $y = f(x)$ and the lines $x = a$, $x = b$ and the X-axis.

Since $t_i \in [x_{i-1}, x_i]$, a choice of t_i is x_{i-1} in which case we get the sum $\sum_{i=1}^{n} f(x_{i-1})(x_i - x_{i-1})$. Another choice of t_i is x_i in which case the sum is $\sum_{i=1}^{n} f(x_i)(x_i - x_{i-1})$. Therefore $\int_a^b f(x)\,dx$ can be regarded as the limit of the sum $\sum_{i=1}^{n} f(x_{i-1})(x_i - x_{i-1})$ or $\sum_{i=1}^{n} f(x_i)(x_i - x_{i-1})$.

7.2.1 Note : If f is continuous on [0, 1] and $P = \left\{0, \frac{1}{n}, \frac{2}{n}, ..., \frac{n-1}{n}, 1\right\}$ is a partition of [0, 1] into n subintervals each of length $\frac{1}{n}$, then from the above discussion, it follows that

$$\int_0^1 f(x)\,dx = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} f\left(\frac{i}{n}\right). \qquad ...(2)$$

More generally, if f is continuous on $[0, p]$ where p is a positive integer then

$$\int_0^p f(x)\,dx = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{np} f\left(\frac{i}{n}\right). \qquad \ldots(3)$$

7.2.2 Example : Let us find $\int_a^b f(x)\,dx$, where $f(x)=x$ in $[a, b]$ as the limit of a sum.

We define f: $[a, b] \to \mathbf{R}$ by $f(x)=x$, $x \in [a, b]$.

Let $P = \left\{a, a+\frac{b-a}{n}, a+\frac{2(b-a)}{n}, \ldots, a+\frac{n(b-a)}{n} = b\right\}$ be a partition of $[a, b]$ into n subintervals so that $\|P\| = \frac{b-a}{n}$. Here $x_0 = a$, $x_i = a + \frac{i(b-a)}{n}$, $i = 1, 2, \ldots, n$. We take $t_i = \frac{x_{i-1}+x_i}{2}$. Hence

$$S(P, f) = \sum_{i=1}^{n} f(t_i)\Delta x_i = \sum_{i=1}^{n}\left(\frac{x_{i-1}+x_i}{2}\right)(x_i - x_{i-1}) = \sum_{i=1}^{n}\frac{x_i^2 - x_{i-1}^2}{2} = \frac{1}{2}(b^2 - a^2).$$

Hence

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(t_i)\Delta x_i = \frac{1}{2}(b^2 - a^2).$$

7.2.3 Example : We find $\int_0^2 (x^2+1)\,dx$ as the limit of a sum.

Here we use the formula (3) of Note 7.2.1 with $p = 2$ and $f(x) = x^2 + 1, x \in [0, 2]$. We observe that f is continuous on $[0, 2]$. Now

$$\int_0^2 (x^2+1)\,dx = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{np} f\left(\frac{i}{n}\right)$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{2n}\left[\left(\frac{i}{n}\right)^2 + 1\right]$$

$$= \lim_{n\to\infty}\frac{1}{n}\left[\left(\frac{1}{n}\right)^2 + 1 + \left(\frac{2}{n}\right)^2 + 1 + \cdots + \left(\frac{2n}{n}\right)^2 + 1\right]$$

$$= \lim_{n\to\infty}\frac{1}{n}\left[\left(\frac{1}{n}\right)^2 + \left(\frac{2}{n}\right)^2 + \cdots + \left(\frac{2n}{n}\right)^2 + 2n\right]$$

$$= \lim_{n\to\infty}\frac{1}{n}\left[\frac{1^2 + 2^2 + \cdots + (2n)^2}{n^2} + 2n\right]$$

$$= \lim_{n\to\infty} \frac{1}{n^3}(1^2 + 2^2 + \cdots + (2n)^2) + 2$$

$$= \lim_{n\to\infty} \frac{1}{n^3}\left[\frac{2n(2n+1)(4n+1)}{6}\right] + 2$$

$$= \lim_{n\to\infty} \frac{1}{3}\left(2+\frac{1}{n}\right)\left(4+\frac{1}{n}\right) + 2$$

$$= \frac{8}{3} + 2 = \frac{14}{3}.$$

7.2.4 Example : We evaluate $\int_0^2 e^x\, dx$ as the limit of a sum.

On using the formula (3) of Note 7.2.1 with $p=2$ and $f(x) = e^x, x \in [0,2]$ we have

$$\int_0^2 e^x\, dx = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{np} f\left(\frac{i}{n}\right) = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{2n} e^{\frac{i}{n}}$$

$$= \lim_{n\to\infty} \frac{1}{n}\left[e^{\frac{1}{n}} + e^{\frac{2}{n}} + \dots + e^{\frac{2n}{n}}\right] = \lim_{n\to\infty} \frac{e^{\frac{1}{n}}}{n}\left[1 + e^{\frac{1}{n}} + \dots + e^{\frac{2n-1}{n}}\right]$$

$$= \lim_{n\to\infty} \frac{e^{\frac{1}{n}}}{n}\left[\frac{(e^{\frac{1}{n}})^{2n} - 1}{e^{\frac{1}{n}} - 1}\right] = \lim_{n\to\infty} \frac{e^{\frac{1}{n}}}{n}\left[\frac{e^2 - 1}{e^{\frac{1}{n}} - 1}\right]$$

$$= \frac{\lim_{n\to\infty} e^{1/n}(e^2 - 1)}{\lim_{n\to\infty}\left(\frac{e^{1/n} - 1}{1/n}\right)}$$

$$= e^2 - 1. \qquad \left(\text{since } \lim_{n\to\infty} e^{\frac{1}{n}} = 1 \text{ and } \lim_{n\to\infty}\left(\frac{e^{1/n} - 1}{1/n}\right) = 1\right)$$

7.2.5 Example : Let us define $f : [0, 1] \to \mathbf{R}$ by

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational}. \end{cases}$$

We show that f is not integrable on $[0, 1]$.

Let $P = \{x_0, x_1, ..., x_n\}$ be a partition of $[0, 1]$. We know that between any two real numbers there lies a rational number, and also an irrational number. For each $i = 1, 2, ..., n$, choose t_i, s_i in $[x_{i-1}, x_i]$ such that t_i is a rational number and s_i is an irrational number.

Let $S_1(P, f) = \sum_{i=1}^{n} f(t_i)\Delta x_i$ and $S_2(P, f) = \sum_{i=1}^{n} f(s_i)\Delta x_i$. Since $f(t_i) = 1$ and $f(s_i) = 0$, we have $S_1(P, f) = 1$ and $S_2(P, f) = 0$.

Hence $S_1(P, f) \to 1$ as $\|P\| \to 0$ and $S_2(P, f) \to 0$ as $\|P\| \to 0$.

Hence $\lim_{\|P\| \to 0} S(P, f)$ does not exist. Thus f is not Riemann integrable on $[0, 1]$.

7.2.6 Definite integral as an area function

Let $f : [a, b] \to \mathbf{R}$ be continuous and $f(x) \geq 0$ for all x in $[a, b]$. In this section, we define $\int_a^b f(x)\, dx$ as the area of the region bounded by the curve $y = f(x)$, the ordinates $x = a$ and $x = b$ and X-axis.

Let x be a point in $[a, b]$. Then $\int_a^x f(t)\, dt$ represents the area of the shaded region in Fig. 7.2. The area of this shaded region depends upon the value of x so that this shaded region is a function of x. We denote this function by $A(x)$ and call A as 'Area function', and it is given by

$$A(x) = \int_a^x f(t)\, dt\,.$$

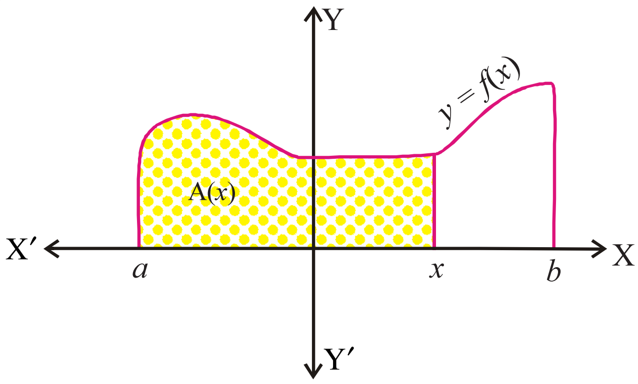


Fig. 7.2

We state without proof the following theorem which gives an important property of Area function.

7.2.7 Theorem (First Fundamental theorem of integral calculus) : Let f be integrable on $[a, b]$. We write $A(x) = \int_a^x f(t)\, dt$, $x \in [a, b]$. Then A is continuous on $[a, b]$. If f is continuous on $[a, b]$ then A is differentiable in $[a, b]$. Further, $A'(x) = f(x)$ for all $x \in [a, b]$.

Exercise 7(a)

I. Evaluate the following integrals as limit of a sum

1. $\int_0^5 (x + 1)\, dx$
2. $\int_0^4 x^2\, dx$

II. Evaluate the following integrals as limit of a sum

1. $\int_0^4 (x+e^{2x})\, dx$ 2. $\int_0^1 (x-x^2)\, dx$

7.3 The Fundamental Theorem of Integral Calculus

In the evaluation of definite integral, the following 'Fundamental Theorem of Integral Calculus' is useful. This theorem is also known as 'Second Fundamental Theorem of Integral Calculus'.

This important theorem is stated without proof. You will learn its proof in higher classes.

7.3.1 Theorem : *If f is integrable on* $[a, b]$ *and if there is a differentiable function F on* $[a, b]$ *such that* $F' = f$, *then* $\int_a^b f(x)\, dx = F(b) - F(a)$.

7.3.2 Note : We write $[F(x)]_a^b$ for $F(b) - F(a)$. $[F(x)]_a^b$ is not dependent on x. Also, we write $[F(x)]_a^b = -[F(x)]_b^a$.

7.3.3 Solved Problems

1. Problem : *Evaluate* $\int_1^2 x^5 dx$.

Solution : $f(x) = x^5$ is continuous on $[1, 2]$ and hence integrable on $[1, 2]$ (Theorem 7.1.4). Also, $F(x) = \frac{x^6}{6}$ is a primitive of f on $[1, 2]$. Hence, from the Fundamental theorem of integral Calculus (Theorem 7.3.1), we have

$$\int_1^2 x^5\, dx = \int_1^2 f(x)\, dx = F(2) - F(1) = \frac{2^6}{6} - \frac{1}{6} = \frac{63}{6} = \frac{21}{2}.$$

We note that in the definite integrals that we deal with hereafter in this book, the integrands are invariably continuous functions on their domains of integration and hence are integrable. So to evaluate $\int_a^b f(x)\, dx$ for a given function f on $[a, b]$, we find a primitive, say F, of f on $[a, b]$ (i.e., we find the indefinite integral $\int f(x)\, dx$) and find $F(b) - F(a)$ which is equal to $\int_a^b f(x)\, dx$.

2. Problem : *Evaluate* $\int_0^{\pi} \sin x \, dx$.

Solution : $\int_0^{\pi} \sin x \, dx = [-\cos x]_0^{\pi}$ (since $-\cos x$ is a primitive of $\sin x$)

$= -\cos \pi - (-\cos 0)$

$= -(-1) - (-1) = 2.$

3. Problem : *Evaluate* $\int_0^{a} \frac{dx}{x^2 + a^2}$.

Solution : $\int_0^{a} \frac{dx}{x^2 + a^2} = \left[\frac{1}{a} \operatorname{Tan}^{-1}\left(\frac{x}{a}\right)\right]_0^{a}$

$= \frac{1}{a}\left[\operatorname{Tan}^{-1}(1) - \operatorname{Tan}^{-1}(0)\right] = \frac{1}{a}\left(\frac{\pi}{4} - 0\right) = \frac{\pi}{4a}.$

7.4 Properties

We now discuss certain properties of definite integrals.

7.4.1 Definition

Let $f : [a, b] \to \mathbf{R}$ *be integrable on* $[a, b]$. *Then, we define* $\int_b^a f(x)\, dx$ *as the negative of* $\int_a^b f(x)\, dx$, *and, for any c in* $[a, b]$, $\int_c^c f(x)\, dx$ *as zero. Thus*

$$\int_b^a f(x)\, dx = -\int_a^b f(x)\, dx, \quad \int_a^a f(x)\, dx = 0 \quad and \quad \int_b^b f(x)\, dx = 0.$$

We state without proof, Theorems 7.4.2 to 7.4.5 which will be used in the subsequent development of the theory and in solving problems.

7.4.2 Theorem : *Suppose that f and g are integrable on* $[a, b]$. *Then*

(i) $f + g$ *is integrable on* $[a, b]$ *and*

$$\int_a^b (f + g)(x)\, dx = \int_a^b f(x)\, dx + \int_a^b g(x)\, dx .$$

(ii) *for any* $\alpha \in$ **R**, αf *is integrable on* $[a, b]$ *and*

$$\int_a^b (\alpha f)(x)\, dx = \alpha \int_a^b f(x)\, dx.$$

7.4.3 Theorem : Let $f : [a, b] \to \mathbf{R}$ *be bounded. Let* $c \in (a, b)$. *Then* f *is integrable on* $[a, b]$ *if and only if it is integrable on* $[a, c]$ *as well as on* $[c, b]$ *and, in this case,*

$$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx.$$

7.4.4 Theorem : *If* $f : [a, b] \to \boldsymbol{R}$ *is continuous, then* $f([a, b])$ *is a closed and bounded interval in* **R**.

7.4.5 Theorem (Method of Substitution) : *Let* $g : [c, d] \to \boldsymbol{R}$ *have continuous derivative on* $[c, d]$. Let $f : g([c, d]) \to \boldsymbol{R}$ *be continuous. Then* $(fog)\, g'$ *is integrable on* $[c, d]$ *and*

$$\int_{g(c)}^{g(d)} f(t)\, dt = \int_c^d f(g(x))\, g'(x)\, dx.$$

Proof : Since g is differentiable in $[c, d]$, it is continuous therein. Hence by Theorem 7.4.4, $g([c, d])$ is a closed and bounded interval of the real line, say, $[\alpha, \beta]$. Define F on $[\alpha, \beta]$ as $F(t) = \int_a^t f(s)\, ds$ for all $t \in [\alpha, \beta]$. Then F is well defined and continuous. Further, it is differentiable on $[\alpha, \beta]$, and $F'(t) = f(t)$ for all t in $[\alpha, \beta]$ (Theorem 7.2.7).

Since F is differentiable in $[\alpha, \beta]$, g is differentiable in $[c, d]$ and $g([c, d]) = [\alpha, \beta]$, it follows that Fog is a differentiable function on $[c, d]$ and

$$(Fog)' = (F'og)\, g' = (fog)\, g'.$$

Hence Fog is a primitive of $(fog)\, g'$ on $[c, d]$. Since f is continuous on $[\alpha, \beta]$ and g' is continuous on $[c, d]$ it follows that $(fog)\, g'$ is continuous on $[c, d]$. Hence $(fog)\, g'$ is integrable on $[c, d]$ (Theorem 7.1.4). Now, from the fundamental theorem of integral calculus (Theorem 7.3.1), it follows that

$$\int_c^d ((fog)g')(x)\, dx = (Fog)(d) - (Fog)(c).$$

That is,

$$\int_c^d f(g(x))\, g'(x)\, dx = F(g(d)) - F(g(c))$$

$$= \int_\alpha^{g(d)} f(t)\, dt - \int_\alpha^{g(c)} \alpha(t)\, dt$$

$$= \int_{g(c)}^{g(d)} f(t)\, dt \quad \text{(by Definition 7.4.1 and Theorem 7.4.3).}$$

7.4.6 Note : Theorem 7.4.5 remains valid if the continuity of f on $g([c, d])$ is replaced by the integrability of f on $g([c, d])$. The proof of the theorem in this general form is beyond the scope of this book.

7.4.7 Solved Problems

1. Problem : *Evaluate* $\int_1^4 x\sqrt{x^2-1}\,dx$.

Solution : Define $g : [1, 4] \to \mathbf{R}$ by $g(x) = x^2 - 1$. Clearly g is differentiable, g' is continuous in $[1, 4]$ and $g'(x) = 2x$. We have $g([1, 4]) = [0, 15]$. Define $f : [0, 15] \to \mathbf{R}$ by $f(t) = \sqrt{t}$. Then f is continuous on $[0, 15]$. Hence from Theorem 7.4.5, we have

$$\int_1^4 f(g(x))\ g'(x)\,dx = \int_{g(1)}^{g(4)} f(t)\ dt.$$

i.e., $$\int_1^4 2x\sqrt{x^2-1}\ dx = \int_0^{15} \sqrt{t}\ dt.$$

i.e., $$2\int_1^4 x\sqrt{x^2-1}\ dx = \left[\frac{2}{3}t^{\frac{3}{2}}\right]_0^{15} = \frac{2}{3}(15)^{\frac{3}{2}}.$$

Thus $$\int_1^4 x\sqrt{x^2-1}\ dx = \frac{1}{3}(15)^{\frac{3}{2}}.$$

The above problem is worked out in a simple way as follows. Usually we adopt this formal procedure in evaluating such integrals.

Put $x^2 - 1 = t$. Then $2x\,dx = dt$.

When $x = 1,\ t = 0$; when $x = 4,\ t = 15$.

Hence $$\int_1^4 x\sqrt{x^2-1}\ dx = \int_0^{15} \frac{\sqrt{t}}{2}\ dt = \left[\frac{1}{2}\cdot\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^{15}$$

$$= \frac{1}{3}\left[t^{\frac{3}{2}}\right]_0^{15} = \frac{1}{3}(15)^{\frac{3}{2}}.$$

When we integrate a function by expressing the given variable in terms of a new variable it is sometimes difficult to translate the result back into the original variable. But, when integrating between limits, we may avoid the process of restoring the original variable by changing the limits corresponding to the new variable. This process is illustrated in the following problem.

2. Problem : *Evaluate* $\int_0^2 \sqrt{4-x^2}\, dx.$

Solution : Define g on $\left[0, \frac{\pi}{2}\right]$ as $g(\theta) = 2\sin\theta$. Then g has continuous derivative, $g'(\theta) = 2\cos\theta \ \forall\ \theta \in \left[0, \frac{\pi}{2}\right]$, $g(0) = 0$, $g\left(\frac{\pi}{2}\right) = 2$ and $g\left(\left[0, \frac{\pi}{2}\right]\right) = [0, 2]$. Define f on $[0, 2]$ as $f(x) = \sqrt{4-x^2} \ \forall\ x \in [0, 2]$. Then f is continuous on $[0, 2]$. Hence by Theorem 7.4.5, it follows that $(fog)\, g'$ is integrable on $\left[0, \frac{\pi}{2}\right]$ and

$$\int_{g(0)}^{g(\frac{\pi}{2})} f(x)\, dx = \int_0^{\frac{\pi}{2}} f(g(\theta))\, g'(\theta)\, d\theta.$$

i.e.,
$$\int_0^2 \sqrt{4-x^2}\, dx = \int_0^{\frac{\pi}{2}} \sqrt{4-4\sin^2\theta}\ .\ 2\cos\theta\, d\theta. \qquad \dots (1)$$

We have
$$\sqrt{4-4\sin^2\theta} = \sqrt{4\cos^2\theta} = 2\cos\theta \ \forall\ \theta \in \left[0, \frac{\pi}{2}\right].$$

Hence
$$\int_0^{\frac{\pi}{2}} \sqrt{4-4\sin^2\theta}\ .\ 2\cos\theta\, d\theta = \int_0^{\frac{\pi}{2}} 4\cos^2\theta\, d\theta$$

$$= 4\int_0^{\frac{\pi}{2}} \left(\frac{1+\cos 2\theta}{2}\right) d\theta$$

$$= 2\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{2}}$$

$$= 2\cdot\frac{\pi}{2} = \pi.$$

Hence, from (1),
$$\int_0^2 \sqrt{4-x^2}\, dx = \pi.$$

The above problem is usually worked out in a formal way by substituting $x = 2\sin\theta \left(0 \le \theta \le \frac{\pi}{2}\right)$, replacing dx by $2\cos\theta\, d\theta$ in the given integral and by taking the lower and upper limits in the integral so obtained as 0 and $\frac{\pi}{2}$ respectively, since $x = 0$ when $\theta = 0$, $x = 2$ when $\theta = \frac{\pi}{2}$ and $x \in [0, 2]$ when $\theta \in \left[0, \frac{\pi}{2}\right]$.

3. Problem : *Evaluate* $\int_0^{16} \frac{x^{\frac{1}{4}}}{1 + x^{\frac{1}{2}}} dx.$

Solution : Put $x = t^4$. Then $dx = 4t^3\, dt$. Also, when $t = 0, x = 0$; when $t = 2, x = 16$; when $x \in [0, 16]$, $t \in [0, 2]$. Hence

$$\int_0^{16} \frac{x^{\frac{1}{4}}}{1 + x^{\frac{1}{2}}} dx = \int_0^2 \frac{t}{1+t^2} \cdot 4t^3 dt. = 4\int_0^2 \left(t^2 - 1 + \frac{1}{1+t^2}\right) dt$$

$$= 4\left[\frac{t^3}{3} - t + \mathrm{Tan}^{-1} t\right]_0^2 = 4\left[\frac{2^3}{3} - 2 + \mathrm{Tan}^{-1} 2\right]$$

$$= 4\left(\frac{2}{3} + \mathrm{Tan}^{-1} 2\right).$$

4. Problem : *Evaluate* $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin|x|\, dx.$

Solution : We have

$$\sin|x| = \begin{cases} \sin(-x), & \text{if } x < 0 \\ \sin x, & \text{if } x \ge 0. \end{cases}$$

Hence
$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin|x|\, dx = \int_{-\frac{\pi}{2}}^{0} \sin|x|\, dx + \int_0^{\frac{\pi}{2}} \sin|x| dx$$

$$= \int_{-\frac{\pi}{2}}^{0} \sin(-x)\, dx + \int_0^{\frac{\pi}{2}} \sin x\, dx = \int_{-\frac{\pi}{2}}^{0} -\sin x\, dx + \int_0^{\frac{\pi}{2}} \sin x\, dx$$

$$= [\cos x]_{-\frac{\pi}{2}}^{0} + [-\cos x]_0^{\frac{\pi}{2}} = 1 - 0 - 0 + 1 = 2.$$

We now prove the following theorems by using the method of substitution in its general form.

7.4.8 Theorem : *Let f be integrable on $[a, b]$. Then the function h, defined on $[a, b]$ as $h(x) = f(a+b-x)$ for all x in $[a, b]$, is integrable on $[a, b]$ and*

$$\int_a^b h(x)\,dx = \int_a^b f(x)\,dx.$$

Proof : Define g on $[a, b]$ by $g(x) = a + b - x$. Then $h = fog$, $g'(x) = -1$, $g(a) = b$, $g(b) = a$ and $g([a, b]) = [a, b]$.

From Note 7.4.6, it follows that $(fog)\,g'$ is integrable on $[a, b]$ and

$$\int_a^h f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(t)\,dt.$$

i.e.,
$$\int_a^b f(a+b-x)\,(-1)\,dx = \int_b^a f(t)\,dt.$$

i.e.,
$$-\int_a^b f(a+b-x)\,dx = \int_b^a f(t)\,dt.$$

Hence
$$\int_a^b f(a+b-x)\,dx = -\int_b^a f(t)\,dt$$

$$= \int_a^b f(t)\,dt \quad \text{(by Definition 7.4.1).}$$

7.4.9 Corollary : *If f is integrable on $[0, a]$, then the function h defined on $[0, a]$ as $h(x) = f(a-x)$ for all x in $[0, a]$ is integrable on $[0, a]$ and*

$$\int_0^a f(a-x)\,dx = \int_0^a h(x)dx = \int_0^a f(x)dx.$$

Proof : Follows from Theorem 7.4.8 by replacing 'a' by '0' and 'b' by 'a'.

7.4.10 Theorem : *Let $f : [-a, a] \to \mathbf{R}$ be integrable on $[0, a]$. Suppose that f is either odd or even. Then f is integrable on $[-a, a]$ and*

$$\int_{-a}^a f(x)\,dx = \begin{cases} 0, & \text{if } f \text{ is odd} \\ 2\int_0^a f(x)\,dx, & \text{if } f \text{ is even.} \end{cases}$$

Proof : Define $g:[-a,0]\to[0,a]$ as $g(x)=-x$ for all x in $[-a,0]$. Then g has continuous derivative on $[-a,0]$ and $g'(x)=-1$ for all x in $[-a,0]$.

Case (i) : f is odd.

Then $$f(x) = -f(-x) = -(fog)(x) = ((fog)\,g')(x) \quad \forall\ x\in[-a,0].$$

Hence, from Note 7.4.6, it follows that f is integrable on $[-a,0]$ and

$$\int_{-a}^{0} f(x)\,dx = \int_{-a}^{0} ((fog)\,g')(x)\,dx$$

$$= \int_{g(-a)}^{g(0)} -f(-t)\,dt = \int_{a}^{0} f(t)\,dt = -\int_{0}^{a} f(t)\,dt.$$

Hence f is integrable on $[-a,a]$ and

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx$$

$$= -\int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(x)\,dx = 0.$$

Case (ii) : f is even.

Then $f(x)=f(-x)=(fog)(x) = -((fog)g')(x)$ for all x in $[-a,0]$. Hence, from Note 7.4.6, it follows that f is integrable on $[-a,0]$ and

$$\int_{-a}^{0} f(x)dx = -\int_{-a}^{0} ((fog)\,g')(x)\,dx$$

$$= \int_{g(-a)}^{g(0)} -f(t)dt = -\int_{-a}^{0} f(t)\,dt = \int_{0}^{a} f(t)\,dt.$$

Hence f is integrable on $[-a,a]$ and

$$\int_{-a}^{a} f(x)dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx$$

$$= \int_{0}^{a} f(x)dx + \int_{0}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx.$$

7.4.11 Theorem : Let $f : [0, 2a] \rightarrow \mathbf{R}$ *be integrable on* $[0, a]$.

(i) *If* $f(2a - x) = f(x)$ *for all* x *in* $[a, 2a]$ *then* f *is integrable on* $[0, 2a]$ *and*

$$\int_0^{2a} f(x)dx = 2\int_0^a f(x)\,dx.$$

(ii) *If* $f(2a - x) = -f(x)$ *for all* x *in* $[a, 2a]$, *then* f *is integrable on* $[0, 2a]$ *and*

$$\int_0^{2a} f(x)dx = 0.$$

Proof : Define g on $[a, 2a]$ as $g(x) = (2a - x)$ for all x in $[a, 2a]$. Then g has continuous derivative, $g'(x) = -1$ for all x in $[a, 2a]$, $g(a) = a$, $g(2a) = 0$ and $g([a, 2a]) = [0, a]$. Hence, from Note 7.4.6, it follows that $((fog)\,g')$ is integrable on $[a, 2a]$ and

$$\int_{g(a)}^{g(2a)} f(t)\,dt = \int_a^{2a} ((fog)g')(x)\,dx. \qquad \text{... (1)}$$

We have $\qquad (fog)\,g' = -(fog)$ on $[a, 2a]$.

Since $(fog)\,g'$ is intergrable on $[a, 2a]$, $g' = -1$, it follows that fog is also integrable on $[a, 2a]$. We have $(fog)\,(x) = f(g(x)) = f(2a - x)$ for all x $[a, 2a]$.

(i) Suppose that $f(2a - x) = f(x)$ for all $x \in [a, 2a]$. Then

$$f = fog \text{ on } [a, 2a].$$

Hence f is integrable on $[a, 2a]$ and

$$\int_a^{2a} f(x)\,dx = \int_a^{2a} (fog)(x)\,dx = -\int_a^{2a} ((fog)g')\,(x)\,dx$$

$$= -\int_{g(a)}^{g(2a)} f(t)\,dt \qquad \text{(from (1))}$$

$$= -\int_a^0 f(t)\,dt = \int_0^a f(t)\,dt.$$

Hence f is integrable on $[0, 2a]$ and

$$\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_a^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx.$$

(ii) Suppose that $f(2a-x) = -f(x)\ \forall\ x \in [a, 2a]$. Then it can be shown as above that f is integrable on $[a, 2a]$ and that

$$\int_a^{2a} f(x)\,dx = -\int_a^a f(t)\,dt$$

Hence f is integrable on $[0, 2a]$ and

$$\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_a^{2a} f(x)\,dx = 0.$$

We state the following theorem without proof.

7.4.12 Theorem : *If f and g are integrable functions on $[a, b]$, then their product fg is integrable on $[a, b]$.*

7.4.13 Theorem (Integration by parts formula)

Let u and v be real valued differentiable functions on $[a, b]$ such that u' and v' are integrable on $[a, b]$. Then uv' and $u'v$ are integrable on $[a, b]$ and

$$\int_a^b (uv')(x)\,dx = \left[(uv)(x)\right]_a^b - \int_a^b (u'v)(x)\,dx$$

That is, $$\int_a^b u(x)\,v'(x)\,dx = u(b)v(b) - u(a)v(a) - \int_a^b u'(x)\,v(x)\,dx.$$

Proof : Since u and v are differentiable on $[a, b]$, they are continuous on $[a, b]$ and hence they are integrable on $[a, b]$. We know that the product of two integrable functions is integrable (Theorem 7.4.12). Since u, v, u' and v' are integrable on $[a, b]$, so are uv' and $u'v$.

Since both u and v are differentiable on $[a, b]$, uv is differentiable on $[a, b]$ and $(uv)' = uv' + u'v$.

Since uv' and $u'v$ are integrable on $[a, b]$, so is $uv' + u'v = (uv)'$ and

$$\int_a^b (uv)'(x)\,dx = \int_a^b (uv')(x)\,dx + \int_a^b (u'v)(x)\,dx. \qquad \text{... (1)}$$

From the fundamental theorem of integral calculus, (Theorem 7.3.1), it follows that

$$\int_a^b (uv)'(x)\,dx = (uv)(b) - (uv)(a). \qquad \text{... (2)}$$

From (1) and (2), we have

$$\int_a^b (uv')(x)\,dx = \left[u(x)\,v(x)\right]_a^b - \int_a^b (u'v)(x)\,dx.$$

7.4.14 Solved Problems

1. Problem : *Show that* $\int_0^{\pi/2} \sin^n x\, dx = \int_0^{\pi/2} \cos^n x\, dx.$

Solution : Suppose that $f(x) = \sin^n x$. Then

$$f\left(\frac{\pi}{2}-x\right) = \sin^n\left(\frac{\pi}{2}-x\right) = \cos^n x.$$

Now, $$\int_0^{\pi/2} \sin^n x\, dx = \int_0^{\pi/2} f(x)\, dx = \int_0^{\pi/2} f\left(\frac{\pi}{2}-x\right) dx$$

$$= \int_0^{\pi/2} \cos^n x\, dx. \qquad \text{(by Corollary 7.4.9)}$$

2. Problem : *Evaluate* $\int_0^{\pi/2} \frac{\cos^{\frac{5}{2}} x}{\sin^{\frac{5}{2}} x + \cos^{\frac{5}{2}} x}\, dx$

Solution : Let $f(x) = \frac{\cos^{\frac{5}{2}} x}{\sin^{\frac{5}{2}} x + \cos^{\frac{5}{2}} x}$ and $A = \int_0^{\pi/2} f(x)\, dx.$

By Corollary 7.4.9, we have $A = \int_0^{\pi/2} f(x)\, dx = \int_0^{\pi/2} f\left(\frac{\pi}{2}-x\right) dx$

$$= \int_0^{\pi/2} \frac{\cos^{\frac{5}{2}}\left(\frac{\pi}{2}-x\right)}{\sin^{\frac{5}{2}}\left(\frac{\pi}{2}-x\right) + \cos^{\frac{5}{2}}\left(\frac{\pi}{2}-x\right)}\, dx = \int_0^{\pi/2} \frac{\sin^{\frac{5}{2}} x}{\sin^{\frac{5}{2}} x + \cos^{\frac{5}{2}} x}\, dx.$$

Hence $$2A = \int_0^{\pi/2} \frac{\cos^{\frac{5}{2}} x}{\sin^{\frac{5}{2}} x + \cos^{\frac{5}{2}} x}\, dx + \int_0^{\pi/2} \frac{\sin^{\frac{5}{2}} x}{\sin^{\frac{5}{2}} x + \cos^{\frac{5}{2}} x}\, dx$$

$$= \int_0^{\pi/2} \frac{\cos^{\frac{5}{2}} x + \sin^{\frac{5}{2}} x}{\sin^{\frac{5}{2}} x + \cos^{\frac{5}{2}} x}\, dx = \int_0^{\pi/2} dx = \frac{\pi}{2}.$$

Hence $$A = \int_0^{\frac{\pi}{2}} \frac{\cos^{\frac{5}{2}} x}{\sin^{\frac{5}{2}} x + \cos^{\frac{5}{2}} x}\, dx = \frac{\pi}{4}.$$

3. Problem : *Show that* $\displaystyle\int_0^{\frac{\pi}{2}} \frac{x}{\sin x + \cos x}\, dx = \frac{\pi}{2\sqrt{2}} \log(\sqrt{2}+1).$

Solution : Let $$A = \int_0^{\frac{\pi}{2}} \frac{x}{\sin x + \cos x}\, dx.$$

Then $$A = \int_0^{\frac{\pi}{2}} \frac{\left(\frac{\pi}{2} - x\right)}{\sin\left(\frac{\pi}{2} - x\right) + \cos\left(\frac{\pi}{2} - x\right)}\, dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\left(\frac{\pi}{2} - x\right)}{\sin x + \cos x}\, dx. \qquad \text{(by Corollary 7.4.9)}$$

Hence $$2A = \int_0^{\frac{\pi}{2}} \left[\frac{x}{\sin x + \cos x} + \frac{\frac{\pi}{2} - x}{\sin x + \cos x}\right] dx$$

$$= \frac{\pi}{2} \int_0^{\frac{\pi}{2}} \frac{dx}{\sin x + \cos x}.$$

Hence $$A = \frac{\pi}{4} \int_0^{\frac{\pi}{2}} \frac{dx}{\sin x + \cos x}.$$

Put $t = \tan\frac{x}{2}$. Then $dt = \frac{1}{2}\sec^2\frac{x}{2}\, dx$, $\sin x = \frac{2t}{1+t^2}$, $\cos x = \frac{1-t^2}{1+t^2}$ and $\sec^2\frac{x}{2} = 1 + t^2$.

When $x = 0$, $t = 0$ and when $x = \frac{\pi}{2}$, $t = 1$. Thus

$$A = \frac{\pi}{4} \int_0^{\frac{\pi}{2}} \frac{\frac{1}{2}\sec^2\frac{x}{2}}{(\sin x + \cos x)\left(\frac{1}{2}\sec^2\frac{x}{2}\right)}\, dx = \frac{\pi}{4} \int_0^1 \frac{2dt}{2t + 1 - t^2}$$

$$= \frac{\pi}{4} \int_0^1 \frac{dt}{(\sqrt{2})^2 - (t-1)^2}$$

$$= \frac{\pi}{2}\left[\frac{1}{2\sqrt{2}}\log\frac{\sqrt{2}+t-1}{\sqrt{2}-t+1}\right]_0^1 = -\frac{\pi}{4\sqrt{2}}\log\left(\frac{\sqrt{2}-1}{\sqrt{2}+1}\right)$$

$$= \frac{\pi}{2\sqrt{2}}\log\sqrt{\frac{\sqrt{2}+1}{\sqrt{2}-1}}. \qquad = \frac{\pi}{2\sqrt{2}}\log(\sqrt{2}+1).$$

4. Problem : *Evaluate* $\int_{\pi/6}^{\pi/3}\frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx.$

Solution : Let A denote the value of the given integral. Put $x = \frac{\pi}{2} - t$. Then $dx = -dt$. When $x = \frac{\pi}{6}$, $t = \frac{\pi}{3}$; when $x = \frac{\pi}{3}$, $t = \frac{\pi}{6}$. Hence

$$A = \int_{\pi/6}^{\pi/3}\frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx = -\int_{\pi/3}^{\pi/6}\frac{\sqrt{\sin\left(\frac{\pi}{2}-t\right)}}{\sqrt{\sin\left(\frac{\pi}{2}-t\right)}+\sqrt{\cos\left(\frac{\pi}{2}-t\right)}}\,dt$$

$$= \int_{\pi/6}^{\pi/3}\frac{\sqrt{\cos t}}{\sqrt{\cos t}+\sqrt{\sin t}}\,dt = \int_{\pi/6}^{\pi/3}\frac{\sqrt{\cos x}}{\sqrt{\cos x}+\sqrt{\sin x}}\,dx.$$

Thus

$$2A = \int_{\pi/6}^{\pi/3}\frac{\sqrt{\sin x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx + \int_{\pi/6}^{\pi/3}\frac{\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx$$

$$= \int_{\pi/6}^{\pi/3}\frac{\sqrt{\sin x}+\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx = \int_{\pi/6}^{\pi/3}dx = [x]_{\pi/6}^{\pi/3} = \frac{\pi}{6}.$$

Hence $A = \frac{\pi}{12}$.

5. Problem : *Find* $\int_{-a}^{a}\left(x^2+\sqrt{a^2-x^2}\right)dx$.

Solution : We have

$$\int_{-a}^{a}\left(x^2+\sqrt{a^2-x^2}\right)dx = \int_{-a}^{a}x^2\,dx + \int_{-a}^{a}\sqrt{a^2-x^2}\,dx.$$

Since x^2 and $\sqrt{a^2-x^2}$ are even functions, by Theorem 7.4.10, we have

$$\int_{-a}^{a} x^2\,dx = 2\int_{0}^{a} x^2\,dx \;;\; \int_{-a}^{a}\sqrt{a^2-x^2}\,dx = 2\int_{0}^{a}\sqrt{a^2-x^2}\,dx.$$

Therefore

$$\int_{-a}^{a}(x^2+\sqrt{a^2-x^2})\,dx = 2\int_{0}^{a} x^2\,dx + 2\int_{0}^{a}\sqrt{a^2-x^2}\,dx$$

$$= 2\left[\frac{x^3}{3}\right]_0^a + 2\left[\frac{a^2}{2}\,\text{Sin}^{-1}\left(\frac{x}{a}\right) + \frac{x\sqrt{a^2-x^2}}{2}\right]_0^a$$

$$= 2\,.\frac{a^3}{3} + 2.\frac{a^2}{2}\,.\,\frac{\pi}{2} = \frac{2}{3}a^3 + \frac{\pi a^2}{2}.$$

6. Problem : *Evaluate* $\int_{0}^{\pi}\frac{x\sin x}{1+\sin x}\,dx.$

Solution : Let $A = \int_{0}^{\pi}\frac{x\sin x}{1+\sin x}\,dx.$

Then $A = \int_{0}^{\pi}\frac{(\pi-x)\sin(\pi-x)}{1+\sin(\pi-x)}\,dx$ (by Corollary 7.4.9)

$$= \int_{0}^{\pi}\frac{(\pi-x)\sin x}{1+\sin x}\,dx.$$

Hence $2A = \int_{0}^{\pi}\frac{x\sin x}{1+\sin x}\,dx + \int_{0}^{\pi}\frac{(\pi-x)\sin x}{1+\sin x}\,dx$

$$= \int_{0}^{\pi}\frac{[x+(\pi-x)]\sin x}{1+\sin x}\,dx = \pi\int_{0}^{\pi}\frac{\sin x}{1+\sin x}\,dx$$

$$= \pi\left[\int_{0}^{\pi}\left(1-\frac{1}{1+\sin x}\right)dx\right] = \pi\int_{0}^{\pi}dx - \pi\int_{0}^{\pi}\frac{1}{1+\sin x}\,dx$$

$$= \pi^2 - \pi\int_{0}^{\pi}\frac{1}{1+\sin x}\,dx. \qquad \ldots(1)$$

Let us now evaluate $\int_0^{\pi} \frac{1}{1+\sin x}dx$.

Since $\sin(\pi - x) = \sin x$, by Theorem 7.4.11 (i) it follows that

$$\int_0^{\pi} \frac{1}{1+\sin x}dx = 2\int_0^{\frac{\pi}{2}} \frac{1}{1+\sin x}\,dx$$

$$= 2\int_0^{\frac{\pi}{2}} \frac{1}{1+\sin\left(\frac{\pi}{2}-x\right)}\,dx \qquad \text{(by Corollary 7.4.9)}$$

$$= 2\int_0^{\frac{\pi}{2}} \frac{1}{1+\cos x}\,dx = 2\int_0^{\frac{\pi}{2}} \frac{dx}{2\cos^2\frac{x}{2}}$$

$$= \int_0^{\frac{\pi}{2}} \sec^2\frac{x}{2}\,dx = \left[2\tan\frac{x}{2}\right]_0^{\pi/2} = 2. \qquad \ldots (2)$$

Hence, from (1) and (2), $2A = \pi^2 - 2\pi$ and $A = \frac{\pi^2}{2} - \pi$.

7. Problem : *Evaluate* $\int_0^{\frac{\pi}{2}} x \sin x\, dx$.

Solution : Let $u(x) = x$ and $v(x) = -\cos x$. Then $u'(x) = 1$; $v'(x) = \sin x$.

Using the formula for integration by parts (7.4.13), we have

$$\int_0^{\frac{\pi}{2}} x \sin x\,dx = \int_0^{\frac{\pi}{2}} x\,(-\cos x)'\,dx = \int_0^{\frac{\pi}{2}} u(x)v'(x)\,dx$$

$$= \left[u(x)\,v(x)\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} u'(x)v(x)\,dx$$

$$= \left[-x\cos x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} (-\cos x)dx$$

$$= \int_0^{\frac{\pi}{2}} \cos x\,dx = \left[\sin x\right]_0^{\frac{\pi}{2}} = 1.$$

8. Problem : *Evaluate* $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n}\left[\frac{n-i}{n+i}\right]$ *by using the method of finding definite integral as the limit of a sum.*

Solution : $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n}\left[\frac{n-i}{n+i}\right]=\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n}\left[\frac{1-\frac{i}{n}}{1+\frac{i}{n}}\right]=\int_0^1\left(\frac{1-x}{1+x}\right)dx$

by using the formula (2) of Note 7.2.1 with $f(x)=\frac{1-x}{1+x}, x\in[0,1]$.

Now

$$\int_0^1\frac{1-x}{1+x}dx=-\int_0^1\frac{x+1-2}{1+x}dx=-\int_0^1 dx+2\int_0^1\frac{1}{1+x}dx \text{ (by Theorem 7.4.2)}$$

$$=-[x]_0^1+2\ln(1+x)\Big|_0^1 \qquad \text{(by Theorem 7.3.1)}$$

$$=-1+2\ln 2.$$

9. Problem : *Evaluate* $\lim_{n\to\infty}\frac{2^k+4^k+6^k+...+(2n)^k}{n^{k+1}}$ *by using the method of finding definite integral as the limit of a sum.*

Solution : $\lim_{n\to\infty}\frac{2^k+4^k+6^k+\cdots+(2n)^k}{n^{k+1}}=\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}2^k\left(\frac{i}{n}\right)^k$

$$=\int_0^1 2^k x^k\,dx,$$

by using the formula (2) of Note 7.2.1 with $f(x)=2^kx^k$, $x\in[0,1]$, k is a fixed real number not equal to -1.

Now

$$\int_0^1 2^k x^k\,dx=2^k\left[\frac{x^{k+1}}{k+1}\right]_0^1=\frac{2^k}{k+1}.$$

Hence

$$\lim_{n\to\infty}\frac{2^k+4^k+6^k+\cdots+(2n)^k}{n^{k+1}}=\frac{2^k}{k+1}.$$

10. Problem : *Evaluate* $\lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right).....\left(1+\frac{n}{n}\right)\right]^{\frac{1}{n}}$.

Solution : Let $\lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right).....\left(1+\frac{n}{n}\right)\right]^{\frac{1}{n}} = l$.

Then

$\ln l = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\ln\left(1+\frac{i}{n}\right)$, since logarithmic function is continuous. Now by using (2) of Note 7.2.1 with $f(x)=\ln(1+x), x\in[0,1]$, we get

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\ln\left(1+\frac{i}{n}\right) = \int_0^1 \ln(1+x)\,dx$$

$$= [(1+x)\ \ln(1+x)-(1+x)]_0^1$$

$$=2\ln 2-2-(-1)$$

$$=\ln 4-1.$$

Hence

$$\ln l=\ln 4-1.$$

i.e., $l=e^{\ln 4-1}=4e^{-1}$.

11. Problem : *Let* $f:\mathbf{R}\to\mathbf{R}$ *be a continuous periodic function and* T *be the period of it. Then prove that for any positive integer n,*

$$\int_0^{nT} f(x)\ dx = n\int_0^{T} f(x)\ dx. \qquad ...(1)$$

Solution : Let k be an integer and define

$g:[kT,(k+1)T]\to[0,T]$ as $g(t)=t-kT$. Then $g'(t)=1$ for all $t\in[kT,(k+1)T]$.
Hence by Theorem 7.4.5, $(fog)\,g'$ is integrable on $[kT,(k+1)T]$ and

$$\int_{kT}^{(k+1)T} f(g(t))\ g'(t)\ dt = \int_0^{T} f(x)\ dx. \qquad ...(2)$$

We have $f(g(t))\ g'(t) = f(t-kT)\ .1 = f(t)$,

since f is periodic with T as the period.

Hence $$\int_{kT}^{(k+1)T} f(g(t))\, g'(t)\, dt = \int_{kT}^{(k+1)T} f(t)\, dt. \qquad \ldots (3)$$

Thus, from (2) and (3),

$$\int_{kT}^{(k+1)T} f(t)\, dt = \int_{0}^{T} f(t)\, dt.$$

Let us now prove formula (1) by using the principle of mathematical induction.

For $n = 1$, clearly (1) is true.

Assume (1) is true for a positive integer m.

Thus $$\int_{0}^{mT} f(x)\, dx = m \int_{0}^{T} f(x)\, dx.$$

Now, by Theorem 7.4.3 and by (4) and (5), we get

$$\int_{0}^{(m+1)T} f(x)\, dx = \int_{0}^{mT} f(x)\, dx + \int_{mT}^{(m+1)T} f(x)\, dx$$

$$= m \int_{0}^{T} f(x)\, dx + \int_{0}^{T} f(x)\, dx$$

$$= (m+1) \int_{0}^{T} f(x)\, dx.$$

Hence formula (1) is true for $n = m+1$.

Thus, formula (1) is true for any positive integer n, by the principle of mathematical induction.

7.4.15 Note : Formula (1) of Problem 7.4.14(11) is valid for any integer n. Further, it remains valid if the continuity of f on R is replaced by integrability on [0, T], in view of Note 7.4.6.

Exercise 7(b)

I. Evaluate the following definite integrals

1. $\int_{0}^{a} (a^2 x - x^3)\, dx$

2. $\int_{2}^{3} \frac{2x}{1+x^2}\, dx$

3. $\int_{0}^{\pi} \sqrt{2 + 2\cos\theta}\, d\theta$

4. $\int_{0}^{\pi} \sin^3 x \cos^3 x\, dx$

5. $\int_{0}^{2} |1 - x|\, dx$

6. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos x}{1+e^x}\, dx$

7. $\int_0^1 \frac{dx}{\sqrt{3-2x}}$ 8. $\int_0^a (\sqrt{a}-\sqrt{x})^2\,dx$ 9. $\int_0^{\frac{\pi}{2}} \sec^4\theta\,d\theta$

10. $\int_0^3 \frac{x}{\sqrt{x^2+16}}\,dx$ 11. $\int_0^1 x\,e^{-x^2}\,dx$ 12. $\int_1^5 \frac{dx}{\sqrt{2x-1}}$

II. Evaluate the following integrals.

1. $\int_0^4 \frac{x^2}{1+x}dx$ 2. $\int_{-1}^2 \frac{x^2}{x^2+2}dx$ 3. $\int_0^1 \frac{x^2}{x^2+1}dx$

4. $\int_0^{\frac{\pi}{2}} x^2\sin x\,dx$ 5. $\int_0^4 |2-x|\,dx$ 6. $\int_0^{\frac{\pi}{2}} \frac{\sin^5 x}{\sin^5 x+\cos^5 x}dx$

7. $\int_0^{\frac{\pi}{2}} \frac{\sin^2 x-\cos^2 x}{\sin^3 x+\cos^3 x}dx$

Evaluate the following limits

8. $\lim_{n\to\infty} \frac{\sqrt{n+1}+\sqrt{n+2}+....+\sqrt{n+n}}{n\sqrt{n}}$ 9. $\lim_{n\to\infty}\left[\frac{1}{n+1}+\frac{1}{n+2}+...+\frac{1}{6n}\right]$

10. $\lim_{n\to\infty}\frac{1}{n}\left[\tan\frac{\Pi}{4n}+\tan\frac{2\Pi}{4n}+...+\tan\frac{n\Pi}{4n}\right]$ 11. $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{i^3}{i^4+n^4}$

12. $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{i}{n^2+i^2}$ 13. $\lim_{n\to\infty}\frac{1+2^4+3^4+....+n^4}{n^5}$

14. $\lim_{n\to\infty}\left[\left(1+\frac{1}{n^2}\right)\left(1+\frac{2^2}{n^2}\right).....\left(1+\frac{n^2}{n^2}\right)\right]^{\frac{1}{n}}$ 15. $\lim_{n\to\infty}\frac{(n!)^{\frac{1}{n}}}{n}$

III. Evaluate the following integrals.

1. $\int_0^{\frac{\pi}{2}} \frac{dx}{4+5\cos x}$ 2. $\int_a^b \sqrt{(x-a)\,(b-x)}\;dx$

3. $\int_0^{\frac{1}{2}} \frac{x\,\text{Sin}^{-1}x}{\sqrt{1-x^2}}\,dx$ 4. $\int_0^{\frac{\pi}{4}} \frac{\sin x+\cos x}{9+16\sin 2x}\,dx$

5. $\int_0^{\frac{\pi}{2}} \frac{a\sin x + b\cos x}{\sin x + \cos x}\,dx$ 6. $\int_0^a x(a-x)^n\,dx$ 7. $\int_0^2 x\sqrt{2-x}\,dx$

8. $\int_0^{\pi} x\sin^3 x\,dx$ 9. $\int_0^{\pi} \frac{x}{1+\sin x}\,dx$ 10. $\int_0^{\pi} \frac{x\sin^3 x}{1+\cos^2 x}\,dx$

11. $\int_0^1 \frac{\log(1+x)}{1+x^2}\,dx$ 12. $\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx$ 13. $\int_0^{\frac{\pi}{2}} \frac{\sin^2 x}{\cos x + \sin x}\,dx$

14. $\int_0^{\pi} \frac{1}{3+2\cos x}\,dx$ 15. $\int_0^{\frac{\pi}{4}} \log(1+\tan x)\,dx$ 16. $\int_{-1}^{\frac{3}{2}} |x\sin \pi x|\,dx$

17. $\int_0^1 \mathrm{Sin}^{-1}\left(\frac{2x}{1+x^2}\right)dx$ 18. $\int_0^1 x\,\mathrm{Tan}^{-1}x\,dx$ 19. $\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx$

20. Suppose that $f:\mathbf{R}\to\mathbf{R}$ is a continuous periodic function and T is the period of it.

Let $a\in\mathbf{R}$. Then prove that for any positive integer n, $\int_a^{a+nT} f(x)\,dx = n\int_a^{a+T} f(x)\,dx$.

7.5 Reduction Formulae

In this section, we derive some reduction formulae for the evaluation of definite integrals of $\sin^n x$, $\cos^n x$ and $\sin^m x\cos^n x$ between 0 and $\frac{\pi}{2}$ for positive integers m, n.

The following theorem gives a useful formula to evaluate the definite integral of $\sin^n x$ between 0 and $\frac{\pi}{2}$, when n is an integer ≥ 2.

7.5.1 Theorem : *Let n be an integer greater than or equal to 2. Then*

$$\int_0^{\frac{\pi}{2}} \sin^n x\,dx = \begin{cases} \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\frac{1}{2}\cdot\frac{\pi}{2}, & \text{if } n \text{ is even.} \\ \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\frac{2}{3}, & \text{if } n \text{ is odd.} \end{cases}$$

Proof: On using the formula for integration by parts (Theorem 7.4.13), we have

$$\int_0^{\frac{\pi}{2}} \sin^n x\,dx = \int_0^{\frac{\pi}{2}} \sin^{n-1} x\,.\,\sin x\;dx = \int_0^{\frac{\pi}{2}} \sin^{n-1} x\,\frac{d}{dx}(-\cos x)\,dx$$

$$= \left[\sin^{n-1} x\,(-\cos x)\right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} (n-1)\sin^{n-2} x \cos^2 x\, dx$$

$$= 0 + (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x\,(1-\sin^2 x)\,dx$$

$$= (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x - (n-1)\int_0^{\frac{\pi}{2}} \sin^n x\, dx.$$

Hence $$n\int_0^{\frac{\pi}{2}} \sin^n x\,dx = (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2} x\, dx.$$

Therefore $$\int_0^{\frac{\pi}{2}} \sin^n x\,dx = \frac{(n-1)}{n}\int_0^{\frac{\pi}{2}} \sin^{n-2} x\, dx. \quad \ldots (1)$$

Hence if we write $I_n = \int_0^{\frac{\pi}{2}} \sin^n x\,dx$, then from (1), we have

$$I_n = \frac{n-1}{n} I_{n-2}.$$

This is the reduction formula.

On applying successively, the formula for integration by parts to the right hand side integral, we get

$$I_n = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\, I_{n-4}$$

$$\vdots$$

$$= \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdot\frac{n-5}{n-4}\cdots A,$$

where $$A = \begin{cases} \int_0^{\pi/2} dx, & \text{if } n \text{ is even} \\ \int_0^{\pi/2} \sin x\, dx, & \text{if } n \text{ is odd} \end{cases}$$

$$= \begin{cases} \frac{\pi}{2}, & \text{if } n \text{ is even} \\ 1, & \text{if } n \text{ is odd}. \end{cases}$$

Therefore, we have $$\int_0^{\frac{\pi}{2}} \sin^n x\,dx = \begin{cases} \dfrac{n-1}{n}\cdot\dfrac{n-3}{n-2}\cdots\dfrac{1}{2}\cdot\dfrac{\pi}{2}, & \text{if } n \text{ is even} \\ \dfrac{n-1}{n}\cdot\dfrac{n-3}{n-2}\cdots\dfrac{2}{3}, & \text{if } n \text{ is odd.} \end{cases}$$

7.5.2 Observation : From problem 7.4.14(1), we have

$$\int_0^{\frac{\pi}{2}} \sin^n x\,dx = \int_0^{\frac{\pi}{2}} \cos^n x\,dx.$$

Therefore from Theorem 7.5.1, for an integer $n \geq 2$, we have

$$\int_0^{\frac{\pi}{2}} \cos^n x\,dx = \begin{cases} \dfrac{n-1}{n}\cdot\dfrac{n-3}{n-2}\cdots\dfrac{1}{2}\cdot\dfrac{\pi}{2}, & \text{if } n \text{ is even} \\ \dfrac{n-1}{n}\cdot\dfrac{n-3}{n-2}\cdots\dfrac{2}{3}, & \text{if } n \text{ is odd.} \end{cases}$$

7.5.3 Solved Problems

1. Problem : *Find* (i) $\int_0^{\frac{\pi}{2}} \sin^4 x\,dx$ (ii) $\int_0^{\frac{\pi}{2}} \sin^7 x\,dx$ (iii) $\int_0^{\frac{\pi}{2}} \cos^8 x\,dx$.

Solution: We solve (i) and (ii) by using Theorem 7.5.1.

(i) $$\int_0^{\frac{\pi}{2}} \sin^4 x\,dx = \frac{4-1}{4}\cdot\frac{4-3}{4-2}\cdot\frac{\pi}{2}$$
$$= \frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{3}{16}\pi.$$

(ii) $$\int_0^{\frac{\pi}{2}} \sin^7 x\,dx = \frac{7-1}{7}\cdot\frac{7-3}{7-2}\cdot\frac{7-5}{7-4}$$
$$= \frac{6}{7}\cdot\frac{4}{5}\cdot\frac{2}{3} = \frac{16}{35}.$$

We solve (iii) by using Observation 7.5.2.

(iii) $$\int_0^{\frac{\pi}{2}} \cos^8 x\,dx = \frac{8-1}{8}\cdot\frac{8-3}{8-2}\cdot\frac{8-5}{8-4}\cdot\frac{8-7}{8-6}\cdot\frac{\pi}{2}$$
$$= \frac{7}{8}\cdot\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}$$
$$= \frac{35}{256}\pi.$$

2. Problem : *Evaluate* $\int_0^a \sqrt{a^2 - x^2}\, dx$.

Solution : Put $x = a\sin\theta$. Then $dx = a\cos\theta\, d\theta$. When $\theta = 0$, $x = 0$ and when $\theta = \frac{\pi}{2}$, $x = a$. When $\theta \in \left[0, \frac{\pi}{2}\right]$, $x \in [0, a]$. Hence, by the method of substitution (Theorem 7.4.5) it follows that

$$\int_0^a \sqrt{a^2 - x^2}\, dx = \int_0^{\frac{\pi}{2}} a\cos\theta \cdot a\cos\theta\, d\theta.$$

$$= a^2 \int_0^{\frac{\pi}{2}} \cos^2\theta\, d\theta.$$

$$= a^2 \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi a^2}{4} \text{ (on using Observation 7.5.2).}$$

Now, the following theorem gives a formula for evaluating the definite integral of $\sin^m x \cos^n x$ between 0 and $\frac{\pi}{2}$, where both m and n are positive integers.

7.5.4 Theorem : *Let m and n be positive integers. Then*

$$\int_0^{\frac{\pi}{2}} \sin^m x \cos^n x\, dx = \begin{cases} \dfrac{1}{m+1}, & \text{if } n = 1 \\ \dfrac{n-1}{m+n} \cdot \dfrac{n-3}{m+n-2} \cdots \dfrac{2}{m+3} \cdot \dfrac{1}{m+1}, & \text{if } 1 \neq n \text{ is odd} \\ \dfrac{n-1}{m+n} \cdot \dfrac{n-3}{m+n-2} \cdots \dfrac{1}{m+2} \cdot \dfrac{m-1}{m} \cdots \dfrac{1}{2} \cdot \dfrac{\pi}{2}, & \text{if } n \text{ is even and } m \text{ is even} \\ \dfrac{n-1}{m+n} \cdot \dfrac{n-3}{m+n-2} \cdots \dfrac{1}{m+2} \cdot \dfrac{m-1}{m} \cdots \dfrac{2}{3}, & \text{if } n \text{ is even, } 1 \neq m \text{ is odd} \\ \dfrac{1}{n+1}, & \text{if } m = 1. \end{cases}$$

Proof : It is easy to see that

$$\int_0^{\frac{\pi}{2}} \sin^m x \cos x\, dx = \frac{1}{m+1}; \quad \text{and} \quad \int_0^{\frac{\pi}{2}} \sin x \cos^n x\, dx = \frac{1}{n+1}.$$

Suppose that $m \geq 2$ and $n \geq 2$. On using the formula for integration by parts (Theorem 7.4.14), we have

$$\int_0^{\frac{\pi}{2}} \sin^m x \cos^n x\, dx = \int_0^{\frac{\pi}{2}} \sin^m x \cos^{n-1} x \cos x\, dx$$

$$= \int_0^{\frac{\pi}{2}} \cos^{n-1} x \frac{d}{dx}\left(\frac{\sin^{m+1} x}{m+1}\right) dx$$

$$= \left[\frac{1}{m+1} \cos^{n-1} x \ \sin^{m+1} x\right]_0^{\frac{\pi}{2}} - \frac{1}{m+1}\int_0^{\frac{\pi}{2}} \sin^{m+1} x \ \frac{d}{dx}(\cos^{n-1} x)\, dx$$

$$= 0 + \frac{n-1}{m+1}\int_0^{\frac{\pi}{2}} \sin^{m+2} x \ \cos^{n-2} x\, dx$$

$$= \frac{n-1}{m+1}\int_0^{\frac{\pi}{2}} \sin^{m} x \ \cos^{n-2} x\,(1-\cos^2 x)\, dx$$

$$= \frac{n-1}{m+1}\int_0^{\frac{\pi}{2}} \sin^{m} x \ \cos^{n-2} x\ dx - \frac{n-1}{m+1}\int_0^{\frac{\pi}{2}} \sin^{m} x \cos^{n} x\ dx.$$

Hence $$\left[1+\frac{n-1}{m+1}\right]\int_0^{\frac{\pi}{2}} \sin^{m} x \ \cos^{n} x\ dx = \frac{n-1}{m+1}\int_0^{\frac{\pi}{2}} \sin^{m} x \cos^{n-2} x\ dx.$$

Therefore $$\int_0^{\frac{\pi}{2}} \sin^{m} x \ \cos^{n} x\ dx = \frac{n-1}{m+n}\int_0^{\frac{\pi}{2}} \sin^{m} x \cos^{n-2} x\ dx. \qquad \text{... (1)}$$

Hence, if we write $\mathrm{I}_{m,n} = \int_0^{\frac{\pi}{2}} \sin^{m} x \ \cos^{n} x\ dx,$ then from (1),

we have $$\mathrm{I}_{m,n} = \frac{n-1}{m+n}\,\mathrm{I}_{m,\,n-2}.$$

This is the required *reduction formula*.

Now, on using the formula for integration by parts to $\mathrm{I}_{m;\,n-2}$, we get

$$\mathrm{I}_{m,\,n} = \frac{n-1}{m+n}\cdot\frac{n-3}{m+n-2}\,\mathrm{I}_{m,\,n-4}.$$

On proceeding like this we obtain

$$\mathrm{I}_{m,\,n} = \frac{n-1}{m+n}\cdot\frac{n-3}{m+n-2}....\mathrm{I}_{m,}\ \text{ or }\ \mathrm{I}_{m,0}$$

according as n is odd or even, where

$$\mathrm{I}_{m,1} = \int_0^{\frac{\pi}{2}} \sin^{m} x \cos x\ dx = \frac{1}{m+1};$$

and $$I_{m,0} = \int_0^{\frac{\pi}{2}} \sin^m x\,dx = \begin{cases} \dfrac{m-1}{m}\cdot\dfrac{m-3}{m-2}\cdots\dfrac{1}{2}\cdot\dfrac{\pi}{2}, & \text{if } m \text{ is even} \\ \dfrac{m-1}{m}\cdot\dfrac{m-3}{m-2}\cdots\dfrac{2}{3}, & \text{if } m \text{ is odd} \\ 1, \text{ if } m=1 \text{ and } m\geq 2. & \end{cases}$$ (Theorem 7.5.1)

Hence $$\int_0^{\frac{\pi}{2}} \sin^m x\cos^n x\,dx = \frac{n-1}{m+n}\cdot\frac{n-3}{m+n-2}\cdots\frac{2}{m+3}\cdot\frac{1}{m+1}, \text{ if } 1\neq n \text{ is odd.}$$

Hence, when n is even, we have

$$\int_0^{\frac{\pi}{2}} \sin^m x\cos^n x\,dx = \begin{cases} \dfrac{n-1}{m+n}\cdot\dfrac{n-3}{m+n-2}\cdots\dfrac{1}{m+2}\cdot\dfrac{m-1}{m}\cdots\dfrac{1}{2}\cdot\dfrac{\pi}{2}, & \text{if } m \text{ is even} \\ \dfrac{n-1}{m+n}\cdot\dfrac{n-3}{m+n-2}\cdots\dfrac{1}{m+2}\cdot\dfrac{m-1}{m}\cdots\dfrac{2}{3}, & \text{if } 1\neq m \text{ is odd}. \end{cases}$$

7.5.5 Solved Problems

1. Problem : *Evaluate the following definite integrals.*

(i) $\int_0^{\frac{\pi}{2}} \sin^4 x\cos^5 x\,dx$ (ii) $\int_0^{\frac{\pi}{2}} \sin^5 x\cos^4 x\,dx$ (iii) $\int_0^{\frac{\pi}{2}} \sin^6 x\cos^4 x\,dx$

Solution : We use Theorem 7.5.4 to solve (i), (ii) and (iii).

(i) $$\int_0^{\frac{\pi}{2}} \sin^4 x\cos^5 x\,dx = \frac{5-1}{4+5}\cdot\frac{5-3}{4+5-2}\cdot\frac{1}{4+1}$$

$$= \frac{4}{9}\cdot\frac{2}{7}\cdot\frac{1}{5} = \frac{8}{315}.$$

(ii) $$\int_0^{\frac{\pi}{2}} \sin^5 x\cos^4 x\,dx = \frac{4-1}{5+4}\cdot\frac{4-3}{5+4-2}\cdot\frac{5-1}{5}\cdot\frac{5-3}{5-2}$$

$$= \frac{3}{9}\cdot\frac{1}{7}\cdot\frac{4}{5}\cdot\frac{2}{3} = \frac{8}{315}.$$

(iii) $$\int_0^{\frac{\pi}{2}} \sin^6 x\cos^4 x\,dx = \frac{4-1}{6+4}\cdot\frac{4-3}{6+4-2}\cdot\frac{6-1}{6}\cdot\frac{6-3}{6-2}\cdot\frac{6-5}{6-4}\cdot\frac{\pi}{2}$$

$$= \frac{3}{10}\cdot\frac{1}{8}\cdot\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{3}{512}\pi.$$

2. Problem : *Find* $\int_0^{2\pi} \sin^4 x \cos^6 x \, dx$.

Solution : Let $f(x) = \sin^4 x \cos^6 x$.

Since $f(2\pi - x) = f(\pi - x) = f(x)$, it follows from Theorem 7.4.11, that

$$\int_0^{2\pi} \sin^4 x \cos^6 x \, dx = 2\int_0^{2\pi} \sin^4 x \cos^6 x \, dx$$

$$= 4 \int_0^{2\pi} \sin^4 x \cos^6 x \, dx$$

$$= 4 \cdot \frac{6-1}{4+6} \cdot \frac{6-3}{4+6-2} \cdot \frac{6-5}{4+6-4} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

$$= \frac{3}{128}\pi.$$

3. Problem : *Find* $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 x \cos^4 x \, dx.$

Solution : Let $f(x) = \sin^2 x \cos^4 x$.

Since f is even, by Theorem, 7.4.10, we have

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(x) \, dx = 2\int_0^{\frac{\pi}{2}} f(x) \, dx.$$

Hence $$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2 x \cos^4 x \, dx = 2\int_0^{\frac{\pi}{2}} \sin^2 x \cos^4 x \, dx$$

$$= 2 \cdot \frac{4-1}{2+4} \cdot \frac{4-3}{2+4-2} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

$$= 2 \cdot \frac{3}{6} \cdot \frac{1}{4} \cdot \frac{\pi}{4} = \frac{\pi}{16}.$$

4. Problem : *Find* $\int_0^{\pi} x \sin^7 x \cos^6 x \, dx$.

Solution : Let $A = \int_0^{\pi} x \sin^7 x \cos^6 x \, dx$. Then

$$A = \int_0^{\pi} (\pi - x) \sin^7 (\pi - x) \cos^6 (\pi - x) \, dx \quad \text{(by Corollary 7.4.9)}$$

$$= \int_0^{\pi} (\pi - x)\sin^7 x \cos^6 x \, dx$$

$$= \int_0^{\pi} \pi \sin^7 x \cos^6 x \, dx - \int_0^{\pi} x \sin^7 x \cos^6 x \, dx.$$

This implies $2A = \pi \int_0^{\pi} \sin^7 x \cos^6 x \, dx$.

Hence $A = \frac{\pi}{2} \int_0^{\pi} \sin^7 x \cos^6 x \, dx$ ... (1)

Let $f(x) = \sin^7 x \cos^6 x$.

Since $f(\pi - x) = \sin^7(\pi - x)\cos^6(\pi - x)$

$= \sin^7 x \cos^6 x = f(x)$,

by Theorem 7.4.11, we have

$$\int_0^{\pi} \sin^7 x \cos^6 x \, dx = 2\int_0^{\frac{\pi}{2}} \sin^7 x \cos^6 x \, dx.$$

Hence from (1), $A = \pi \int_0^{\frac{\pi}{2}} \sin^7 x \cos^6 x \, dx$

$$= \pi \cdot \frac{6-1}{7+6} \cdot \frac{6-3}{7+6-2} \cdot \frac{6-5}{7+6-4} \cdot \frac{7-1}{7} \cdot \frac{7-3}{5} \cdot \frac{7-5}{3}$$

$$= \frac{16}{3003}\pi.$$

5. Problem : *Find* $\int_{-a}^{a} x^2 (a^2 - x^2)^{3/2} dx.$

Solution : Since $f(x) = x^2(a^2 - x^2)^{3/2}$ is an even function, by Theorem 7.4.10, it follows that

$$\int_{-a}^{a} x^2 (a^2 - x^2)^{3/2} dx = 2\int_0^{a} x^2 (a^2 - x^2)^{3/2} dx.$$

Put $x = a\sin\theta$. Then $dx = a\cos\theta \, d\theta$. When $\theta = 0, x = 0$; when $\theta = \frac{\pi}{2}$, $x = a$; when $\theta \in \left[0, \frac{\pi}{2}\right]$, $x \in [0, a]$.

Hence, by Theorem 7.4.5, it follows that

$$2\int_0^a x^2(a^2-x^2)^{3/2}\,dx = 2\int_0^{\frac{\pi}{2}} a^2\sin^2\theta\cdot a^3\cos^3\theta\cdot a\cos\theta\,d\theta$$

$$= 2a^6\int_0^{\frac{\pi}{2}}\sin^2\theta\cos^4\theta\,d\theta$$

$$= 2a^6\cdot\frac{\pi}{32} = \frac{\pi a^6}{16}.$$

6. Problem : *Find* $\int_0^1 x^{\frac{3}{2}}\sqrt{1-x}\,dx.$

Solution : Put $x=\sin^2\theta$. Then $dx = 2\sin\theta\cos\theta\,d\theta$.

When $\theta=0,\ x=0;$ when $\theta=\frac{\pi}{2},\ x=1.$

When $\theta\in\left[0,\frac{\pi}{2}\right],\ x\in[0,1]$. Hence by Theorem 7.4.5, it follows that

$$\int_0^1 x^{\frac{3}{2}}\sqrt{1-x}\,dx = \int_0^{\frac{\pi}{2}}\sin^3\theta\cos\theta\cdot 2\sin\theta\cos\theta\,d\theta$$

$$= 2\int_0^{\frac{\pi}{2}}\sin^4\theta\cos^2\theta\,d\theta$$

$$= 2\cdot\frac{\pi}{32}$$

$$= \frac{\pi}{16}.$$

Exercise 7(c)

I. Find the values of the following integrals.

1. $\int_0^{\frac{\pi}{2}}\sin^{10}x\,dx$
2. $\int_0^{\frac{\pi}{2}}\cos^{11}x\,dx$
3. $\int_0^{\frac{\pi}{2}}\cos^7 x\,\sin^2 x\,dx$
4. $\int_0^{\frac{\pi}{2}}\sin^4 x\,\cos^4 x\,dx$
5. $\int_0^{\pi}\sin^3 x\,\cos^6 x\,dx$
6. $\int_0^{2\pi}\sin^2 x\,\cos^4 x\,dx$

7. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2\theta \ \cos^7\theta\, d\theta$

8. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^3\theta \ \cos^3\theta\, d\theta$

9. $\int_0^a x(a^2 - x^2)^{\frac{7}{2}}\ dx$

10. $\int_0^2 x^{\frac{3}{2}}\ \sqrt{2-x}\ dx$

II. Evaluate the following integrals.

1. $\int_0^1 x^5(1-x)^{\frac{5}{2}}\ dx$

2. $\int_0^4 (16-x^2)^{\frac{5}{2}}\ dx$

3. $\int_{-3}^{3} (9-x^2)^{3/2} x\ dx$

4. $\int_0^5 x^3(25-x^2)^{7/2} dx$

5. $\int_{-\pi}^{\pi} \sin^8 x \cos^7 x\ dx$

6. $\int_3^7 \sqrt{\frac{7-x}{x-3}}\ dx$

7. $\int_2^6 \sqrt{(6-x)(x-2)}\, dx$

8. $\int_0^{\frac{\pi}{2}} \tan^5 x \cos^8 x\ dx$

III. Evaluate the following integrals.

1. $\int_0^1 x^{7/2}(1-x)^{5/2}\ dx$

2. $\int_0^{\pi} (1+\cos x)^3\ dx$

3. $\int_4^9 \frac{dx}{\sqrt{(9-x)(x-4)}}$

4. $\int_0^5 x^2(\sqrt{5-x})^7\ dx$

5. $\int_0^{2\pi} (1+\cos x)^5(1-\cos x)^3\, dx$

7.6 Applications of definite integral to areas

In the previous sections of this chapter, we observed that if $y=f(x)$ is a non-negative continuous function defined on $[a, b]$ then the area under the graph of f between the ordinates $x=a$, $x=b$ and the X-axis is given by the value of the definite integral $\int_b^a f(x)\ dx$. In the following, we give different possible ways of calculating such areas depending on the nature of the integrand.

7.6.1 Areas under curves

(i) If $f:[a, b]\to[0, \infty)$ is continuous, then the area A of the region bounded by the curve $y=f(x)$, the X-axis and the lines $x=a$ and $x=b$ is given by $A=\int_a^b f(x)\,dx$ which is shown in Fig. 7.3 graphically.

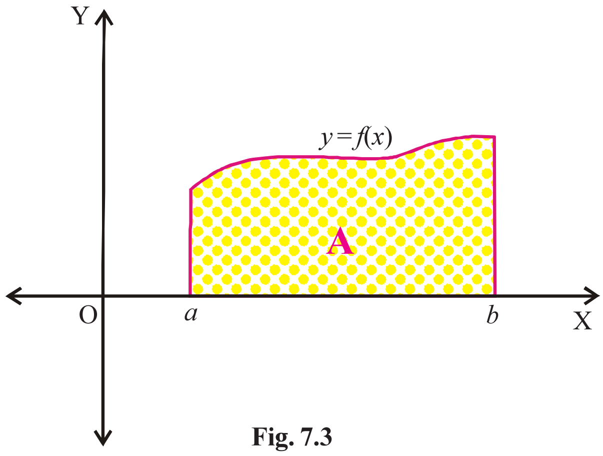


Fig. 7.3

(ii) Let $f:[a, b]\to(-\infty, 0]$ be continuous. Then the graphs of $y=f(x)$ and $y=-f(x)$ on $[a, b]$ are symmetric about the X-axis. So, the area bounded by the graph of $y=f(x)$, the X-axis and the lines $x=a,\ x=b$ is same as the area bounded by the graph of $y=-f(x)$, the X-axis and the lines $x=a$, $x=b$ which is, hence, given by $A=-\int_a^b f(x)\,dx$. This is shown graphically in the Fig. 7.4.

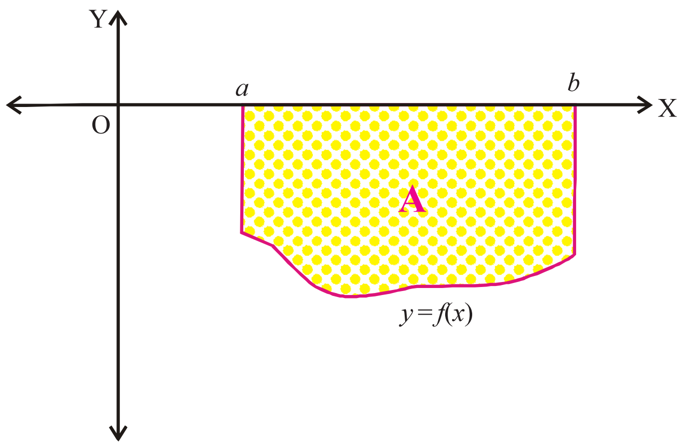


Fig. 7.4

From (i) and (ii), we observe that $A=\left|\int_a^b f(x)\,dx\right|$.

(iii) Let $f:[a,b]\to\mathbf{R}$ be continuous and $f(x)\geq 0$ for all $x\in[a, c]$ and $f(x)\leq 0$ for all $x\in[c, b]$ where $a<c<b$. Then the area of the region bounded by the curve $y=f(x)$, the X-axis, and the lines $x=a,\ x=b$ is given by

$$A = \int_a^c f(x)dx + \int_c^b -f(x)dx$$

$$= \int_a^c f(x)dx - \int_c^b f(x)dx \qquad \text{... (1)}$$

which is shown in Fig.7.5 graphically.

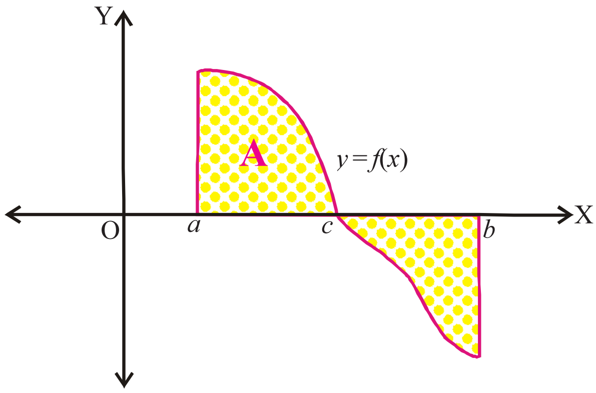


Fig. 7.5

Note : (1) can be written as $A = \left|\int_a^c f(x)dx\right| + \left|\int_c^b f(x)dx\right|$

(iv) Let $f:[a,b] \to \mathbf{R}$ and $g:[a,b] \to \mathbf{R}$ be continuous and $f(x) \le g(x)$ for all $x \in [a, b]$. Then the area of the region bounded by the curves $y = f(x)$, $y = g(x)$ and the lines $x = a$, $x = b$ is given by

$A = \int_a^b g(x)dx - \int_a^b f(x)dx$ (See Fig.7.6).

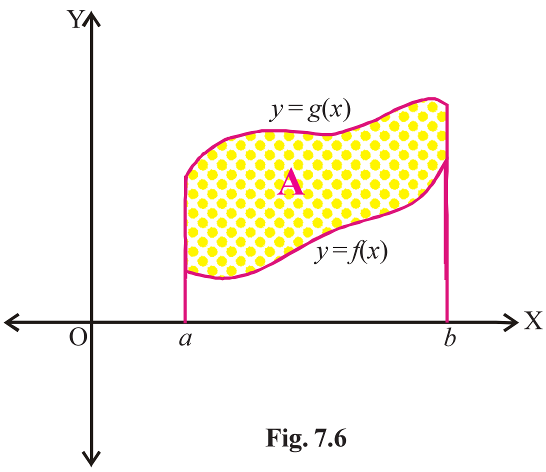


Fig. 7.6

In case $g(x) \le f(x)$ for all $x \in [a, b]$

then the area A is given by

$$A = \int_a^b f(x)dx - \int_a^b g(x)dx.$$

Hence, in either case the area A is given by

$$A = \left|\int_a^b f(x)dx - \int_a^b g(x)dx\right|.$$

(v) Let f and g be two continuous real valued functions on $[a, b]$ and $c \in (a, b)$ such that $f(x) < g(x)$ for all $x \in [a, c)$ and $g(x) < f(x)$ for all $x \in (c, b]$ with $f(c) = g(c)$. Then the area A of the region bounded by $y = f(x)$, $y = g(x)$, and the lines $x = a$, $x = b$ is given by

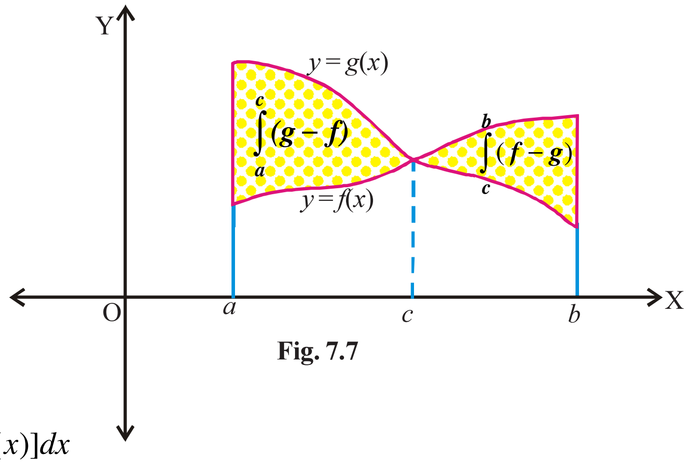


Fig. 7.7

$$A = \int_a^c [g(x) - f(x)]dx + \int_c^b [f(x) - g(x)]dx$$

$$= \left|\int_a^c (f(x) - g(x))\, dx\right| + \left|\int_c^b (f(x) - g(x))\, dx\right|$$

which is shown in Fig. 7.7 graphically.

(vi) Let $f:[a,b] \to \mathbf{R}$, and $g:[a,b] \to \mathbf{R}$ be two continuous functions. Suppose that, there exist points $x_1, x_2 \in (a, b)$ with $x_1 < x_2$ such that $f(x_1) = g(x_1)$ and $f(x_2) = g(x_2)$ and $f(x) \geq g(x)$ for all $x \in (x_1, x_2)$. Then the area A of the region bounded by the curves $y = f(x), y = g(x)$ and the lines $x = x_1$, $x = x_2$ is given by

$$A = \int_{x_1}^{x_2} (f(x) - g(x))dx \text{ (See Fig.7.8).}$$

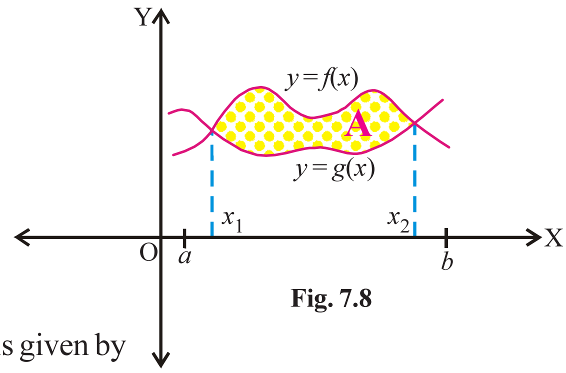


Fig. 7.8

In case $f(x) \leq g(x)$ for all $x \in (x_1, x_2)$, then the area A is given by

$$A = \int_{x_1}^{x_2} (g(x) - f(x))\, dx.$$

Hence, in either case, the area A is given by

$$A = \left|\int_{x_1}^{x_2} (f(x) - g(x))dx\right|.$$

7.6.2 Note

(i) Some regions are best treated by regarding x as a function of y. If a region is bounded by the curve $x = g(y)$ where g is a non-negative continuous function on $[c, d]$, the Y-axis and the lines $y = c$, and $y = d$, then its area is given by

$$A = \int_c^d g(y)dy$$

and this is illustrated in Fig. 7.9.

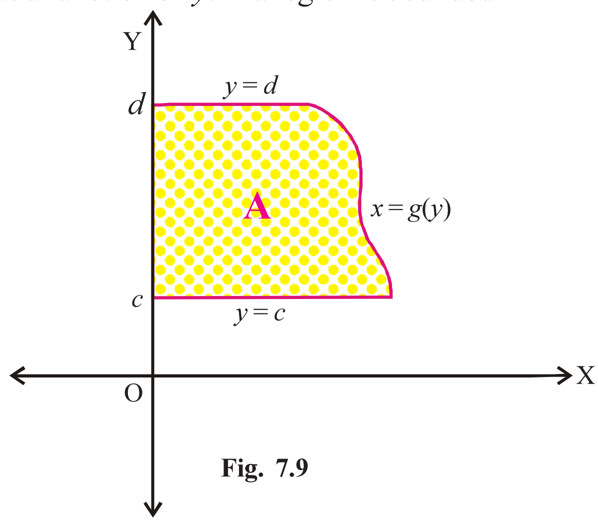


Fig. 7.9

(ii) Similarly, if $g : [c, d] \to (-\infty, 0]$ is continuous, then the area A of the region bounded by the curve $x = g(y)$, the Y-axis and the lines $y = c$, $y = d$ is

$$A = \int_c^d -g(y)dy.$$

This is shown graphically in Fig. 7.10.

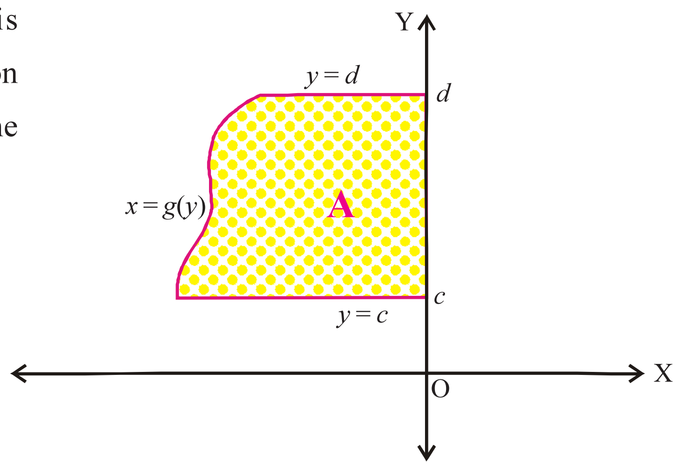


Fig. 7.10

Combining (i) and (ii), we may write $A = \left|\int_c^d g(y)\, dy\right|$.

(iii) If a region is bounded by the curves with equations $x = f(y)$, $x = g(y)$, $y = c$ and $y = d$, where f and g are continuous and $f(y) \geq g(y)$ for $c \leq y \leq d$ (See Fig. 7.11), then its area is

$$A=\int_c^d [f(y)-g(y)]\,dy.$$

If we write x_R for the right boundary and x_L for the left boundary, then the above area A is given by

$$A=\int_c^d (x_R-x_L)\,dy.$$

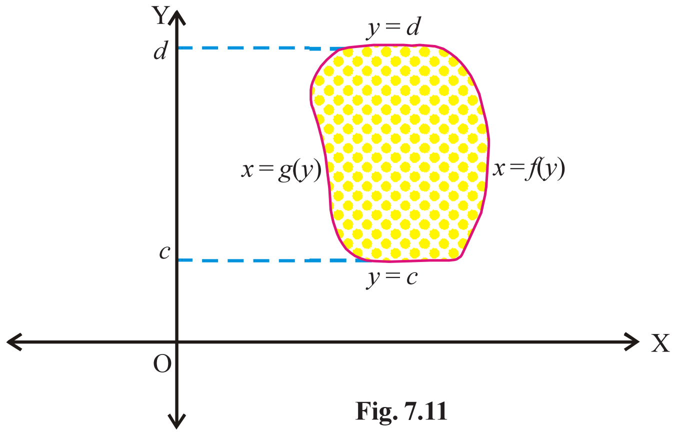


Fig. 7.11

In general, when $f(x)\geq g(y)$ on $[c, d]$ or $f(y)\leq g(y)$ on $[c, d]$, we have

$$A=\left|\int_c^d f(y)-g(y)\,dx\right|$$

7.6.3 Solved Problems

1. Problem: *Find the area under the curve* $f(x)=\sin x$ *in* $[0, 2\pi]$.

Solution: Consider the graph of the function $f(x)=\sin x$ with respect to the interval $[0, 2\pi]$. We know that $\sin x\geq 0$ for all $x\in[0,\pi]$ and $\sin x\leq 0$ for all $x\in[\pi, 2\pi]$. The graph of f will be as shown in Fig. 7.12.

Hence, the area of the region enclosed by the curve $y=\sin x$ in $[0, 2\pi]$ is given by

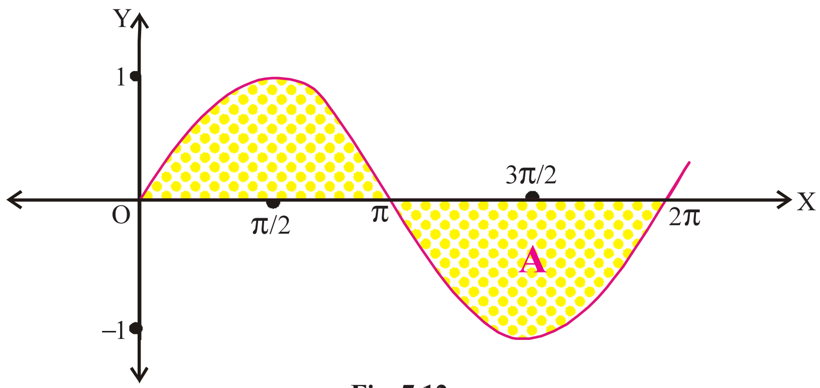


Fig. 7.12

$$A=\int_0^{\pi}\sin x\,dx+\int_{\pi}^{2\pi}(-\sin x)\,dx = -\cos x\Big]_0^{\pi}+\cos x\Big]_{\pi}^{2\pi} = 2+2=4.$$

2. Problem: *Find the area under the curve* $f(x) = \cos x$ *in* $[0, 2\pi]$.

Solution: We know that $\cos x \geq 0$ for all $x \in \left[0, \frac{\pi}{2}\right] \cup \left[\frac{3\pi}{2}, 2\pi\right]$ and $\cos x \leq 0$ for all $x \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$. The graph of $\cos x$ in $[0, 2\pi]$ is therefore as shown in Fig. 7.13. Hence, the area A under the curve $f(x) = \cos x$ in $[0, 2\pi]$ is given by

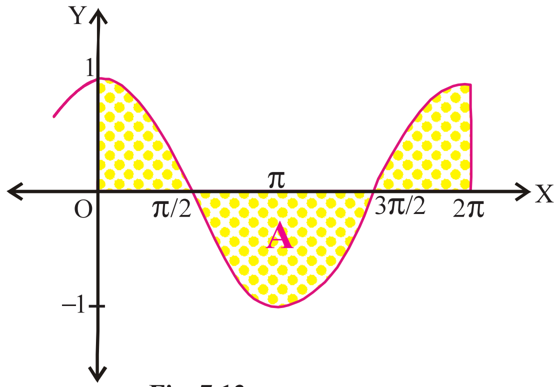


Fig. 7.13

$$A = \int_0^{\pi/2} \cos x \, dx + \int_{\pi/2}^{3\pi/2} (-\cos x) \, dx + \int_{3\pi/2}^{2\pi} \cos x \, dx$$

$$= \sin x\Big]_0^{\pi/2} - \sin x\Big]_{\pi/2}^{3\pi/2} + \sin x\Big]_{3\pi/2}^{2\pi} = 1 + 2 + 1 = 4.$$

3. Problem: *Find the area bounded by the parabola* $y = x^2$, *the X-axis and the lines* $x = -1$, $x = 2$.

Solution: The region bounded by the parabola $y = x^2$, the X-axis and the ordinates $x = -1$, $x = 2$ is as shown in Fig. 7.14.

The required area A is given by

$$A = \int_{-1}^{2} x^2 \, dx = \frac{x^3}{3}\Bigg]_{-1}^{2}$$

$$= \frac{8}{3} - \left(-\frac{1}{3}\right) = 3.$$

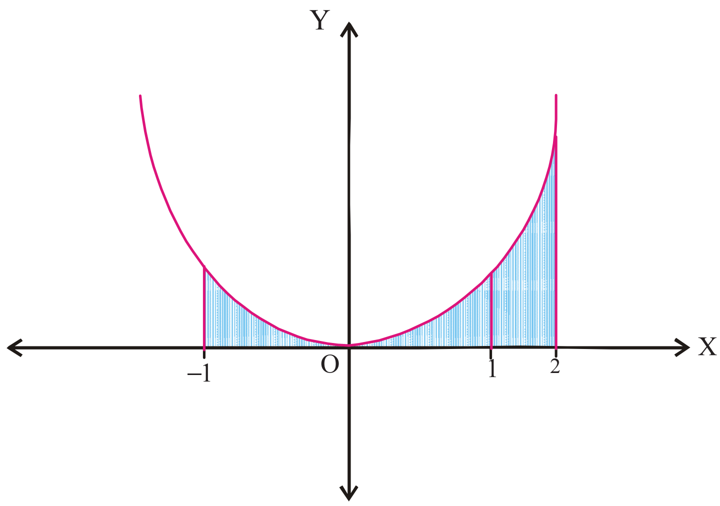


Fig. 7.14

4. Problem: *Find the area cut off between the line* $y = 0$ *and the parabola* $y = x^2 - 4x + 3$.

Solution: First we find the points of intersection of the line $y = 0$ and the parabola

$y = x^2 - 4x + 3$. The abscissae of these points of intersection are given by $x^2 - 4x + 3 = 0$ i.e., $x = 1, 3$. That is, the parabola cuts the X-axis at $x = 1$ and at $x = 3$. Hence the graph of the parabola and the region bounded by the parabola and the X-axis are as shown in Fig. 7.15.

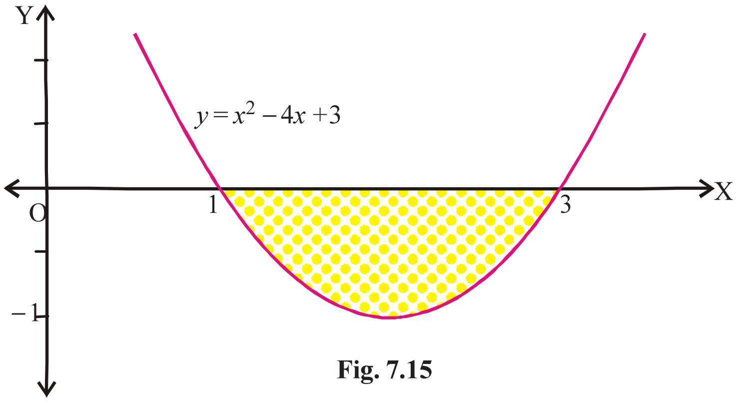


Fig. 7.15

The required area A is therefore given by

$$A = \int_1^3 -(x^2 - 4x + 3)dx = \int_1^3 [1 - (x-2)^2]dx$$

$$= 2 - \frac{(x-2)^3}{3}\Bigg]_1^3 = 2 - \frac{2}{3} = \frac{4}{3}..$$

5. Problem: *Find the area bounded by the curves $y = \sin x$ and $y = \cos x$ between any two consecutive points of intersection.*

Solution: Two consecutive points of intersection of the curves $y = \sin x$ and $y = \cos x$ are $x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$. Also, we have $\sin x \geq \cos x$ for all $x \in \left[\frac{\pi}{4}, \frac{5\pi}{4}\right]$. Hence, the area bounded by these curves and the ordinates $x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$ is as shown in Fig. 7.16.

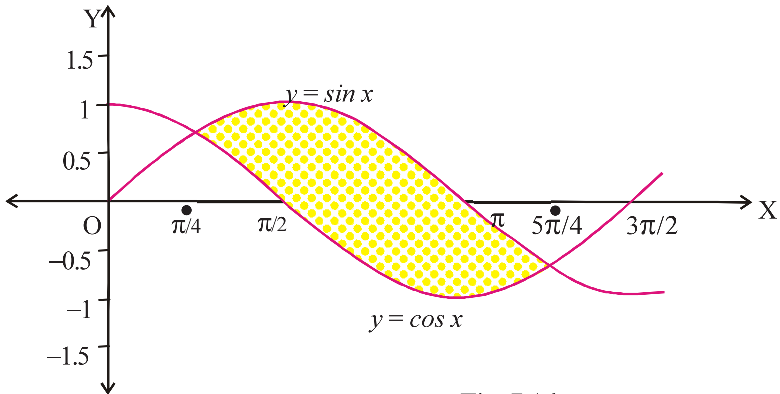


Fig. 7.16

The area A required is therefore given by

$$A = \int_{\pi/4}^{5\pi/4} (\sin x - \cos x)\, dx = (-\cos x - \sin x\Big]_{\pi/4}^{5\pi/4}$$

$$= \sqrt{2} + \sqrt{2} = 2\sqrt{2}.$$

6. Problem: *Find the area of one of the curvilinear triangles bounded by $y = \sin x$, $y = \cos x$ and X-axis.*

Solution: OAB is one of the curvilinear triangles bounded by $y = \sin x$, $y = \cos x$ and the X-axis.

The area of this curvilinear triangle is as shown in Fig. 7.17. Since $\cos x \geq \sin x$ for $x \in \left[0, \frac{\pi}{4}\right]$ and $\cos x \leq \sin x$ for $x \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right]$, the required area A is given by

$$A = \int_0^{\pi/4} \sin x \, dx + \int_{\pi/4}^{\pi/2} \cos x \, dx = -\cos x\Big]_0^{\pi/4} + \sin x\Big]_{\pi/4}^{\pi/2}$$

$$= \left(1 - \frac{1}{\sqrt{2}}\right) + \left(1 - \frac{1}{\sqrt{2}}\right) = (2 - \sqrt{2}).$$

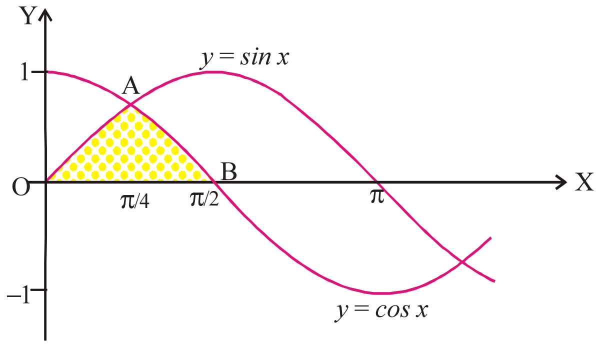


Fig. 7.17

7. Problem: *Find the area of the right angled triangle with base b and altitude h, using the fundamental theorem of integral calculus.*

Solution: Let OAB be a right angled triangle and $\underline{|B} = 90^0$. Choose O as the origin and $\overrightarrow{OB}$ as the positive X-axis. If $OB = b$ and $AB = h$ then $A = (b, h)$. (See Fig. 7.18). So, the equation of $\overrightarrow{OA}$ is $y = \left(\frac{h}{b}\right)x$.

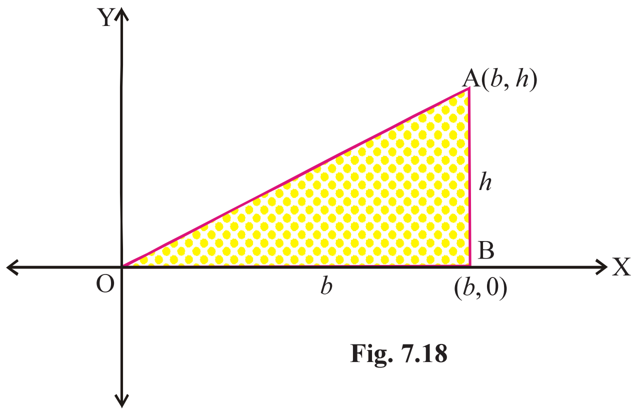


Fig. 7.18

Hence, the area A of the triangle is equal to the area bounded by the lines $x = 0$, $y = h$ and $y = \left(\frac{h}{b}\right)x$.

$$\text{So, } A = \int_0^h \frac{hx}{b} dx = \frac{h}{b}\left(\frac{x^2}{2}\right)\Bigg]_0^h = \frac{1}{2}bh.$$

8. Problem: *Find the area bounded between the curves* $y^2 - 1 = 2x$ *and* $x = 0$.

Solution: The parabola $y^2 - 1 = 2x$ meets the X-axis at $x = -\frac{1}{2}$ and Y-axis at $y = 1$ and $y = -1$. As the curve is symmetric about the X-axis, the area bounded by the curve and the Y-axis is as shown in Fig. 7.19.

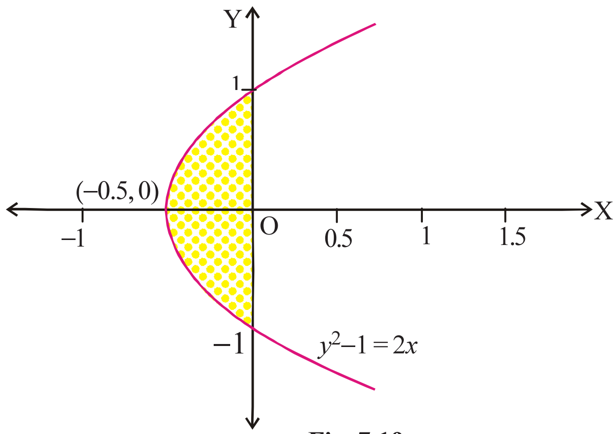


Fig. 7.19

Hence, the required area

$$A = \int_{-1}^{1} (-x)\,dy = \int_{-1}^{1} -\left(\frac{y^2 - 1}{2}\right)dy = \int_{0}^{1} -\left(y^2 - 1\right)dy$$

$$= 1 - \frac{1}{3} = \frac{2}{3}.$$

9. Problem: *Find the area enclosed by the curves* $y = 3x$ *and* $y = 6x - x^2$.

Solution: The straight line $y = 3x$ meets the parabola $y = 6x - x^2$ at the points whose x-coordinates are given by $3x = 6x - x^2$ i.e., $x = 0, 3$. Since $3x \le 6x - x^2$ for all $x \in [0, 3]$, the area enclosed by the parabola and the given line is as shown in Fig.7.20.

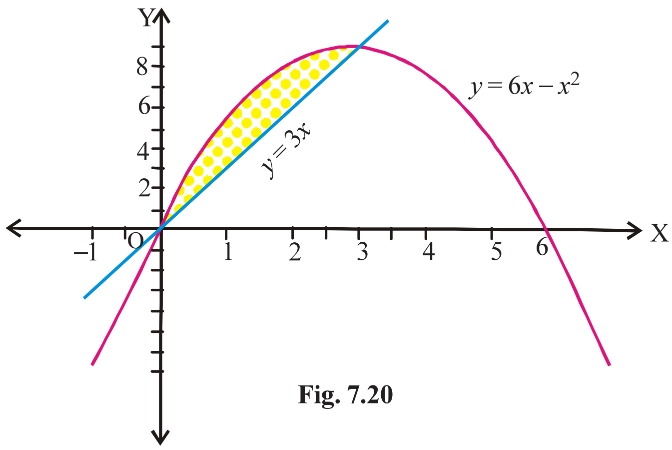


Fig. 7.20

Hence, the required area A is given by.

$$A = \int_{0}^{3} \left[\left(6x - x^2\right) - 3x\right]dx = \int_{0}^{3} \left(3x - x^2\right)dx$$

$$= \left(\frac{3x^2}{2} - \frac{x^3}{3}\right)\Bigg]_0^3 = \frac{27}{2} - \frac{27}{3} = \frac{27}{6} = \frac{9}{2}.$$

10. Problem: *Find the area enclosed between* $y = x^2 - 5x$ *and* $y = 4 - 2x$.

Solution: The points of intersection of the parabola and straight line are given by $y = x^2 - 5x = 4 - 2x$ (i.e.,) $x^2 - 3x - 4 = 0$. Hence $x = 4, -1$.

Also $-1 \le x \le 4 \Rightarrow (x+1)(x-4) \le 0$

$\Rightarrow x^2 - 3x - 4 \le 0$

$\Rightarrow x^2 - 5x \le 4 - 2x$.

So, the area enclosed by the parabola and the straight line is as shown in Fig. 7.21.

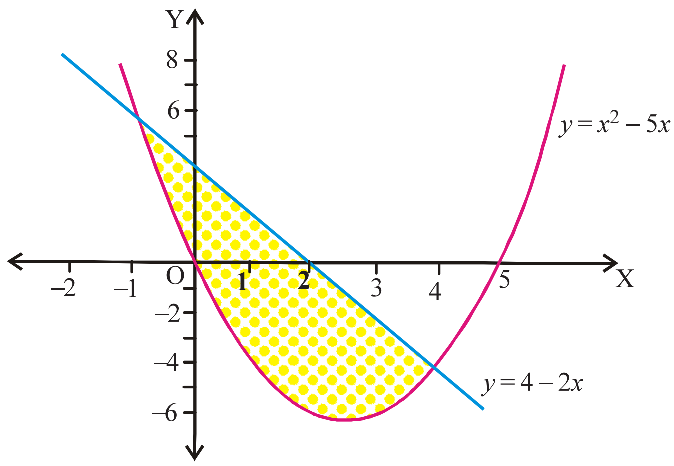


Fig. 7.21

Hence, the required area

$$A = \int_{-1}^{4} [(4-2x) - (x^2 - 5x)]\,dx$$

$$= \int_{-1}^{4} (4 + 3x - x^2)\,dx = \left[4x + 3\left(\frac{x^2}{2}\right) - \frac{x^3}{3}\right]_{-1}^{4}$$

$$= 20 + \frac{45}{2} - \frac{65}{3} = \frac{125}{6}.$$

11. Problem: *Find the area bounded between the curves* $y = x^2$, $y = \sqrt{x}$.

Solution: $x^2 = \sqrt{x} \Leftrightarrow x^4 = x$

$\Leftrightarrow x(x^3 - 1) = 0$

$\Leftrightarrow x = 0$ or 1.

Hence, the two curves intersect at $(0, 0)$ and $(1, 1)$. Also $y = \sqrt{x}$ is the branch of the parabola $y^2 = x$ that lies in the first quadrant. Further $\sqrt{x} \ge x^2 \;\; \forall x \in [0, 1]$. So the area enclosed by the two curves is as shown in Fig. 7.22.

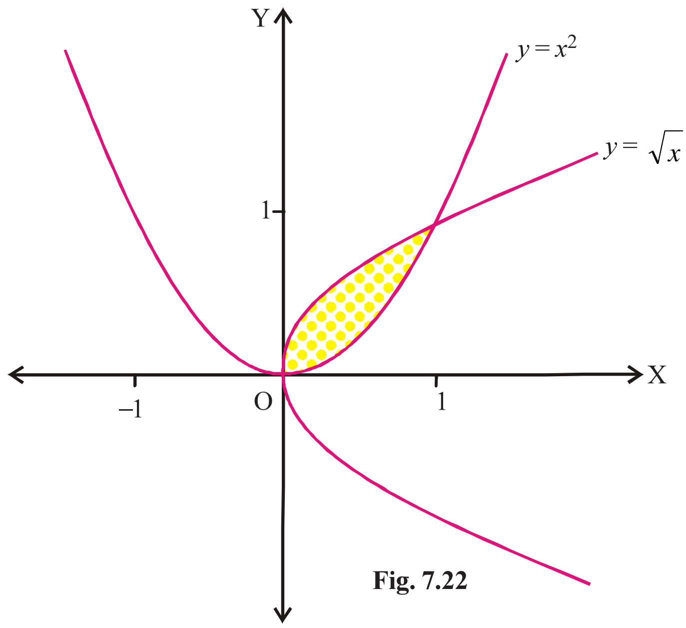


Fig. 7.22

The required area

$$A = \int_0^1 (\sqrt{x} - x^2)\,dx = \left[\frac{2}{3}x^{\frac{3}{2}} - \frac{x^3}{3}\right]_0^1 = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}.$$

12. Problem: *Find the area bounded between the curves* $y^2 = 4ax,\ x^2 = 4by\ (a > 0, b > 0)$.

Solution: First we find the points of intersection of the given curves.

$$\left(\frac{x^2}{4b}\right)^2 = 4ax \Leftrightarrow x = 0 \text{ or } x^3 = 64ab^2$$

$$\Leftrightarrow x = 0 \text{ or } x = 4a^{1/3}b^{2/3}$$

Therefore, the two curves intersect in $(0, 0)$, $\left(4a^{1/3}b^{2/3},\ 4a^{2/3}b^{1/3}\right)$. Further $2\sqrt{ax} \geq \frac{x^2}{4b}$ $\forall x \in [0, 4a^{1/3}b^{2/3}]$. So the area enclosed by the two parabolas is as shown in the Fig. 7.23.

$$\text{Required area A} = \int_0^{4a^{\frac{1}{3}}b^{\frac{2}{3}}} \left(2\sqrt{a}\sqrt{x} - \frac{x^2}{4b}\right)dx$$

$$= \left[(2\sqrt{a})\frac{2}{3}x^{3/2} - \frac{x^3}{12b}\right]_0^{4a^{1/3}b^{2/3}}$$

$$= \frac{32ab}{3} - \frac{16ab}{3}$$

$$= \frac{16ab}{3}.$$

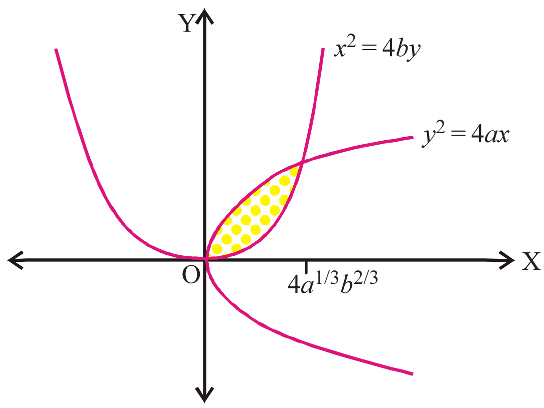


Fig. 7.23

Exercise 7(d)

I. Find the area of the region enclosed by the given curves.

1. $y = \cos x,\ y = 1 - \frac{2x}{\pi}$.
2. $y = \cos x,\ y = \sin 2x,\ x = 0,\ x = \frac{\pi}{2}$.
3. $y = x^3 + 3,\ y = 0,\ x = -1,\ x = 2$.
4. $y = e^x,\ y = x,\ x = 0,\ x = 1$.
5. $y = \sin x,\ y = \cos x,\ x = 0,\ x = \frac{\pi}{2}$.
6. $x = 4 - y^2,\ x = 0$.
7. Find the area enclosed within the curve $|x| + |y| = 1$.

II. 1. $x = 2 - 5y - 3y^2,\ x = 0$.

2. $x^2 = 4y,\ x = 2,\ y = 0$.

3. $y^2 = 3x,\ x = 3$.

4. $y = x^2,\ y = 2x$.

5. $y = \sin 2x,\ y = \sqrt{3}\sin x,\ x = 0,\ x = \frac{\pi}{6}$.

6. $y = x^2,\ y = x^3$.

7. $y = 4x - x^2,\quad y = 5 - 2x$.

8. Find the area in sq. units bounded by the X-axis, part of the curve $y = 1 + \frac{8}{x^2}$ and the ordinates $x = 2$ and $x = 4$.

9. Find the area of the region bounded by the parabolas $y^2 = 4x$ and $x^2 = 4y$.

10. Find the area bounded by the curve $y = ln\, x$, the X-axis and the straight line $x = e$.

III. 1. $y = x^2 + 1,\ y = 2x - 2,\ x = -1,\ x = 2$.

2. $y^2 = 4x,\ y^2 = 4(4 - x)$.

3. $y = 2 - x^2,\ y = x^2$.

4. Show that the area enclosed between the curves $y^2 = 12(x+3)$ and $y^2 = 20(5-x)$ is $64\sqrt{\frac{5}{3}}$.

5. Find the area of the region $\{(x, y) : x^2 - x - 1 \le y \le -1\}$.

6. The circle $x^2 + y^2 = 8$ is divided into two parts by the parabola $2y = x^2$. Find the area of both the parts.

7. Show that the area of the region bounded by $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (ellipse) is πab. Also deduce the area of the circle $x^2 + y^2 = a^2$.

8. Find the area of the region enclosed by the curves $y = \sin \pi x,\ y = x^2 - x,\ x = 2$.

9. Let AOB be the positive quadrant of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with $OA = a,\ OB = b$. Then show that the area bounded between the chord AB and the arc AB of the ellipse is $\frac{(\pi - 2)ab}{4}$.

10. Prove that the curves $y^2 = 4x$ and $x^2 = 4y$ divide the area of the square bounded by the lines $x = 0,\ x = 4,\ y = 4$ and $y = 0$ into three equal parts.

Key Concepts

- **The Definite Integral :** Let $f : [a, b] \to \mathbf{R}$ be a bounded function. Let $P = \{x_0, x_1, ..., x_{i-1}, x_i, x_{i+1}, ..., x_n\}$ be a partition of $[a, b]$ and let $t_i \in [x_{i-1}, x_i]$ for each $i = 1, 2, ..., n$. A sum of the form

$$S(P, f) = \sum_{i=1}^{n} f(t_i)\Delta x_i$$

is called a *Riemann sum of f*.

We say that f is *Riemann integrable on* $[a, b]$ (or simply integrable on $[a, b]$) if there exists a real number A such that $S(P, f)$ approaches A as $\|P\|$ approaches zero. Such an A, if exists, is unique and is denoted by $\int_a^b f(x)\,dx$. We call $\int_a^b f(x)\,dx$, *the definite integral* of f from a to b.

- If f is continuous on $[0, 1]$ and $P = \left\{0, \frac{1}{n}, \frac{2}{n}, ..., \frac{n-1}{n}, 1\right\}$ is a partition of $[0, 1]$ into n subintervals each of length $\frac{1}{n}$, then $\int_0^1 f(x)\,dx = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} f\left(\frac{i}{n}\right)$

More generally, if f is continuous on $[0, p]$ where p is a positive integer then,

$$\int_0^p f(x)\,dx = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{np} f\left(\frac{i}{n}\right).$$

- **First Fundamental Theorem of Integral Calculus :** Let f be integrable on $[a, b]$. We write $A(x) = \int_a^x f(t)\,dt$, $x \in [a, b]$. Then A is continuous on $[a, b]$. If f is continuous on $[a, b]$ then A is differentiable in $[a, b]$. Further, $A'(x) = f(x)$ for all $x \in [a, b]$.

- **The Fundamental Theorem of Integral Calculus :** If f is integrable on $[a, b]$ and if there is a differentiable function F on $[a, b]$ such that $F' = f$ then

$$\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a).$$

- If $f : [a, b] \to \mathbf{R}$ is integrable, then $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$ and $\int_c^c f(x)\,dx = 0 \;\forall\; c \in [a, b]$.

- If f and g are integrable on $[a, b]$, then $f+g$ is integrable on $[a, b]$ and

$$\int_a^b (f+g)(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

- Let $f:[a,b] \to \mathbf{R}$ be integrable and $\alpha \in \mathbf{R}$. Then αf is integrable on $[a, b]$ and

$$\int_a^b (\alpha f)(x)\,dx = \alpha \int_a^b f(x)\,dx.$$

- Let $f:[a,b] \to \mathbf{R}$ be bounded. Let $c \in (a, b)$. Then f is integrable on $[a, b]$ if and only if it is integrable on $[a, c]$ as well as on $[c, b]$ and, in this case,

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

- **Method of Substitution :** Let $g:[c,d] \to \mathbf{R}$ have continuous derivative on $[c, d]$. Let $f: g([c,d]) \to \mathbf{R}$ be continuous. Then $(fog)g'$ is integrable on $[c, d]$ and

$$\int_{g(c)}^{g(d)} f(t)\,dt = \int_c^d f(g(x))\,g'(x)\,dx.$$

- Let f be integrable on $[a, b]$. Then the function g defined on $[a, b]$ as

  $g(x) = f(a+b-x)$ for all x in $[a, b]$, is integrable on $[a, b]$ and

$$\int_a^b g(x)\,dx = \int_a^b f(x)\,dx.$$

- If f is integrable on $[0, a]$, then the function g defined on $[0, a]$ as $g(x) = f(a-x)$ for all x in $[0, a]$ is integrable on $[0, a]$ and

$$\int_0^a f(a-x)\,dx = \int_0^a g(x)\,dx = \int_0^a f(x)\,dx.$$

- Let $f:[-a,a] \to \mathbf{R}$ be integrable on $[0, a]$. Suppose that f is either odd or even. Then f is integrable on $[-a, a]$ and

$$\int_{-a}^{a} f(x)\,dx = \begin{cases} 0, \text{ if } f \text{ is odd.} \\ 2\int_0^a f(x)dx, \text{ if } f \text{ is even.} \end{cases}$$

- Let $f:[0,2a] \to \mathbf{R}$ be integrable on $[0, a]$. Suppose that $f(2a-x) = f(x)$ for all x in $[a, 2a]$. Then f is integrable on $[0, 2a]$ and

$$\int_0^{2a} f(x)\, dx = 2\int_0^{a} f(x)\, dx.$$

❖ Let $f : [0, 2a] \to \mathbf{R}$ be integrable on $[0, a]$. Suppose that $f(2a-x) = -f(x)$ for all x in $[a, 2a]$. Then f is integrable on $[0, 2a]$ and

$$\int_0^{2a} f(x)\, dx = 0.$$

❖ **Integration by parts formula :** Let u and v be real valued differentiable functions on $[a, b]$ such that u' and v' are integrable on $[a, b]$. Then uv' and $u'v$ are integrable on $[a, b]$ and

$$\int_a^b u(x)\, v'(x)\, dx = u(b)\, v(b) - u(a)\, v(a) - \int_a^b u'(x)\, v(x)\, dx.$$

❖ If n is an integer ≥ 2, then

$$\int_0^{\frac{\pi}{2}} \sin^n x\, dx = \begin{cases} \dfrac{n-1}{n}\cdot\dfrac{n-3}{n-2}\cdots\dfrac{1}{2}\cdot\dfrac{\pi}{2}, & \text{if } n \text{ is even} \\[2mm] \dfrac{n-1}{n}\cdot\dfrac{n-3}{n-2}\cdots\dfrac{2}{3}, & \text{if } n \text{ is odd.} \end{cases}$$

❖ $\displaystyle\int_0^{\frac{\pi}{2}} \sin^n x\, dx = \int_0^{\frac{\pi}{2}} \cos^n x\, dx$, n is a positive integer.

❖ If m and n are positive integers then

$$\int_0^{\frac{\pi}{2}} \sin^m x \cos^n x\, dx = \begin{cases} \dfrac{1}{m+1}, & \text{if } n = 1 \\[2mm] \dfrac{n-1}{m+n}\cdot\dfrac{n-3}{m+n-2}\cdots\dfrac{2}{m+3}\cdot\dfrac{1}{m+1}, & \text{if } 1 \neq n \text{ is odd} \\[2mm] \dfrac{n-1}{m+n}\cdot\dfrac{n-3}{m+n-2}\cdots\dfrac{1}{m+2}\cdot\dfrac{m-1}{m}\cdots\dfrac{1}{2}\cdot\dfrac{\pi}{2}, & \text{if } n \text{ is even and } m \text{ is even} \\[2mm] \dfrac{n-1}{m+n}\cdot\dfrac{n-3}{m+n-2}\cdots\dfrac{1}{m+2}\cdot\dfrac{m-1}{m}\cdots\dfrac{2}{3}, & \text{if } n \text{ is even and } 1 \neq m \text{ is odd} \\[2mm] \dfrac{1}{n+1}, & \text{if } m = 1. \end{cases}$$

❖ Area A of the region bounded by the curve $y = f(x)$, the X-axis, $x = a$ and $x = b$ is given by

$$A = \begin{cases} \int_a^b f(x)dx, \text{ if } f(x) \geq 0 \ \forall x \in [a, b] \\ -\int_a^b f(x)dx, \text{ if } f(x) \leq 0 \ \forall x \in [a, b] \end{cases}$$

❖ Area A of the region bounded by the curve $x = g(y)$, the Y-axis and the lines $y = c,\ y = d$ is given by

$$A = \begin{cases} \int_c^d x\,dy \text{ or } \int_c^d g(y)dy, \text{ if } g(y) \geq 0 \ \forall y \in [c, d] \\ -\int_c^d x\,dy \text{ or } -\int_c^d g(y)dy, \text{ if } g(y) \leq 0 \ \forall y \in [c, d] \end{cases}$$

❖ The area of the region enclosed between the curves $y = f(x),\ y = g(x)$ and the lines $x = a,\ x = b$ is given by the formula : Area $= \int_a^b [f(x) - g(x)]dx$ if $f(x) \geq g(x)$ for all $x \in [a, b]$.

❖ If $f(x) \geq g(x)$ in $[a, c]$ and $f(x) \leq g(x)$ in $[c, b]$, $a < c < b$, then area bounded by the two curves $y = f(x),\ y = g(x)$ and the lines $x = a,\ x = b$ is

$$= \int_a^c [f(x) - g(x)]dx + \int_c^b [g(x) - f(x)]dx.$$

Historical Note

Integral Calculus is the study of the definition, properties and applications of two related concepts : the indefinite integral and the definite integral.

The indefinite integral is the antiderivative. The definite integral inputs a function and outputs a number. The technical definition of the definite integral is the limit of a sum of areas of rectangles, called *Riemann* sum, as propounded by *Riemann*.

The symbol for indefinite integration was introduced by *Leibnitz.* The notation for definite integral *i.e.,* $\int_a^b f(x)\ dx$ was proposed by *Fourier* and *Cauchy* immediately adopted and popularised it.

Answers

Exercise 7(a)

I. 1. $\frac{35}{2}$ 2. $\frac{64}{3}$

II. 1. $\frac{15+e^8}{2}$ 2. $\frac{1}{6}$

Exercise 7(b)

I. 1. $\frac{a^4}{4}$ 2. $\log 2$ 3. 4 4. 0

5. 1 6. 1 7. $\sqrt{3}-1$ 8. $\frac{a^2}{6}$

9. $\frac{4}{3}$ 10. 1 11. $\frac{e-1}{2e}$ 12. 2

II. 1. $4+\log 5$ 2. $3+\sqrt{2}\left[\text{Tan}^{-1}\left(-\frac{1}{\sqrt{2}}\right)-\text{Tan}^{-1}(\sqrt{2})\right]$ 3. $1-\frac{\pi}{4}$

4. $\pi-2$ 5. 4 6. $\frac{\pi}{4}$ 7. 0

8. $\frac{2(2\sqrt{2}-1)}{3}$ 9. $\ln 6$ 10. $\frac{2}{\pi}\ln 2$ 11. $\frac{1}{4}\ln 2$

12. $\ln\sqrt{2}$ 13. $\frac{1}{5}$ 14. $2e^{\frac{\pi-4}{2}}$ 15. $\frac{1}{e}$

III. 1. $\frac{1}{3}\log 2$ 2. $\frac{\pi}{8}(b-a)^2$ 3. $\frac{1}{2}\left[1-\frac{\sqrt{3}}{6}\pi\right]$ 4. $\frac{1}{20}\log 3$

5. $(a+b)\frac{\pi}{4}$ 6. $\frac{a^{n+2}}{(n+1)(n+2)}$ 7. $\frac{16\sqrt{2}}{15}$ 8. $\frac{2}{3}\pi$

9. π 10. $\frac{\pi}{2}(\pi-2)$ 11. $\frac{\pi}{8}(\log 2)$ 12. $\frac{\pi^2}{4}$

13. $\frac{1}{\sqrt{2}}\log(\sqrt{2}+1)$ 14. $\frac{\pi}{\sqrt{5}}$ 15. $\frac{\pi}{8}\log 2$ 16. $\frac{3}{\pi}+\frac{1}{\pi^2}$

17. $\frac{\pi}{2}-\log 2$ 18. $\frac{\pi}{4}-\frac{1}{2}$ 19. $\frac{\pi^2}{4}$

Exercise 7(c)

I. 1. $\frac{63}{512}\pi$ 2. $\frac{256}{693}$ 3. $\frac{16}{315}$ 4. $\frac{3}{256}\pi$

5. $\frac{4}{63}$ 6. $\frac{\pi}{8}$ 7. $\frac{32}{315}$ 8. 0

9. $\frac{a^9}{9}$ 10. $\frac{\pi}{2}$

II. 1. $\frac{512}{153153}$ 2. $640\,\pi$ 3. $\frac{243\pi}{8}$ 4. $\frac{2.5^{11}}{99}$

5. 0 6. 2π 7. 2π 8. $\frac{1}{24}$

III. 1. $\frac{5\pi}{2048}$ 2. $\frac{5\pi}{2}$ 3. π 4. $\frac{16(5^{13/2})}{1287}$

5. 0

Exercise 7(d)

I. 1. $2-\frac{\pi}{2}$ 2. $\frac{1}{2}$ 3. $\frac{51}{4}$ 4. $e-\frac{3}{2}$

5. $2\left(\sqrt{2}-1\right)$ 6. $\frac{32}{3}$ 7. 2

II. 1. $\frac{343}{54}$ 2. $\frac{2}{3}$ 3. 12 4. $\frac{4}{3}$

5. $\frac{7}{4}-\sqrt{3}$ 6. $\frac{1}{12}$ 7. $\frac{32}{3}$ 8. 4

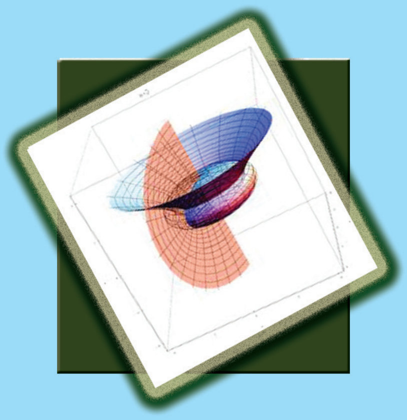
9. $\frac{16}{3}$ 10. 1

III. 1. 9 2. $\frac{32\sqrt{2}}{3}$ 3. $\frac{8}{3}$ 5. $\frac{1}{6}$

6. $2\pi+\frac{4}{3},\ 6\pi-\frac{4}{3}$. 8. $\frac{5}{6}+\frac{2}{\pi}$

Chapter 8

Differential Equations



"Among all the mathematical disciplines, the theory of differential equations is the most important. It furnishes the explanation of all those elementary manifestations of nature which involve time" - ***Sophus Lie***

Introduction

Differential equations have applications in many branches of physics, physical chemistry etc.

In this chapter we study some basic concepts of differential equations and learn how to solve simple differential equations.

8.1 Formation of differential equations - Degree and order of an ordinary differential equation

The present section is aimed at defining an ordinary differential equation, forming such an equation from a given firmly of curves or surfaces. We also define two concepts, namely order and degree of an ordinary differential equation.

8.1.1 Definition

An equation involving one dependent variable and its derivatives w.r.t. one or more independent variables is called a differential equation.

Claude Alexis Clairaut
(1713 - 1765)

Clairaut was a French mathematician and thinker. He was a prodigy at the age of twelve and wrote a memoir on four geometrical curves, and at the age of sixteen he wrote a treatise on tortuous curves which on publication in 1731 got him admission into French Academy of Sciences. He wrote various papers on the orbits of moon and planets particularly on the path of Halley's Comet.

If a differential equation contains only one independent variable, then it is called an *ordinary differential equation* and if it contains more than one independent variable, then it is called a *partial differential equation*. Hence an ordinary differential equation contains only ordinary derivatives whereas a partail differential equation contains partial derivatives.

Since derivative is a rate of change, it is only natural that differential equations arise in the description of change in state or motion. Differential equations occur in problems of radioactive decay, Newton's Law of cooling, chemical reactions, the motion of a particle or a planet, the motion of springs to electric circuits and population dynamics.

8.1.2 Examples

(i) $\dfrac{dy}{dx} + 5x = \cos x.$

(ii) $\dfrac{dy}{dx} = kx$ (k being a constant)

(iii) $\left(\dfrac{d^2y}{dx^2}\right)^2 - 3\left(\dfrac{dy}{dx}\right)^3 - e^x = 4.$

(iv) $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = z.$

(v) $\dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} + \dfrac{\partial^2 w}{\partial z^2} = 0.$

8.1.3 Note

1. (i), (ii) and (iii) of 8.1.2 are examples of ordinary differential equations in which y is the dependent variable and x is the independent variable.

2. (iv) and (v) of 8.1.2 are examples of partial differential equations. In (iv), z is the dependent variable and x and y are independent variables whereas in (v) w is the dependent variable and x, y and z are independent variables.

   In this chapter, we study only ordinary differential equations.

8.1.4 Order and Degree of a differential equation

Order : The **order** of a differential equation is the order of the highest order derivative occuring in it.

Degree : The **degree** of a differential equation is the highest power of the highest order derivative appearing in the equation after the equation is written free from radicals and fractions as far as derivatives are concerned.

8.1.5 Examples

1. Order and degree of $\frac{dy}{dx} = \frac{x^{1/2}}{y^{1/2}(1+x^{1/2})}$ are 1 and 1.

2. $\frac{d^2y}{dx^2} = \left[1+\left(\frac{dy}{dx}\right)^2\right]^{5/3}$ has order 2 and degree 3, since the equation can be expressed as a polynomial equation in the derivatives as $\left(\frac{d^2y}{dx^2}\right)^3 = \left[1+\left(\frac{dy}{dx}\right)^2\right]^5$. In this the exponent of the highest order derivative $\frac{d^2y}{dx^2}$ is 3 and hence the degree is 3.

3. The order and degree of $1+\left(\frac{d^2y}{dx^2}\right)^2 = \left[2+\left(\frac{dy}{dx}\right)^2\right]^{3/2}$ are 2 and 4 respectively, since it can be expressed in the from $\left[1+\left(\frac{d^2y}{dx^2}\right)^2\right]^2 = \left[2+\left(\frac{dy}{dx}\right)^2\right]^3$.

4. Order of $\frac{d^2y}{dx^2}+2\frac{dy}{dx} + y = \log\left(\frac{dy}{dx}\right)$ is 2 and degree is not defined since the equation cannot be expressed as a polynomial equation in the derivatives.

5. Order and degree of $\left[\left(\frac{dy}{dx}\right)^{1/2} + \left(\frac{d^2y}{dx^2}\right)^{1/3}\right]^{1/4} = 0$ are 2 and 2, since the equation can be written as $\left(\frac{dy}{dx}\right)^3 = \left(\frac{d^2y}{dx^2}\right)^2$.

8.1.6 Note : The general form of an ordinary differential equation of n^{th} order is

$$F\left(x,\ y,\ \frac{dy}{dx},\ ...,\ \frac{d^n y}{dx^n}\right) = 0$$

or $$F(x, y, y^{(1)}, ..., y^{(n)}) = 0.$$

8.1.7 Formation of a differential equation : Suppose that an equation

$$y = \phi(x,\ \alpha_1,\ \alpha_2,\ ...,\ \alpha_n) \qquad ...(1)$$

where $\alpha_1, \alpha_2, ..., \alpha_n$ are parameters (or arbitrary constants), representing a family of curves is given. Then by successively differentiating (1), a differential equation of the form

$$F(x, y, y^{(1)}, ..., y^{(n)}) = 0 \qquad ... (2)$$

can be formed by eliminating the parameters $\alpha_1, \alpha_2, ..., \alpha_n$. This process is called the formation of differential equation (2) satisfying the family of curves (1). For example, we know that

$$y = mx \qquad ... (3)$$

represents a family of straight lines passing through the origin. This can be represented by $\phi(x, m) = y$, where $\phi(x, m) = mx$. Infact for different values of the parameters m, we get different straight lines of the family. Hence by eliminating m from (3), that is, $y = \phi(x, m)$ we get the required differential equation. Differentiating (3) w.r.t. x, we get

$$\frac{dy}{dx} = m. \qquad ... (4)$$

Substituting the value of m from (4) in (3), we get

$$y = \left(\frac{dy}{dx}\right)x$$

i.e., $$F\left(x, y, \frac{dy}{dx}\right) = y - \left(\frac{dy}{dx}\right)x = 0. \qquad ... (5)$$

Hence (5) is a differential equation whose solution set represents the family of straight lines (3).

8.1.8 Solved Problems

1. Problem: *Find the order and degree of the differential equation* $\frac{d^2y}{dx^2} = -p^2 y.$

Solution : The given equation is a polynomial equation in $\frac{d^2y}{dx^2}$. Hence the degree is 1. Since $\frac{d^2y}{dx^2}$ is the highest derivative occuring in the equation, its order is 2.

2. Problem: *Find the order and degree of* $\left(\frac{d^3y}{dx^3}\right)^2 - 3\left(\frac{dy}{dx}\right)^2 - e^x = 4.$

Solution: The equation is a polynomial equation in $\frac{dy}{dx}$ and $\frac{d^3y}{dx^3}$. The exponent of $\frac{d^3y}{dx^3}$ is 2.

Hence the degree is 2, since $\frac{d^3y}{dx^3}$ is the highest order derivative occuring in the equation, the order of the equation is 3.

3. Problem: $x^{1/2}\left(\frac{d^2y}{dx^2}\right)^{1/3} + x\frac{dy}{dx} + y = 0$ *has order 2 and degree 1. Prove*

Solution: The given equation can be written as

$$x^{3/2}\frac{d^2y}{dx^2} = -\left(x\frac{dy}{dx}+y\right)^3,$$

the order and degree of the equation are 2 and 1 respectively.

4. Problem: *Find the order and degree of* $\left[\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^3\right]^{6/5} = 6y.$

Solution: The given equation can be written as

$$\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 = (6y)^{5/6}.$$

Hence the order and degree of the equation are 2 and 1 respectively.

5. Problem : *Find the order of the differential equation corresponding to $y = c(x-c)^2$, where c is an arbitrary constant.*

Solution : The differential equation of $y = c(x-c)^2$ is obtained by eliminating c from

$y = c(x-c)^2$ and $\frac{dy}{dx} = 2c(x-c)$.

Hence the order of the differential equation is 1.

6. Problem : *Find the order of the differential equation corresponding to $y = Ae^x + Be^{3x} + Ce^{5x}$ (A, B, C being parameters) is a solution.*

Solution: Required differential equation is obtained by eliminating A, B, C, from y,

$$\frac{dy}{dx}, \frac{d^2y}{dx^2} \text{ and } \frac{d^3y}{dx^3}.$$

Of these the highest order derivative is $\frac{d^3y}{dx^3}$.

Hence the order of the differential equation is 3.

7. Problem : *Form the differential equation corresponding to $y = cx - 2c^2$, where c is a parameter.*

Solution: We have $y = cx - 2c^2$ ... (1)

Differentiating (1) w.r.t. x, we get

$$\frac{dy}{dx} = c \quad \text{... (2)}$$

Substituting the value of c from (2) in (1), we get

$$y = x\left(\frac{dy}{dx}\right) - 2\left(\frac{dy}{dx}\right)^2 \quad \text{... (3)}$$

Hence (3) is the differential equation corresponding to (1).

8. Problem : *Form the differential equation corresponding to $y = A\cos 3x + B\sin 3x$, where A and B are parameters.*

Solution: We have $y = \mathrm{A}\cos 3x + \mathrm{B}\sin 3x$ ... (1)

Differentiating (1) w.r.t. x, we get

$$\frac{dy}{dx} = -3\mathrm{A}\sin 3x + 3\mathrm{B}\cos 3x \quad \text{... (2)}$$

Again differentiating (2) w.r.t. x, we get

$$\frac{d^2y}{dx^2} = -9\mathrm{A}\cos 3x - 9\,\mathrm{B}\sin 3x$$
$$= -9(\mathrm{A}\cos 3x + \mathrm{B}\sin 3x)$$
$$= -9y.$$

That is, $$\frac{d^2y}{dx^2} + 9y = 0. \quad \text{... (3)}$$

Hence (3) is the required differential equation.

Alternate Method

Eliminating A, B from the equations

$$y = \mathrm{A}\cos 3x + \mathrm{B}\sin 3x$$
$$\frac{dy}{dx} = -3\mathrm{A}\sin 3x + 3\mathrm{B}\cos 3x$$

and $$\frac{d^2y}{dx^2} = -9\,\mathrm{A}\cos 3x - 9\mathrm{B}\sin 3x,$$

we get

$$\begin{vmatrix} y & -\cos 3x & -\sin 3x \\ \frac{dy}{dx} & 3\sin 3x & -3\cos 3x \\ \frac{d^2y}{dx^2} & 9\cos 3x & 9\sin 3x \end{vmatrix} = 0$$

(ii) $$[27\sin^2 3x + 27\cos^2 3x]y - [-9\sin 3x\cos 3x + 9\sin 3x\cos 3x]\frac{dy}{dx} + [3\cos^2 3x + 3\sin^2 3x]\frac{d^2y}{dx^2} = 0$$

that is, $$27y + 3\frac{d^2y}{dx^2} = 0 \text{ (or) } \frac{d^2y}{dx^2} + 9y = 0.$$

This is the required differential equation.

9. Problem: *Form the differential equation corresponding to the family of circles of radius r given by* $(x-a)^2 + (y-b)^2 = r^2$, *where a and b are parameters.*

Solution: We have $(x-a)^2+(y-b)^2=r^2$ ... (1)

Differentiating (1) w.r.t. x, we get

$$(x-a)+(y-b)\frac{dy}{dx} = 0 \quad \text{... (2)}$$

Again differentiating (2) w.r.t. x, we get

$$1+(y-b)\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2 = 0 \quad \text{... (3)}$$

Eliminating a from (1) and (2), we get

$$(y-b)^2\left[\left(\frac{dy}{dx}\right)^2+1\right] = r^2 \quad \text{... (4)}$$

Eliminating b from (3) and (4), we get

$$r^2\left(\frac{d^2y}{dx^2}\right)^2 = \left[1+\left(\frac{dy}{dx}\right)^2\right]^3$$

which is the required differential equation.

10. Problem : *Form the differential equation corresponding to the family of circles passing through the origin and having centres on Y-axis.*

Solution : The equation of the family of circles passing through the origin and having centres on the Y-axis is

$$x^2+y^2+2hy=0 \quad \text{... (1)}$$

where h is a parameter.

Now differentiating (1) w.r.t. x, we get

$$x+y\frac{dy}{dx}+h\frac{dy}{dx}=0 \quad \text{... (2)}$$

Eliminating h from (1) and (2), we get

$$(x^2-y^2)\frac{dy}{dx}-2xy=0$$

and this is the differential equation corresponding to (1).

8.1.9 Solution of a differential equation

Solution : A solution of a differential equation is a relation between dependent variable, independent variables and along with some arbitrary constants satisfying the differential equation.

General solution : A solution of a differential equation in which the number of arbitrary constants is equal to the order of the differential equation is called the general solution.

Particular solution : A particular solution of a differential equation is a solution obtained by giving particular values to the arbitrary constants in the general solution.

8.1.10 Note

1. If the equation of a given family of curves contains n parameters, then we have to differentiate n times successively to eliminate all the n parameters from it. Hence the order of the differential equation correpsonding to an equation having n parameters is n.
2. We have seen that a differential equation can be formed corresponding to a family of curves. Conversly, the solutions of a differential equation are equations of curves. It is obvious that the solutions of a differential equation of order n contains a family of curves having n parameters. This equation of the family of curves having n parameters is called the general or primitive or complete solution of the given differential equation of order n. By giving particular values to the parameters in the general solution, we get different members of the family of curves. Hence a particular integral or a particular solution is obtained by giving particular values to the parameters in the general solution.
3. (i) We have seen in 8.1.7 that $y = mx$, where m is a parameter, is the general solution of the differential equation $y = \left(\frac{dy}{dx}\right)x$. Hence by giving a particular value to m say $m = 2$, we get that $y = 2x$ is a particular solution of the above differential equation.

   (ii) We get from 8 problem of 8.1.8 that $y = \text{A}\cos 3x + \text{B}\sin 3x$ is the general solution of $\frac{d^2y}{dx^2} + 9y = 0$ whereas $y = \cos 3x + \sin 3x$ is a particular solution which is obtained by taking A = 1 and B = 1 in the general solution.

Exercise 8(a)

I. 1. Find the order of the differential equation obtained by eliminating the arbitrary constants b and c from $xy = c\,e^x + b\,e^{-x} + x^2$.

2. Find the order of the differential equation of the family of all circles with their centres at the origin.

II. 1. Form the differential equations of the following family of curves where parameters are given in brackets.

(i) $y = c(x - c)^2$; (c) (ii) $xy = ae^x + be^{-x}$; (a, b)

(iii) $y = (a + bx)\,e^{kx}$; (a, b) (iv) $y = a\cos(nx + b)$; (a, b)

2. Obtain the differential equation which corresponds to each of the following family of curves.

   (i) The rectangular hyperbolas which have the coordinate axes as asymptotes.

   (ii) The ellipses with centres at the origin and having coordinate axes as axes.

III. 1. Form the differential equations of the following family of curves where parameters are given in brackets:

(i) $y = ae^{3x} + be^{4x}$; (a, b) (ii) $y = ax^2 + bx$; (a, b)

(iii) $ax^2 + by^2 = 1$; (a, b) (iv) $xy = ax^2 + b/x$; (a, b)

2. Obtain the differential equation which corresponds to each of the following family of curves.

(i) The circles which touch the Y-axis at the origin.

(ii) The parabolas each of which has a latus rectum $4a$ and whose axes are parallel to X-axis.

(iii) The parabolas having their focii at the origin and axis along the X-axis.

8.2 Solving Differential Equations

In this section we discuss *methods* to solve some first order first degree differential equations.

Since a first order first degree differential equation contains terms like $\frac{dy}{dx}$ and some terms involving x and y, a general first order first degree differential equation is of the form

$$\frac{dy}{dx} = F(x, y), \text{ where F is a function of } x \text{ and } y.$$

Throughout our discussion in the rest of the chapter, unless otherwise mentioned, a differential equation means a first order first degree ordinary differential equation.

8.2(a) Variables separable method

If a given differential equation can be put in the form

$$f(x)\, dx + g(y)\, dy = 0 \qquad \dots (1)$$

then its solution can be obtained by integrating each term. This method of solving the differential equation is called variables separable method.

For example, consider the differential equation, $x\,dy - y\,dx = 0$. This can be written as $\frac{dx}{x} = \frac{dy}{y}$ so that the given equation is one in which the variables are separable. Integrating w.r.t. x, we get $\int \frac{dx}{x} = \int \frac{dy}{y}$.

Therefore, $\log x = \log y + \log c = \log(yc)$, $c \in \mathbf{R}$ is arbitrary.

That is, $x = yc$ which is the required solution.

Hence if the given differential equation can be reduced to the form (1), then its solution can be obtained by variables separable method.

8.2(a)(i) Solved Problems

1. Problem : *Express the following differential equations in the form f(x) dx + g(y) dy = 0*

(i) $\frac{dy}{dx} = \frac{1 + y^2}{1 + x^2}$ (ii) $y - x\frac{dy}{dx} = a\left(y^2 + \frac{dy}{dx}\right)$

(iii) $\frac{dy}{dx} = e^{x-y} + x^2e^{-y}$ (iv) $\frac{dy}{dx} + x^2 = x^2e^{3y}$

Solution

(i) $\frac{dx}{1+x^2} - \frac{dy}{1+y^2} = 0.$

(ii) The given equation can be written as $y - ay^2 = (x+a)\frac{dy}{dx}$ so that $\frac{dx}{x+a} = \frac{dy}{y-ay^2}.$

(iii) Multiplying both sides by e^y, we get $e^y\frac{dy}{dx} = e^x + x^2$ so that $(e^x+x^2)\,dx - e^y\,dy = 0.$

(iv) The given equation can be written as $\frac{dy}{dx} = x^2(e^{3y}-1)$ so that

$$x^2\,dx + \frac{1}{(1-e^{3y})}dy = 0.$$

2. Problem : *Find the general solution of* $x + y\frac{dy}{dx} = 0.$

Solution: The given equation can be written as $x\,dx + y\,dy = 0.$

Hence $\int x\,dx + \int y\,dy = c$

i.e., $x^2+y^2 = 2c$ is the required solution.

3. Problem : *Find the general solution of* $\frac{dy}{dx} = e^{x+y}.$

Solution: The given equation can be written as $e^x\,dx - e^{-y}\,dy = 0.$

Hence $\int e^x\,dx - \int e^{-y}\,dy = c$ so that $e^x + e^{-y} = c$ is the required solution.

4. Problem : *Solve* $y^2 - x\frac{dy}{dx} = a\left(y + \frac{dy}{dx}\right).$

Solution : Given equation can be written as $y^2 - ay = (x+a)\frac{dy}{dx}$

so that $\frac{dx}{x+a} = \frac{dy}{y(y-a)} = \left[-\frac{1}{ay} + \frac{1}{a(y-a)}\right]dy$ (using partial fractions).

Hence $\int\frac{dx}{x+a} = -\int\frac{dy}{ay} + \int\frac{dy}{a(y-a)}.$

Therefore, $\log(x+a) = -\frac{1}{a}\log(ay) + \frac{1}{a}\log(y-a) + \log c$

i.e., $\log(x+a) = \log\left[c\left(\frac{y-a}{ay}\right)^{1/a}\right].$

Hence $x+a = c\left(\frac{y-a}{ay}\right)^{1/a}$ so that $y = \frac{c^a}{a}\frac{(y-a)}{(x+a)^a}$

which is the required solution of the differential equation.

5. Problem : *Solve* $\frac{dy}{dx} = \frac{y^2+2y}{x-1}$.

Solution: From the given equation $\frac{dy}{y^2+2y} = \frac{dx}{x-1}$. On using partial fractions, we get

$$\left[\frac{1}{2y} - \frac{1}{2(y+2)}\right]dy = \frac{dx}{x-1} \quad \text{i.e.,} \quad \frac{dy}{2y} - \frac{dy}{2(y+2)} = \frac{dx}{x-1}.$$

Therefore, $\int\frac{dy}{2y} - \frac{1}{2}\log(y+2) = \log(x-1) + \log c$

so that $\frac{1}{2}\log y - \frac{1}{2}\log(y+2) = \log(x-1) + \log c$

i.e., $\log\left(\frac{y}{y+2}\right)^{1/2} = \log[c(x-1)].$

Hence $\left(\frac{y}{y+2}\right)^{1/2} = c(x-1)$ or $y = c^2(x-1)^2\,(y+2)$

which is the required solution.

6. Problem : *Solve* $\frac{dy}{dx} = \frac{x(2\log x+1)}{\sin y + y\cos y}$.

Solution: From the given equation $(\sin y + y\cos y)dy = x(2\log x + 1)\,dx$.

Hence, $\int \sin y\; dy + \int y\cos y\; dy = \int 2x\log x\, dx + \int x\; dx$.

Using integration by parts,

$$\int \sin y\; dy + y\sin y - \int \sin y\, dy = x^2\log x - \int x^2\frac{1}{x}\, dx + \int x\, dx + c$$

so that $y\sin y = x^2\log x + c$. This is the required solution.

7. Problem : *Find the equation of the curve whose slope, at any point (x, y), is* $\frac{y}{x^2}$ *and which satisfies the condition* $y = 1$ *when* $x = 3$.

Solution: We know that the slope at any point (x, y) on the curve is $\frac{dy}{dx}$.

Hence, by hypothesis, $\frac{dy}{dx} = \frac{y}{x^2}$ i.e., $\frac{dy}{y} = \frac{dx}{x^2}$.

Hence $\int \frac{dy}{y} = \int \frac{dx}{x^2}$ so that $\log y = -\frac{1}{x} + c$. ... (1)

Therefore, (1) is the equation of the family of curves whose slope at any point is $\frac{y}{x^2}$. Taking $y = 1, x = 3$ in (1) we get $c = \frac{1}{3}$. Hence, $\log y = -\frac{1}{x} + \frac{1}{3}$, that is $y = e^{\frac{x-3}{3x}}$.

This is the required solution.

8. Problem : *Solve* $y(1 + x)dx + x(1 + y)\, dy = 0$.

Solution: The given equation can be written as $\frac{1+x}{x}dx + \frac{1+y}{y}dy = 0$.

Therefore, $\int \frac{1+x}{x}dx + \int \frac{1+y}{y}dy = 0$ or $\log x + x + \log y + y = c$

i.e. $x + y + \log(xy) = c$, which is the required solution.

9. Problem : *Solve* $\frac{dy}{dx} = \sin(x + y) + \cos(x + y)$.

Solution: Put $x + y = t$. Then $1 + \frac{dy}{dx} = \frac{dt}{dx}$.

Therefore, the given equation becomes $\frac{dt}{dx} - 1 = \sin t + \cos t$

i.e., $$dx = \frac{dt}{1 + \cos t + \sin t}$$

i.e., $$dx = \frac{dt}{2\cos^2 \frac{t}{2} + 2\sin\frac{t}{2}\cos\frac{t}{2}} = \frac{\frac{1}{2}\sec^2\frac{t}{2}}{1 + \tan\frac{t}{2}}dt.$$

Therefore, $\int dx = \int \dfrac{\frac{1}{2}\sec^2\frac{t}{2}}{1 + \tan\frac{t}{2}}\, dt.$

Hence $x = \log\left(\left|1+\tan\dfrac{t}{2}\right|\right) + c.$

Since $t = x + y$, we get $x = \log\left[\left|1 + \tan\dfrac{(x+y)}{2}\right|\right] + c$

which is the required solution.

10. Problem : *Solve* $(x-y)^2 \dfrac{dy}{dx} = a^2.$

Solution : Put $x - y = t$. Then $1 - \dfrac{dy}{dx} = \dfrac{dt}{dx}$

(or) $\dfrac{dy}{dx} = 1 - \dfrac{dt}{dx}$ so that the given equation becomes $t^2\left(1 - \dfrac{dt}{dx}\right) = a^2.$

Hence, $\dfrac{dt}{dx} = 1 - \dfrac{a^2}{t^2} = \dfrac{t^2 - a^2}{t^2}$ so that $dx = \dfrac{t^2}{t^2 - a^2}dt = \left[1 + \dfrac{a^2}{t^2 - a^2}\right]dt.$

Therefore, $\int dx = \int dt + \int \dfrac{a^2}{t^2 - a^2}dt$ so that $x = t + a^2 \cdot \dfrac{1}{2a}\log\left|\dfrac{t-a}{t+a}\right| + c.$

Since $t = x - y$, we get $x = x - y + \dfrac{a}{2}\log\left|\dfrac{x-y-a}{x-y+a}\right| + c$

i.e., $y = \dfrac{a}{2}\log\left|\dfrac{x-y-a}{x-y+a}\right| + c$, which is the required solution.

11. Problem : *Solve* $\sqrt{1+x^2}\ \sqrt{1+y^2}\,dx + xy\,dy = 0.$

Solution: The given equation can be written as $\dfrac{\sqrt{1+x^2}}{x}dx + \dfrac{1}{2}\dfrac{2y}{\sqrt{1+y^2}}dy = 0.$

Therefore, $\int \dfrac{\sqrt{1+x^2}}{x}dx + \dfrac{1}{2}\int \dfrac{2y}{\sqrt{1+y^2}}dy = c$

i.e., $\int \dfrac{\sqrt{1+x^2}}{x}dx + \dfrac{1}{2}2\sqrt{1+y^2}\,dy = c$ ... (1)

Now, we evaluate $\int \dfrac{\sqrt{1+x^2}}{x}\,dx.$

$$\int \frac{\sqrt{1+x^2}}{x}dx = \int \frac{1+x^2}{x\sqrt{1+x^2}}\,dx$$

$$= \int \frac{dx}{x\sqrt{1+x^2}} + \int \frac{x}{\sqrt{1+x^2}}\,dx$$

$$= \int \frac{dx}{x\sqrt{1+x^2}} + \frac{1}{2}\int \frac{2x}{\sqrt{1+x^2}}\,dx = \int \frac{dx}{x\sqrt{1+x^2}} + \frac{1}{2}\cdot 2\sqrt{1+x^2}$$

(putting $x=\frac{1}{t}$ in the first integral on the right)

$$= -\int \frac{dt}{\sqrt{1+t^2}} + \sqrt{1+x^2} = -\log\left|t+\sqrt{1+t^2}\right| + \sqrt{1+x^2}$$

$$= -\log\left|\frac{1}{x} + \frac{\sqrt{1+x^2}}{x}\right| + \sqrt{1+x^2}.$$

Hence from (1), required solution is

$$-\log\left|\frac{1}{x} + \frac{\sqrt{1+x^2}}{x}\right| + \sqrt{1+x^2} + \sqrt{1+y^2} = c$$

i.e., $\log|x| - \log\left|1 + \sqrt{1+x^2}\right| + \sqrt{1+x^2} + \sqrt{1+y^2} = c.$

12. Problem : *Solve* $\frac{dy}{dx} = \frac{x-2y+1}{2x-4y}$.

Solution : Put $x-2y=t$. Then $1-2\frac{dy}{dx} = \frac{dt}{dx}$ so that the given equation becomes

$$\frac{1}{2}\left(1-\frac{dt}{dx}\right) = \frac{t+1}{2t}$$

i.e., $\frac{dt}{dx} = 1 - \frac{t+1}{t} = -\frac{1}{t}$

i.e., $t\,dt = -dx$.

Hence, $\int t\,dt = \int -dx$ so that $\frac{t^2}{2} = -x+c$. Since $t = x-2y$, we get

$(x-2y)^2+2x = c$, which is the required solution.

13. Problem : *Solve* $\frac{dy}{dx} = \sqrt{y-x}$.

Solution : Put $y-x = t^2$. Then $\frac{dy}{dx} - 1 = 2t\frac{dt}{dx}$ so that the given equation becomes $2t\frac{dt}{dx} + 1 = t$

i.e., $\frac{2t}{t-1}dt = dx$.

Therefore, $\int \frac{2t}{t-1}dt = \int dx$

i.e., $\int \left(2 + \frac{2}{t-1}\right)dt = x + c$.

Therefore, $2t + 2\log(t-1) = x + c$.

Since $t = \sqrt{y-x}$, we get $2\sqrt{y-x} + 2\log(\sqrt{y-x} - 1) = x + c$

which is the required solution.

14. Problem : *Solve* $\frac{dy}{dx} + 1 = e^{x+y}$.

Solution : Put $x + y = t$. Then $1 + \frac{dy}{dx} = \frac{dt}{dx}$ so that the given equation becomes

$\frac{dt}{dx} = e^t$ and hence $\int \frac{dt}{e^t} = \int dx$. That is, $-e^{-t} = x + c$

i.e., $-e^{-(x+y)} = x + c$ (by substituting for t)

(or) $e^{-(x+y)} + x + c = 0$.

This is the required solution of the given equation.

15. Problem : *Solve* $\frac{dy}{dx} = (3x + y + 4)^2$.

Solution : Put $3x + y + 4 = t$. Then $\frac{dy}{dx} = \frac{dt}{dx} - 3$ so that the given equation becomes $\frac{dt}{dx} - 3 = t^2$

(or) $\frac{dt}{t^2+3} = dx$. Hence, $\int \frac{dt}{t^2+3} = \int dx$ so that $\frac{1}{\sqrt{3}}\text{Tan}^{-1}\left(\frac{t}{\sqrt{3}}\right) = x + c$

(or) $\frac{1}{\sqrt{3}}\text{Tan}^{-1}\left(\frac{3x+y+4}{\sqrt{3}}\right) = x + c$ (by substituting for t)

which is the required solution.

16. Problem : *Solve* $\frac{dy}{dx} - x\tan(y-x) = 1.$

Solution : Put $y - x = t$ so that $\frac{dy}{dx} - 1 = \frac{dt}{dx}.$

Therefore, the given equation becomes

$$1 + \frac{dt}{dx} - x\tan t = 1$$

(or) $$\frac{dt}{dx} = x\tan t.$$

Therefore, $\cot t\, dt = x\, dx$ so that $\int \cot t\ dt = \int x\ dx$.

Hence, $$\log|\sin t| = \frac{x^2}{2} + c$$

i.e., $\log|\sin(y-x)| = \frac{x^2}{2} + c$ which is the required solution.

Exercise 8(b)

I 1. Find the general solution of $\sqrt{1-x^2}\,dy + \sqrt{1-y^2}\,dx = 0.$

2. Find the general solution of $\frac{dy}{dx} = \frac{2y}{x}.$

II. Solve the following differential equations.

1. $\frac{dy}{dx} = \frac{1+y^2}{1+x^2}$
2. $\frac{dy}{dx} = e^{y-x}$
3. $(e^x+1)y\,dy + (y+1)dx = 0$
4. $\frac{dy}{dx} = e^{x-y} + x^2e^{-y}$
5. $\tan y\, dx + \tan x\, dy = 0$
6. $\sqrt{1+x^2}\, dx + \sqrt{1+y^2}\, dy = 0$
7. $y - x\frac{dy}{dx} = 5\left(y^2 + \frac{dy}{dx}\right)$
8. $\frac{dy}{dx} = \frac{xy+y}{xy+x}.$

III. Solve the following differential equations.

1. $\frac{dy}{dx} = \frac{1+y^2}{(1+x^2)xy}$
2. $\frac{dy}{dx} + x^2 = x^2e^{3y}$
3. $(xy^2+x)dx + (yx^2+y)dy = 0$
4. $\frac{dy}{dx} = 2y\tanh x$

5. $\mathrm{Sin}^{-1}\left(\frac{dy}{dx}\right) = x + y.$

6. $\frac{dy}{dx} + \frac{y^2+y+1}{x^2+x+1} = 0.$

7. $\frac{dy}{dx} = \tan^2(x+y).$

8.2(b) Homogeneous Differential Equation

8.2(b)(i) Definition

A function $f(x, y)$ of two variables x and y is said to be a homogeneous function of degree α if $f(kx, ky) = k^{\alpha} f(x, y)$ for all values of k for which both sides of the above equation are meaningful.

8.2(b)(ii) Note

If $f(x,y)$ is a homogeneous function of degree α, then $f(x,y)$ can be written as $f(x,y) = x^{\alpha}\phi\left(\frac{y}{x}\right)$.

For, $f(x,y)$ is a homogeneous function of degree α implies that

$$f(kx, ky) = k^{\alpha} f(x, y) \ \forall\ k.$$

Taking $k = \frac{1}{x}$, we get

$$f\left(1, \frac{y}{x}\right) = f\left(\frac{1}{x}\cdot x, \frac{1}{x}y\right) = \frac{1}{x^{\alpha}} f(x, y).$$

Hence $f(x,y) = x^{\alpha} f\left(1, \frac{y}{x}\right)$

that is, $f(x,y) = x^{\alpha}\phi\left(\frac{y}{x}\right)$, $\left(\text{if we write } \phi\left(\frac{y}{x}\right) = f\left(1, \frac{y}{x}\right)\right)$.

8.2(b)(iii) Examples

(i) $f(x,y) = 4x^2y + 2xy^2$ is a homogeneous function of degree 3, since

$$f(kx, ky) = 4k^2x^2\, ky + 2kx\, k^2y^2$$
$$= k^3(4x^2y + 2xy^2) = k^3 f(x, y) \ \forall\ k.$$

Note that

$$f(x,y) = x^3\left(4\frac{y}{x} + 2\frac{y^2}{x^2}\right) = x^3\phi\left(\frac{y}{x}\right).$$

(ii) $g(x,y) = xy^{\frac{1}{2}} + yx^{\frac{1}{2}}$ is a homogeneous function of degree $\frac{3}{2}$ since

$$g(kx,ky) = k^{\frac{3}{2}}(xy^{\frac{1}{2}} + yx^{\frac{1}{2}}) = k^{\frac{3}{2}} g(x,\, y)\ \forall k.$$

Note that, $g(x,y) = x^{\frac{3}{2}}\left(\dfrac{y^{\frac{1}{2}}}{x^{\frac{1}{2}}} + \dfrac{y}{x}\right)$

$$= x^{\frac{3}{2}}\left[\left(\frac{y}{x}\right)^{\frac{1}{2}} + \left(\frac{y}{x}\right)\right]$$

$$= x^{\frac{3}{2}}\phi\left(\frac{y}{x}\right).$$

(iii) $h(x,y) = \dfrac{x^2+y^2}{x^3+y^3}$ is a homogeneous function of degree -1, since

$$h(kx,\, ky) = \frac{k^2x^2+k^2y^2}{k^3x^3+k^3y^3} = \frac{1}{k}\left(\frac{x^2+y^2}{x^3+y^3}\right) = k^{-1}h(x,\, y)\ \forall k \neq 0.$$

Note that, $h(x,y) = \dfrac{1}{x}\left(\dfrac{x^3+xy^2}{x^3+y^3}\right) = \dfrac{1}{x}\left[\dfrac{1+\dfrac{y^2}{x^2}}{1+\dfrac{y^3}{x^3}}\right]$

$$= \frac{1}{x}\phi\left(\frac{y}{x}\right) = x^{-1}\phi\left(\frac{y}{x}\right).$$

8.2(b)(iv) Definition : Homogeneous Differential Equation

A differential equation of the form

$$\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$$

where $f(x,\, y)$ and $g(x,\, y)$ are homogeneous functions of x and y of the same degree is called a homogeneous equation.

8.2(b)(v) Method of solving a homogeneous differential equation

Consider the homogeneous equation

$$\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)} \qquad \text{... (1)}$$

where $f(x, y)$ and $g(x, y)$ are homogeneous functions of the same degree, say α. Then in view of Note 8.2(b)(ii), $f(x, y)$ and $g(x, y)$ can be written as

$$f(x, y) = x^{\alpha}\phi\left(\frac{y}{x}\right) \text{ and } g(x, y) = x^{\alpha}\psi\left(\frac{y}{x}\right).$$

Hence, (1) becomes $\dfrac{dy}{dx} = \dfrac{\phi\left(\frac{y}{x}\right)}{\psi\left(\frac{y}{x}\right)} = F\left(\dfrac{y}{x}\right)$. ... (2)

Put $y = vx$. Then $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$. ... (3)

From (2) and (3), we get

$$v + x\frac{dv}{dx} = \frac{\phi(v)}{\psi(v)} \text{ so that } x\frac{dv}{dx} = \frac{\phi(v) - v\psi(v)}{\psi(v)}$$

(or) $$\frac{\psi(v)}{\phi(v) - v\psi(v)}dv = \frac{dx}{x}.$$

This can be solved by variables separable method.

Note : If $\dfrac{dx}{dy} = F\left(\dfrac{x}{y}\right)$, then we put $x = vy$ and $\dfrac{dx}{dy} = v + y\dfrac{dv}{dy}$.

This gives a differential equation in v and y in which the variables are separable. We solve this equation and put $v = \dfrac{x}{y}$ to obtain the required solution.

8.2(b)(vi) Solved problems

1. Problem : *Show that $f(x, y) = 1 + e^{x/y}$ is a homogeneous function of x and y.*

Solution : $f(kx, ky) = 1 + e^{kx/ky} = 1 + e^{x/y} = f(x, y)$ for all $k (\neq 0)$. Hence $f(x, y)$ is a homogeneous function of degree 0.

2. Problem : *Show that $f(x, y) = x\sqrt{x^2 + y^2} - y^2$ is a homogeneous function of x and y.*

Solution: Now, for $k > 0$

$$f(kx, ky) = kx\sqrt{k^2x^2 + k^2y^2} - k^2y^2$$

$$= k^2\left[x\sqrt{x^2 + y^2} - y^2\right] = k^2 f(x, y).$$

Hence $f(x, y)$ is a homogeneous function of degree 2.

3. Problem : *Show that $f(x,y) = x - y\log y + y\log x$ is a homogeneous function of x and y.*

Solution: Now, for $k > 0$

$$\begin{aligned} f(kx, ky) &= kx - ky\log(ky) + ky\log(kx) \\ &= k[x - y\log(ky) + y\log(kx)] \\ &= k[x - y\log k - y\log y + y\log k + y\log x] \\ &= k[x - y\log y + y\log x] = kf(x,y) \end{aligned}$$

so that $f(x, y)$ is a homogeneous function of degree 1.

4. Problem : *Express $(1 + e^{x/y})dx + e^{x/y}\left(1 - \frac{x}{y}\right)dy = 0$ in the form $\frac{dx}{dy} = F\left(\frac{x}{y}\right)$.*

Solution : The given equation can be written as $\frac{dx}{dy} = \frac{e^{x/y}\left(\frac{x}{y} - 1\right)}{1 + e^{x/y}} = F\left(\frac{x}{y}\right)$ which is in the required form.

5. Problem : *Express $(x\sqrt{x^2 + y^2} - y^2)dx + xy\,dy = 0$ in the form $\frac{dy}{dx} = F\left(\frac{y}{x}\right)$.*

Solution: From the given equation

$$\frac{dy}{dx} = \frac{y^2 - x\sqrt{x^2 + y^2}}{xy} = \frac{\frac{y^2}{x^2} - \frac{x\sqrt{x^2 + y^2}}{x^2}}{\frac{xy}{x^2}}$$

$$= \frac{\left(\frac{y}{x}\right)^2 - \sqrt{1 + \left(\frac{y}{x}\right)^2}}{\left(\frac{y}{x}\right)} = F\left(\frac{y}{x}\right).$$

6. Problem : *Express $\frac{dy}{dx} = \frac{y}{x + ye^{\frac{-2x}{y}}}$ in the form $\frac{dx}{dy} = F\left(\frac{x}{y}\right)$.*

Solution: From the given equation

$$\frac{dx}{dy} = \frac{x + ye^{\frac{-2x}{y}}}{y}$$

$$= \frac{x}{y} + e^{-2(x/y)} = F\left(\frac{x}{y}\right).$$

7. Problem : *Solve* $\frac{dy}{dx} = \frac{y^2 - 2xy}{x^2 - xy}$.

Solution: The given equation is a homogeneous equation, since both the numerator and denominator are homogeneous functions each of degree 2.

Now put $y = vx$. Then $\frac{dy}{dx} = v + x\frac{dv}{dx}$

so that the given equation becomes $v + x\frac{dv}{dx} = \frac{v^2 - 2v}{1 - v}$.

Hence, $x\frac{dv}{dx} = \frac{2v^2 - 3v}{1 - v}$ so that $\frac{1 - v}{2v^2 - 3v}dv = \frac{dx}{x}$.

Therefore, $\int \frac{1 - v}{2v^2 - 3v}dv = \int \frac{dx}{x}$.

Hence, $-\frac{1}{3}\int\left(\frac{1}{v} + \frac{1}{2v - 3}\right)dv = \log x - \log c$

so that $-\frac{1}{3}\left[\log v + \frac{1}{2}\log(2v - 3)\right] = \log x - \log c$

that is, $-\frac{1}{3}\log(v\sqrt{2v - 3}) = \log x - \log c$

that is, $\log(v\sqrt{2v - 3}) = -3\log x + 3\log c = -\log x^3 + \log c^3$

that is, $\log(x^3 v\sqrt{2v - 3}) = \log c^3$.

Hence $x^3 v(\sqrt{2v - 3}) = c^3$.

Put $v = \frac{y}{x}$. Then $x^3\frac{y}{x}\sqrt{\frac{2y}{x} - 3} = c^3$

that is, $x^2 y\sqrt{\frac{2y}{x} - 3} = c^3$ (or) $xy\sqrt{2xy - 3x^2} = c^3$.

This is the general solution of the given equation.

8. Problem : *Solve* $(x^2 + y^2)dx = 2xy\,dy$.

Solution : The given equation can be written as

$$\frac{dy}{dx} = \frac{x^2 + y^2}{2xy} \quad \text{... (1)}$$

which is a homogeneous equation, since the numerator and denominator on the right are homogeneous functions each of degree 2. Put $y = vx$. Then $\frac{dy}{dx} = v + x\frac{dv}{dx}$.

Therefore, (1) becomes

$$v + x\frac{dv}{dx} = \frac{x^2(1+v^2)}{2x^2 v} = \frac{1+v^2}{2v} \text{ so that } x\frac{dv}{dx} = \frac{1-v^2}{2v}.$$

Hence, $\frac{2v}{1-v^2}dv = \frac{dx}{x}$ so that $\int \frac{2v}{1-v^2}dv = \int \frac{dx}{x}$

that is, $-\log(1-v^2) = \log x + \log c$

so that $\log[xc(1-v^2)] = 0 = \log 1.$

Hence $xc(1-v^2) = 1$

that is, $c(x^2 - y^2) = x$ $\left(\text{since } v = \frac{y}{x}\right)$

which is the general solution of the given equation.

9. Problem : *Solve* $xy^2 dy - (x^3 + y^3)dx = 0.$

Solution : The given equation can be written as

$$\frac{dy}{dx} = \frac{x^3 + y^3}{xy^2} \quad \text{... (1)}$$

which is a homogeneous equation.

Put $y = vx$. Then $\frac{dy}{dx} = v + x\frac{dv}{dx}$.

Therefore, (1) becomes $v + x\frac{dv}{dx} = \frac{1+v^3}{v^2}$

so that $x\frac{dv}{dx} = \frac{1}{v^2}$ (or) $v^2\, dv = \frac{dx}{x}$.

Therefore, $\int v^2 dv = \int \frac{dx}{x}$ so that $\frac{v^3}{3} = \log x + \log c$

that is, $\frac{y^3}{3x^3} = \log x + \log c$ (or) $y^3 = 3x^3 \log(cx)$

which is the general solution of the given equation.

10. Problem : *Solve* $\dfrac{dy}{dx} = \dfrac{x^2+y^2}{2x^2}$. ... (1)

Solution : The given equation is a homogeneous equation. Put $y = vx$. Then $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$ so that (1) becomes $v + x\dfrac{dv}{dx} = \dfrac{1+v^2}{2}$,

that is, $2x\,dv = (1+v^2-2v)dx$. Separating variables, we have

$$\frac{2dv}{(v-1)^2} = \frac{dx}{x}.$$

Integrating, we get

$$\frac{-2}{v-1} = \log x + c.$$

But $v = \dfrac{y}{x}$, so

$$-\frac{2}{v-1} = \frac{-2}{\frac{y}{x}-1} = \frac{-2x}{y-x} = \frac{2x}{x-y}.$$

Hence $\dfrac{2x}{x-y} = \log x + c$

so that $2x = (x-y)(\log x + c)$ which is the general solution of the given equation.

11. Problem : *Solve* $x\sec\left(\dfrac{y}{x}\right)\cdot(y\,dx + x\,dy) = y\,\mathrm{cosec}\left(\dfrac{y}{x}\right)\cdot(x\,dy - y\,dx)$

Solution: The given equation can be written as

$$x\sec\left(\frac{y}{x}\right)\cdot\left(y + x\frac{dy}{dx}\right) = y\,\mathrm{cosec}\left(\frac{y}{x}\right)\cdot\left(x\frac{dy}{dx} - y\right)$$

so that $x\dfrac{dy}{dx}\cdot\left(x\sec\left(\dfrac{y}{x}\right) - y\,\mathrm{cosec}\left(\dfrac{y}{x}\right)\right) = -y\left(y\,\mathrm{cosec}\left(\dfrac{y}{x}\right) + x\sec\left(\dfrac{y}{x}\right)\right)$

that is, $$\frac{dy}{dx} = \frac{y}{x}\,\frac{y\,\mathrm{cosec}\left(\frac{y}{x}\right) + x\sec\left(\frac{y}{x}\right)}{-x\sec\left(\frac{y}{x}\right) + y\,\mathrm{cosec}\left(\frac{y}{x}\right)} \qquad \text{... (1)}$$

which is a homogeneous equation, since the numerator and denominator on the right are homogeneous functions each of degree 2.

Put $y = vx$. Then $\frac{dy}{dx} = v + x\frac{dv}{dx}$.

Therefore, (1) becomes

$$v + x\frac{dv}{dx} = v\left(\frac{v\operatorname{cosec} v + \sec v}{v\operatorname{cosec} v - \sec v}\right) = v\left(\frac{v\cos v + \sin v}{v\cos v - \sin v}\right).$$

Hence, $x\frac{dv}{dx} = \frac{2v\sin v}{v\cos v - \sin v}$

that is, $\left(\frac{v\cos v - \sin v}{v\sin v}\right)dv = 2\frac{dx}{x}$.

Integrating, we get

$$\int \frac{v\cos v - \sin v}{v\sin v}dv = 2\int \frac{dx}{x}$$

that is, $\int \frac{\cos v}{\sin v}dv - \int \frac{dv}{v} = 2\log x + \log c$.

Therefore, $\log\sin v - \log v = \log x^2 + \log c$

that is, $\log\left(\frac{\sin v}{v}\right) = \log(cx^2)$ so that $\frac{\sin v}{v} = cx^2$

that is, $\sin\left(\frac{y}{x}\right) = cxy$ $\left(\text{since } v = \frac{y}{x}\right)$ which is the general solution of the given equation.

12. Problem : *Give the solution of* $x\sin^2\frac{y}{x}dx = ydx - xdy$ *which passes through the point* $\left(1, \frac{\pi}{4}\right)$.

Solution: The given equation can be written as

$$\left(x\sin^2\frac{y}{x} - y\right)dx = -x\,dy \text{ so that } \frac{dy}{dx} = \frac{y - x\sin^2\frac{y}{x}}{x} \qquad \ldots (1)$$

which is a homogeneous equation, since both numerator and denominator are homogeneous functions each of degree 1.

Put $y = vx$. Then $\frac{dy}{dx} = v + x\frac{dv}{dx}$

so that (1) becomes $v + x\frac{dv}{dx} = v - \sin^2 v$.

Therefore, $x\frac{dv}{dx} = -\sin^2 v$

that is, $-\frac{dv}{\sin^2 v} = \frac{dx}{x}$.

Integrating, we get $$\int -\frac{dv}{\sin^2 v} = \int \frac{dx}{x}$$

that is, $\int -\operatorname{cosec}^2 v\, dv = \log x + c.$

Hence, $\cot v = \log x + c$ or $\cot\left(\frac{y}{x}\right) = \log x + c$ ... (2)

Since this passes through the point $\left(1, \frac{\pi}{4}\right)$, $\cot\frac{\pi}{4} = c$ so that $c = 1$.

Therefore from (2), the required particular solution is $\cot\left(\frac{y}{x}\right) = \log x + 1.$

13. Problem : *Solve* $(x^3 - 3xy^2)dx + (3x^2y - y^3)dy = 0.$

Solution: The given equation can be written as

$$\frac{dy}{dx} = \frac{x^3 - 3xy^2}{y^3 - 3x^2y}. \quad \text{... (1)}$$

Therefore, the given equation is a homogeneous equation.

Put $y = vx$. Then $v + x\frac{dv}{dx} = \frac{dy}{dx}$ so that (1) becomes

$$v + x\frac{dv}{dx} = \frac{1 - 3v^2}{v^3 - 3v}. \quad \text{... (2)}$$

Therefore, $x\frac{dv}{dx} = \frac{1 - v^4}{v^3 - 3v} = \frac{v^4 - 1}{3v - v^3}$ so that $\frac{3v - v^3}{(v+1)(v-1)(v^2+1)}dv = \frac{dx}{x}$

that is $\left[\frac{1}{2(v+1)} + \frac{1}{2(v-1)} - \frac{2v}{v^2+1}\right]dv = \frac{dx}{x}$ (by partial fractions).

Integrating, we get

$$\frac{1}{2}\log(v+1) + \frac{1}{2}\log(v-1) - \log(v^2+1) = \log x + \log c$$

that is, $\log\left[\frac{\sqrt{v+1}\sqrt{v-1}}{v^2+1}\right] = \log(cx)$ so that $\frac{\sqrt{v^2-1}}{v^2+1} = cx$

(or) $\frac{v^2 - 1}{(v^2+1)^2} = c^2x^2.$

Since $y = \frac{v}{x}$, $y^2 - x^2 = c^2(y^2 + x^2)$

which is the required general solution.

Exercise 8(c)

I. 1. Express $x\,dy - y\,dx = \sqrt{x^2+y^2}\,dx$ in the form $F\left(\frac{y}{x}\right) = \frac{dy}{dx}$.

2. Express $\left(x - y\operatorname{Tan}^{-1}\frac{y}{x}\right)dx + x\operatorname{Tan}^{-1}\frac{y}{x}dy = 0$ in the form $F\left(\frac{y}{x}\right) = \frac{dy}{dx}$.

3. Express $x\frac{dy}{dx} = y(\log y - \log x + 1)$ in the form $F\left(\frac{y}{x}\right) = \frac{dy}{dx}$.

II. Solve the following differential equations.

1. $\frac{dy}{dx} = \frac{x-y}{x+y}$
2. $(x^2+y^2)dy = 2xy\,dx$
3. $\frac{dy}{dx} = \frac{-(x^2+3y^2)}{3x^2+y^2}$
4. $y^2dx + (x^2 - xy)dy = 0$
5. $\frac{dy}{dx} = \frac{(x+y)^2}{2x^2}$
6. $(x^2-y^2)dx - xy\,dy = 0.$
7. $(x^2y - 2xy^2)dx = (x^3 - 3x^2y)dy$
8. $y^2\,dx + (x^2 - xy + y^2)dy = 0$
9. $(y^2 - 2xy)dx + (2xy - x^2)dy = 0$
10. $\frac{dy}{dx} + \frac{y}{x} = \frac{y^2}{x^2}$
11. $x\,dy - y\,dx = \sqrt{x^2+y^2}\,dx$
12. $(2x-y)dy = (2y-x)dx$
13. $(x^2-y^2)\frac{dy}{dx} = xy$
14. $2\frac{dy}{dx} = \frac{y}{x} + \frac{y^2}{x^2}$

III. 1. Solve: $(1+e^{x/y})dx + e^{x/y}\left(1 - \frac{x}{y}\right)dy = 0.$

2. Solve: $x\sin\frac{y}{x}\cdot\frac{dy}{dx} = y\sin\frac{y}{x} - x.$

3. Solve: $x\,dy = \left(y + x\cos^2\frac{y}{x}\right)dx.$

4. Solve: $(x - y\log y + y\log x)dx + x(\log y - \log x)dy = 0.$

5. Solve: $(y\,dx + x\,dy)x\cos\frac{y}{x} = (x\,dy - y\,dx)y\sin\frac{y}{x}.$

6. Find the equation of a curve whose gradient is $\frac{dy}{dx} = \frac{y}{x} - \cos^2\frac{y}{x}$, where $x > 0,\ y > 0$ and which passes through the point $\left(1, \frac{\pi}{4}\right)$.

8.2(c) Non-Homogeneous Differential Equations

Differential equations of the form

$$\frac{dy}{dx} = \frac{ax+by+c}{a'x+b'y+c'} \qquad \text{... (1)}$$

where a, b, c, a', b', c' are constants and c and c' are not both zero are called **non-homogeneous equations**. We reduce (1) to a homogeneous equation by suitable substitutions for x and y.

We explain three methods (in case (i), case (ii) and case (iii)) of solving (1) depending on the nature of coefficients of x and y in the numerator and denominator of the R.H.S. of (1).

Case(i)

Suppose that $b = -a'$. Then (1) becomes $\dfrac{dy}{dx} = \dfrac{ax - a'y + c}{a'x + b'y + c'}$.

Therefore, $(a'x + b'y + c')dy - (ax - a'y + c)dx = 0$

that is, $a'(x\,dy + y\,dx) + b'y\,dy - ax\,dx + c'dy - c\,dx = 0$

that is, $a'd(xy) + b'd\left(\dfrac{y^2}{2}\right) - ad\left(\dfrac{x^2}{2}\right) + c'dy - c\,dx = 0.$

Integrating, we get $a'xy + b'\dfrac{y^2}{2} - a\dfrac{x^2}{2} + c'y - cx = k$

which is the required solution.

8.2(c)(i) Note : In the above case solution can be obtained by integrating each term after regrouping.

8.2(c)(ii) Example : Let us solve $\dfrac{dy}{dx} = \dfrac{3x - y + 7}{x - 7y - 3}$.

Here $b = -1 = -a'$. Hence we can solve by case(i). Now $(x - 7y - 3)dy - (3x - y + 7)dx = 0$.

Therefore, $(x\,dy + y\,dx) - 7y\,dy - 3\,dy - 3x\,dx - 7dx = 0$

that is, $d(xy) - 7d\left(\dfrac{y^2}{2}\right) - 3dy - 3d\left(\dfrac{x^2}{2}\right) - 7dx = 0.$

Integrating, we get

$$xy - \frac{7y^2}{2} - 3y - 3\frac{x^2}{2} - 7x = c$$

(or) $$2xy - 7y^2 - 6y - 3x^2 - 14x = 2c$$

which is the required solution.

Case(ii) : Suppose that $\dfrac{a}{a'} = \dfrac{b}{b'} = m(\text{say})$.

Then (1) becomes

$$\frac{dy}{dx} = \frac{ax + by + c}{\frac{1}{m}(ax + by) + c'} \qquad \text{... (2)}$$

Put $ax + by = v$. Then $a + b\dfrac{dy}{dx} = \dfrac{dv}{dx}$.

Therefore, $$\frac{dy}{dx} = \frac{1}{b}\left(\frac{dv}{dx} - a\right)$$

so that (2) becomes $$\frac{1}{b}\left(\frac{dv}{dx} - a\right) = \frac{v + c}{\frac{v}{m} + c'}.$$

Therefore, $$\frac{dv}{dx} = \frac{bm(v + c)}{v + c'm} + a$$

that is, $$\frac{v + c'm}{bm(v + c) + a(v + c'm)}\, dv = dx$$

which can be solved by variables separable method.

8.2(c)(iii) Example : We shall solve $\dfrac{dy}{dx} = \dfrac{x - y + 3}{2x - 2y + 5}$.

Here $a = 1$, $b = -1$ $a' = 2$, $b' = -2$ and hence

$$\frac{a}{a'} = \frac{b}{b'} = \frac{1}{2}.$$

Therefore, we can solve the equation by case(ii).

Put $x - y = v$. Then $1 - \dfrac{dy}{dx} = \dfrac{dv}{dx}$

so that the given equation becomes

$$1 - \frac{dv}{dx} = \frac{v + 3}{2v + 5}$$

that is, $$\frac{dv}{dx} = \frac{v + 2}{2v + 5}$$

so that $dx = \frac{2v+5}{v+2}dv = \left(2 + \frac{1}{v+2}\right)dv$.

Integrating, we get

$$x = 2v + \log(v+2) + c$$

that is, $$x = 2(x-y) + \log(x-y+2) + c$$

which is the required solution.

8.2(c)(iv) Note : If $b = -a'$ with $\frac{a}{a'} = \frac{b}{b'}$, then the given equation can be solved easily by using case (i) rather than case (ii).

Case(iii) : Suppose that $b \neq -a'$ and $\frac{a}{a'} \neq \frac{b}{b'}$.

Then taking $x = X + h$, $y = Y + k$, where X and Y are variables and h and k are constants, we get $\frac{dy}{dx} = \frac{dY}{dX}$.

Hence (1) becomes

$$\frac{dY}{dX} = \frac{a(X+h) + b(Y+k) + c}{a'(X+h) + b'(Y+k) + c'}$$

that is, $$\frac{dY}{dX} = \frac{aX + bY + (ah + bk + c)}{a'X + b'Y + (a'h + b'k + c')} \quad \text{... (i)}$$

Now choose constants h and k such that

$$ah + bk + c = 0 \quad \text{... (ii)}$$

and $$a'h + b'k + c' = 0 \quad \text{... (iii)}$$

Since $\frac{a}{a'} \neq \frac{b}{b'}$, we can solve (ii) and (iii) for h and k. Hence (1) becomes

$$\frac{dY}{dX} = \frac{aX + bY}{a'X + b'Y}$$

which is a homogeneous equation in X and Y and hence can be solved by homogeneous equation method, that is by putting Y = VX.

8.2(c)(v) Example : We shall solve $(2x + y + 3)dx = (2y + x + 1)dy$.

The given equation can be written as

$$\frac{dy}{dx} = \frac{2x + y + 3}{2y + x + 1} \quad \text{... (i)}$$

Here $a = 2$, $b = 1$, $a' = 1$, $b' = 2$. Hence, $b \neq -a'$ and $\frac{a}{a'} \neq \frac{b}{b'}$.

Therefore, the given equation can be solved by case (iii).

Put $x = X + h,\ y = Y + k$ in (i). Then $\dfrac{dy}{dx} = \dfrac{dY}{dX}$ and $\dfrac{dY}{dX} = \dfrac{2X + Y + 2h + k + 3}{2Y + X + 2k + h + 1}$ ... (ii)

Now choose h and k such that

$$2h + k + 3 = 0 \text{ and } h + 2k + 1 = 0.$$

Solving them for h and k, we get $h = -\dfrac{5}{3},\ k = \dfrac{1}{3}$.

Hence (ii) becomes $$\frac{dY}{dX} = \frac{2X + Y}{2Y + X} \quad \text{... (iii)}$$

which is a homogeneous equation.

Put $Y = VX$. Then $\dfrac{dY}{dX} = V + X\dfrac{dV}{dX}$.

Therefore, (iii) becomes $V + X\dfrac{dV}{dX} = \dfrac{2 + V}{2V + 1}$

that is, $X\dfrac{dV}{dX} = \dfrac{2(1 - V^2)}{2V + 1}$ and hence

$$\frac{2V + 1}{(1 + V)(1 - V)}dV = \frac{2dX}{X}.$$

that is, $\dfrac{3}{2(1 - V)}dV - \dfrac{1}{2(1 + V)}dV = \dfrac{2dX}{X}$.

Integrating, we get $-\dfrac{3}{2}\log(1 - V) - \dfrac{1}{2}\log(1 + V) = 2\log X - \log c$

that is, $3\log(1 - V) + \log(1 + V) + 4\log X = 2\log c$

(or) $\log[(1 - V)^3 (1 + V) X^4] = \log c^2$

so that $X^4(1 - V)^3 (1 + V) = c^2$.

Since $V = \dfrac{Y}{X}$, we get $X^4\left(1 - \dfrac{Y}{X}\right)^3 \left(1 + \dfrac{Y}{X}\right) = c^2$

that is, $(X + Y)(X - Y)^3 = c^2$.

Substituting for X and Y, we get,

$$\left(x+\frac{5}{3}+y-\frac{1}{3}\right)\left(x+\frac{5}{3}-y+\frac{1}{3}\right)^3 = c^2 \text{ (or)} \left(x+y+\frac{4}{3}\right)(x-y+2)^3 = c^2$$

which is the required solution.

Exercise 8(d)

I. Solve the following differential equations.

1. $\dfrac{dy}{dx} = -\dfrac{(12x+5y-9)}{5x+2y-4}$
2. $\dfrac{dy}{dx} = \dfrac{-3x-2y+5}{2x+3y+5}$
3. $\dfrac{dy}{dx} = \dfrac{-3x-2y+5}{2x+3y-5}$
4. $2(x-3y+1)\dfrac{dy}{dx} = 4x-2y+1$
5. $\dfrac{dy}{dx} = \dfrac{x-y+2}{x+y-1}$
6. $\dfrac{dy}{dx} = \dfrac{2x-y+1}{x+2y-3}$

II. Solve the following differential equations.

1. $(2x+2y+3)\dfrac{dy}{dx} = x+y+1$
2. $\dfrac{dy}{dx} = \dfrac{4x+6y+5}{3y+2x+4}$
3. $(2x+y+1)dx+(4x+2y-1)dy=0$
4. $\dfrac{dy}{dx} = \dfrac{2y+x+1}{2x+4y+3}$
5. $(x+y-1)dy=(x+y+1)dx$

III. Solve the following differential equations.

1. $\dfrac{dy}{dx} = \dfrac{3y-7x+7}{3x-7y-3}$
2. $\dfrac{dy}{dx} = \dfrac{6x+5y-7}{2x+18y-14}$
3. $\dfrac{dy}{dx} + \dfrac{10x+8y-12}{7x+5y-9} = 0$
4. $(x-y-2)dx+(x-2y-3)dy=0$
5. $(x-y)dy=(x+y+1)dx$
6. $(2x+3y-8)dx = (x+y-3)dy$
7. $\dfrac{dy}{dx} = \dfrac{x+2y+3}{2x+3y+4}$
8. $\dfrac{dy}{dx} = \dfrac{2x+9y-20}{6x+2y-10}$

8.2(d) Linear Differential Equations

A differential equation is said to be linear if the dependent variable and its derivatives appear only in first degree and their products do not occur in the equation.

8.2(d) (i) A linear differential equation of n^{th} order is of the form $\dfrac{d^n y}{dx^n} + P_1\dfrac{d^{n-1}y}{dx^{n-1}} + P_2\dfrac{d^{n-2}y}{dx^{n-2}} + \dots + P_n y = Q$,

where $P_1, P_2, \dots, P_n$, Q are either constants or functions x.

Any linear differential equation of first order is of the form

$$\frac{dy}{dx} + Py = Q,$$

where P, Q are constants or functions of x only.

8.2(d)(ii) Method of solving a linear differential equation of first order

Consider any linear differential equation of first order

$$\frac{dy}{dx} + Py = Q \qquad \dots(1)$$

Multiplying both sides of (1) by $e^{\int P dx}$, we get

$$\frac{dy}{dx}\left(e^{\int P dx}\right) + Py\, e^{\int P dx} = Q\, e^{\int P dx}$$

from which it follows that

$$\frac{d}{dx}\left(y\, e^{\int P dx}\right) = Q\, e^{\int P dx}.$$

Integrating, we get

$$\int \frac{d}{dx}\left(y\, e^{\int P dx}\right) dx = \int Q\, e^{\int P dx}\, dx$$

Therefore, $\quad y\, e^{\int P dx} = \int Q\, e^{\int P dx}\, dx + c$

The solution of $\frac{dy}{dx} + Py = Q$ is $y\, e^{\int P dx} = \int Q\, e^{\int P dx}\, dx + c$.

8.2(d)(iii) Note

1. The function $e^{\int P dx}$ of x which makes the L.H.S. of (1) as the differential coefficient of $y\, e^{\int P dx}$ is called the integrating factor and is usually denoted by I.F. The above solution can also be written in terms of I.F. as $y(\text{I.F}) = \int (Q \times \text{I.F})\, dx + c$.

2. Note that differential equations of the from $\frac{dx}{dy} + Px = Q$, where P and Q are constants or functions of y only are also linear differential equations of first order in x. For such equations I.F. $= e^{\int P dy}$ and the solution is

$$x(\text{I.F.}) = \int (Q \times \text{I.F})\, dy + c.$$

8.2(d)(iv) Solved Problems

Transform the following two differential equations into linear form

1. Problem : $x \log x \dfrac{dy}{dx} + y = 2\log x$.

Solution : Dividing both sides by $x \log x$, we get $\dfrac{dy}{dx} + \dfrac{1}{x \log x} y = \dfrac{2}{x}$ which is in the form $\dfrac{dy}{dx} + \text{P}y = \text{Q}$.

2. Problem : $(x+2y^3)\dfrac{dy}{dx} = y$.

Solution : The given equation can be written as

$$\frac{dx}{dy} = \frac{x+2y^3}{y} = \frac{x}{y} + 2y^2 \text{ that is, } \frac{dx}{dy} - \frac{1}{y}x = 2y^2 \text{ which is in the form } \frac{dx}{dy} + \text{P}x = \text{Q (linear in } x).$$

Find I.F. of the following two differential equations by transforming them into linear form :

3. Problem : $(\cos x)\dfrac{dy}{dx} + y\sin x = \tan x$.

Solution : The above equation can be written as $\dfrac{dy}{dx} + (\tan x)y = \sec x . \tan x$.

Therefore, $\text{P} = \tan x$ and hence $\int \text{P}dx = \int \tan x\, dx = \log \sec x$ so that

$$\text{I.F} = e^{\int \text{P}dx} = e^{\log \sec x} = \sec x.$$

4. Problem : $(2x - 10y^3)\dfrac{dy}{dx} + y = 0$.

Solution : The given equation can be written as

$$\frac{dx}{dy} = \frac{10y^3 - 2x}{y} = \frac{-2}{y}x + 10y^2$$

(or) $$\frac{dx}{dy} + \frac{2}{y} . x = 10y^2 \text{ (linear in } x).$$

Therefore, $\text{I.F} = e^{\int \text{P}dy} = e^{\int \frac{2}{y}dy} = y^2$.

5. Problem : *Solve* $(1+x^2)\dfrac{dy}{dx} + 2xy - 4x^2 = 0$.

Solution : The given equation can be written as

$$\frac{dy}{dx} + \frac{2x}{1+x^2} y = \frac{4x^2}{1+x^2}. \qquad \text{... (1)}$$

Here $\quad P = \frac{2x}{1+x^2}, \quad Q = \frac{4x^2}{1+x^2}$

Hence $\quad$ I.F. $= e^{\int P dx} = e^{\int \frac{2x}{1+x^2} dx} = e^{\log(1+x^2)} = 1 + x^2$.

General solution of (1) is given by $y(\text{I.F}) = \int (Q \times \text{I.F})\, dx + c$ $\quad$ (by 8.2(d)(iii))

Therefore $\quad y(1+x^2) = \int \left(\frac{4x^2}{1+x^2} \right)(1+x^2)\, dx + c = \frac{4x^3}{3} + c$

that is, $3y(1+x^2) = 4x^3 + 3c$ which is the required solution.

6. Problem : *Solve* $\frac{1}{x}\frac{dy}{dx} + ye^x = e^{(1-x)e^x}$.

Solution : The given equation can be written as

$$\frac{dy}{dx} + xe^x y = x\, e^{(1-x)e^x} \quad \text{... (1)}$$

Here $\quad P = xe^x$ and $Q = x\, e^{(1-x)e^x}$.

Therefore, $\quad$ I.F. $= e^{\int x e^x dx} = e^{(x-1)e^x}$.

General solution of (1) is given by $y(\text{I.F}) = \int (Q \times \text{I.F})\, dx + c$ $\quad$ (by 8.2(d)(iii))

$$y\, e^{(x-1)e^x} = \int x dx + c$$

that is, $\quad y\, e^{(x-1)e^x} = \frac{x^2}{2} + c \quad$ (or) $\quad 2y\, e^{(x-1)e^x} = x^2 + 2c$

which is the required solution.

7. Problem : *Solve* $\sin^2 x . \frac{dy}{dx} + y = \cot x$.

Solution: The given equation can be written as

$$\frac{dy}{dx} + y \operatorname{cosec}^2 x = \operatorname{cosec}^2 x . \cot x \quad \text{... (1)}$$

Here $P = \operatorname{cosec}^2 x$ and $Q = \operatorname{cosec}^2 x . \cot x$.

Therefore, $\quad$ I.F $= e^{\int \operatorname{cosec}^2 x dx} = e^{-\cot x}$.

General solution of (1) is given by $y(\text{I.F}) = \int (\text{Q} \times \text{I.F})\, dx + c$ (by 8.2(d)(iii))

$$y\, e^{-\cot x} = \int e^{-\cot x} \operatorname{cosec}^2 x \cot x\, dx + c. \qquad \text{... (2)}$$

Put $-\cot x = t$. Then $dt = \operatorname{cosec}^2 x\, dx$.

Therefore, (2) becomes

$$y\, e^t = -\int t\, e^t\, dt + c$$
$$= -(t-1)e^t + c.$$

Hence $y\, e^{-\cot x} = -(-\cot x - 1)\, e^{-\cot x} + c$

$$= (1 + \cot x)\, e^{-\cot x} + c$$

which is the required solution.

8. Problem : *Find the solution of the equation*

$$x(x-2)\frac{dy}{dx} - 2(x-1)y = x^3(x-2)$$

which satisfies the condition that $y = 9$ *when* $x = 3$.

Solution : The given equation can be written as

$$\frac{dy}{dx} - 2\frac{(x-1)}{x(x-2)}\, y = x^2 \qquad \text{...(1)}$$

Here $\text{P} = -2\frac{(x-1)}{x(x-2)}$ and $\text{Q} = x^2$.

Therefore, $\text{I.F} = e^{\int -2\frac{(x-1)}{x(x-2)} dx}$

$$= e^{\log \frac{1}{x(x-2)}} = \frac{1}{x(x-2)}.$$

General solution of (1) is given by $y(\text{I.F}) = \int (\text{Q} \times \text{I.F})\, dx + c$ (by 8.2(d)(iii))

$$\frac{y}{x(x-2)} = \int \frac{x^2}{x(x-2)}\, dx + \text{C} \qquad = \int \frac{x}{x-2}\, dx + c$$

$$= \int \left(1 + \frac{2}{x-2}\right) dx + \text{C} = x + 2\log(x-2) + c.$$

Therefore, $\frac{y}{x(x-2)} = x + 2\log(x-2) + c$ ... (2)

which is the general solution of the given equation. Taking $x = 3$ and $y = 9$ in (2), we get $\frac{9}{3} = 3 + c$ so that $c = 0$. Hence $y = [x(x-2)]\,[x + 2\log(x-2)]$ is the required solution.

9. Problem : *Solve* $(1+y^2)dx = (\text{Tan}^{-1}y - x)\,dy$

Solution : The given equation can be written as

$$\frac{dx}{dy} + \frac{x}{1+y^2} = \frac{\text{Tan}^{-1}y}{1+y^2} \qquad \text{... (1)}$$

which is linear in x.

Here $\quad P = \dfrac{1}{1+y^2},\; Q = \dfrac{\text{Tan}^{-1}y}{1+y^2}$ so that

$$\text{I.F.} = e^{\int \frac{dy}{1+y^2}} = e^{\text{Tan}^{-1}y}.$$

General solution of (1) is given by $y(\text{I.F}) = \int (Q \times \text{I.F})\,dx + c$ (by 8.2(d)(iii))

$$x\,e^{\text{Tan}^{-1}y} = \int e^{\text{Tan}^{-1}y}\,\frac{\text{Tan}^{-1}y}{1+y^2}\,dy + c \qquad \text{... (2)}$$

Now put $\text{Tan}^{-1}y = t$. Then $\dfrac{1}{1+y^2}\,dy = dt$.

Hence (2) becomes

$$xe^t = \int t\,e^t\,dt + c = e^t(t-1) + c$$

so that $\quad x\,e^{\text{Tan}^{-1}y} = e^{\text{Tan}^{-1}y}(\text{Tan}^{-1}y - 1) + c$

which is the required solution.

Exercise 8(e)

I. Find the I.F. of the following differential equations by transforming them into linear form.

1. $x\dfrac{dx}{dy} - y = 2x^2\sec^2 2x$.

2. $y\dfrac{dx}{dy} - x = 2y^3$

II. Solve the following differential equations.

1. $\dfrac{dy}{dx} + y\tan x = \cos^3 x$

2. $\dfrac{dy}{dx} + y\sec x = \tan x$

3. $\dfrac{dy}{dx} - y\tan x = e^x\sec x$.

4. $x\dfrac{dy}{dx} + 2y = \log x$

5. $(1+x^2)\dfrac{dy}{dx} + y = e^{\text{Tan}^{-1}x}$.

6. $\dfrac{dy}{dx} + \dfrac{2y}{x} = 2x^2$

7. $\dfrac{dy}{dx} + \dfrac{4x}{1+x^2}y = \dfrac{1}{(1+x^2)^2}$

8. $x\dfrac{dy}{dx} + y = (1+x)e^x$

9. $\frac{dy}{dx}+\frac{3x^2}{1+x^3}y=\frac{1+x^2}{1+x^3}$

10. $\frac{dy}{dx}-y=-2e^{-x}$

11. $(1+x^2)\frac{dy}{dx}+y=\mathrm{Tan}^{-1}x$

12. $\frac{dy}{dx}+y\tan x=\sin x$.

III. Solve the following differential equations.

1. $\cos x\,.\,\frac{dy}{dx}+y\sin x=\sec^2 x$

2. $\sec x\,.\,dy=(y+\sin x)dx$

3. $x\log x\,.\,\frac{dy}{dx}+y=2\log x$.

4. $(x+y+1)\frac{dy}{dx}=1$

5. $x(x-1)\frac{dy}{dx}-y=x^3(x-1)^3$

6. $(x+2y^3)\frac{dy}{dx}=y$

7. $(1-x^2)\frac{dy}{dx}+2xy=x\sqrt{1-x^2}$

8. $x(x-1)\frac{dy}{dx}-(x-2)y=x^3(2x-1)$

9. $\frac{dy}{dx}(x^2y^3+xy)=1$

10. $\frac{dy}{dx}+x\sin 2y=x^3\cos^2 y$

11. $y^2+\left(x-\frac{1}{y}\right).\frac{dy}{dx}=0$.

Key Concepts

❖ **Variables Separable Method**

If the differential equation is of the form

$$f(x)\,dx+g(y)\,dy=0,$$

then its solution is $\int f(x)\,dx+\int g(y)\,dy=0$.

❖ **Homogeneous Equations**

If the differential equation is of the form $\frac{dy}{dx}=\frac{f(x,y)}{g(x,y)}$, where f and g are homogeneous functions of x and y of same degree, then we put $y=vx$ and bring it to the form

$\phi(v)\,dv=\frac{dx}{x}$ and then we integrate.

❖ **Non Homogeneous Equations**

Let the differential equation be of the form $\frac{dy}{dx}=\frac{ax+by+c}{a'x+b'y+c'}$,

where $a,b,c,a',\ b',\ c'$ are constants.

Case(i) : If $b = -a'$, then its solution can be obtained by integrating term by term after regrouping.

Case (ii) : If $\frac{a}{a'} = \frac{b}{b'} = m$, then we put $ax + by = v$ and bring it to the form

$\phi(v)\,dv = dx$ and then we integrate.

Case (iii) : If $\frac{a}{a'} \neq \frac{b}{b'}$, then we put $x = X + h$, $y = Y + k$ (h, k are obtained by solving

$ah + bk + c = 0$, $a'h + b'k + c' = 0$) and bring it to the form $\frac{dY}{dX} = \frac{aX + bY}{a'X + b'Y}$.

Then we put $Y = VX$ and bring it to the form $\phi(V)\,dV = \frac{dX}{X}$ and then we integrate.

❖ Linear Differential Equations

If the differential equation is of the form

$$\frac{dy}{dx} + Py = Q,$$

then its solution is $y\, e^{\int P dx} = C + \int Q\, e^{\int P dx}\, dx$.

Historical Note

The study of differential equations began soon after the invention of the Differential and Integral calculus, to which it forms a natural sequel. A differential equation occurred for the first time in 1693 in the work of *Leibnitz* (whose account of the differential calculus was published in 1684).

In the next few years the progress was rapid. In 1694-97, *Johann Bernoulli* explained the method of "*Separating the Variables*", and he showed how to reduce a homogeneous differential equation of the first order to one in which the varibles are separable. He applied these methods to problems on orthogonal trajectories. Integrating Factors were introduced by *Euler* in 1734 and (independently of him) by *Fontaine* and *Clairaut*, though some attribute them to *Leibnitz*.

Answers

Exercise 8(a)

I. 1. 2   2. 1

II. 1. (i) $\left(\frac{dy}{dx}\right)^3 - 4xy\frac{dy}{dx} + 8y^2 = 0$ (ii) $x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - xy = 0$

(iii) $\frac{d^2y}{dx^2} - 2k\frac{dy}{dx} + k^2y = 0$ (iv) $\frac{d^2y}{dx^2} + n^2y = 0$

2. (i) $x\frac{dy}{dx} + y = 0$ (ii) $xy\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 - y\frac{dy}{dx} = 0$

III. 1. (i) $\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 12y = 0$ (ii) $x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0$

(iii) $xy\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 - y\frac{dy}{dx} = 0$ (iv) $x^2\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} - 2y = 0$

2. (i) $y^2 - x^2 = 2xy\frac{dy}{dx}$ (ii) $2a\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 = 0$

(iii) $y\left(\frac{dy}{dx}\right)^2 + 2x\frac{dy}{dx} = y$

Exercise 8(b)

I. 1. $\text{Sin}^{-1}x + \text{Sin}^{-1}y = c$

2. $x^2 = cy$

II. 1. $\text{Tan}^{-1}y = \text{Tan}^{-1}x + \text{Tan}^{-1}c$ 2. $e^{-y} = e^{-x} + c$

3. $e^y = k(y+1)(1+e^{-x})$ 4. $e^y = e^x + \frac{x^3}{3} + c$

5. $\sin x \sin y = c$

6. $x\sqrt{1+x^2} + y\sqrt{1+y^2} + \log\left[(x+\sqrt{1+x^2})\ (y+\sqrt{1+y^2})\right] = c$

7. $5+x=\dfrac{cy}{1-5y}$

8. $y-x=\log\left|\dfrac{kx}{y}\right|$

III. 1. $(1+x^2)(1+y^2)=cx^2$

2. $1-e^{-3y}=e^{x^3}.\,k$

3. $(x^2+1)(y^2+1)=c$

4. $cy=\cosh^2 x$

5. $\tan(x+y)-\sec(x+y)=x+c$

6. $\text{Tan}^{-1}\left(\dfrac{2x+1}{\sqrt{3}}\right)+\text{Tan}^{-1}\left(\dfrac{2y+1}{\sqrt{3}}\right)=c$

7. $x-y-\dfrac{1}{2}\sin[2(x+y)]=c.$

Exercise 8(c)

I. 1. $\dfrac{dy}{dx}=\dfrac{y}{x}+\sqrt{1+\left(\dfrac{y}{x}\right)^2}$

2. $\dfrac{dy}{dx}=\dfrac{\dfrac{y}{x}\text{Tan}^{-1}\left(\dfrac{y}{x}\right)-1}{\text{Tan}^{-1}\left(\dfrac{y}{x}\right)}$

3. $\dfrac{dy}{dx}=\dfrac{y}{x}\left[\log\left(\dfrac{y}{x}\right)+1\right]$

II. 1. $x^2-2xy-y^2=\text{A}$

2. $k(x^2-y^2)=y$

3. $\log\left(\dfrac{x+y}{c}\right)=\dfrac{-2xy}{(x+y)^2}$

4. $ky=e^{\frac{y}{x}}$

5. $\log x=2\text{Tan}^{-1}\left(\dfrac{y}{x}\right)+c$

6. $x^2(x^2-2y^2)=k$

7. $ky^3\,e^{x/y}=x^2$

8. $y=c\,e^{\text{Tan}^{-1}\left(\frac{y}{x}\right)}$

9. $xy(y-x)=c$

10. $y-2x=kx^2y$

11. $cx^2=y+\sqrt{x^2+y^2}$

12. $(x+y)^3=c(x-y)$

13. $x^2+2y^2(c+\log y)=0$

14. $xy^2=c(x-y)^2$

III. 1. $x + y\,e^{x/y} = k$

2. $kx = e^{\cos\left(\frac{y}{x}\right)}$

3. $\tan\left(\frac{y}{x}\right) = \log x + c$

4. $y \log y + (x - y) \log x = y + cx$

5. $xy \cos\left(\frac{y}{x}\right) = c$

6. $\tan\left(\frac{y}{x}\right) = 1 - \log|x|$

Exercise 8(d)

I. 1. $y^2 + 6x^2 + 5xy - 4y - 9x = c$

2. $4xy + 3(x^2 + y^2) - 10(x - y) = k$

3. $4xy + 3(x^2 + y^2) - 10(x + y) = k$

4. $2xy + 2y - 3y^2 - x - 2x^2 = k$

5. $y^2 - x^2 + 2xy - 2y - 4x = c$

6. $y^2 - x^2 + xy - 3y - x = c$

II. 1. $6y - 3x + \log(3x + 3y + 4) = c$

2. $y - 2x + \frac{3}{8}\log(24y + 16x + 23) = k$

3. $2y + x + \log(2x + y - 1) = k$

4. $(8y - 4x) + \log(4x + 8y + 5) = c$

5. $x - y + \log(x + y) = c$

III. 1. $(y - x + 1)^2 (y + x - 1)^5 = k$

2. $(3y - 2x - 1)^2 (2y + x - 2) = c$

3. $(x + y - 1)^2 (y + 2x - 3)^3 = c$

4. $(x^2 - 2y^2 - 2x - 4y - 2) = c\left[\frac{x - y\sqrt{2} - \sqrt{2} - 1}{x + y\sqrt{2} + \sqrt{2} - 1}\right]^{\frac{1}{\sqrt{2}}}$

5. $2\,\mathrm{Tan}^{-1}\left(\frac{2y+1}{2x+1}\right) = \log\left|c^2\left(x^2 + y^2 + x + y + \frac{1}{2}\right)\right|$

6. $\sqrt{3}\log(Y^2 - 2XY - 2X^2) + 2\log\left\{\frac{Y - (1+\sqrt{3})X}{Y - (1-\sqrt{3})X}\right\} = c$

where $X = x - 1, Y = y - 2$.

7. $\frac{2+\sqrt{3}}{2}\log(\sqrt{3}Y - X) - \frac{2-\sqrt{3}}{2}\log(\sqrt{3}Y + X) = \sqrt{3}C$, where $X = x - 1$, $Y = y + 2$

8. $(2x - y)^2 = c(x + 2y - 5)$.

Exercise 8(e)

I. 1. $\frac{1}{x}$ 2. $\frac{1}{y}$

II. 1. $2y = x\cos x + \sin x\cos^2 x + c\cos x$

2. $y(\sec x + \tan x) = \sec x + \tan x - x + c$

3. $y\cos x = e^x + c$ 4. $yx^2 = \frac{x^2}{2}\log x - \frac{x^2}{4} + c$

5. $2y\, e^{\text{Tan}^{-1}x} = e^{2\text{Tan}^{-1}x} + c$ 6. $yx^2 = \frac{2x^5}{5} + c$

7. $y(x^2+1)^2 = x + c$ 8. $xy = xe^x + c$

9. $y(1+x^3) = x + \frac{x^3}{3} + c$ 10. $y = e^{-x} + c\,e^x$

11. $y = \text{Tan}^{-1}x - 1 + c\, e^{-\text{Tan}^{-1}x}$ 12. $y\sec x = \log\sec x + c$

III. 1. $y\sec x = \tan x + \frac{1}{3}\tan^3 x + c$ 2. $y = -(\sin x + 1) + c\, e^{\sin x}$

3. $y\log x = (\log x)^2 + c$ 4. $x = k\,e^y - (y+2)$

5. $\frac{xy}{x-1} = \frac{x^5}{5} - \frac{x^4}{4} + c$ 6. $x = y(y^2 + c)$

7. $y = \sqrt{1-x^2} + c(1-x^2)$ 8. $y(x-1) = x^2(x^2 - x + c)$

9. $1 + x(y^2 - 2 + c\,e^{-y^2/2}) = 0$ 10. $\tan y = \frac{1}{2}(x^2-1) + ce^{-x^2}$

Reference Books

- The Elements of Coordinate Geometry; S.L. Loney; Macmillan & Co., London; 1975.
- Calculus - Vol - I & II; Lipman Bers; IBH publishing Co., Mumbai; 1973.
- A Course of Pure Mathematics; G.H. Hardy; Cambridge University Press (ELBS); 1976.
- A First Course in Calculus; Serge Lang; Addison Wesley Publication Co. Inc., 1965.
- Calculus - Schaum's Outline series; Frank Ayres; McGra -Hill; Education (India) Ltd.; 2007.
- Differential Equations -Schaum's Outline series; Richard Bronson; McGraw - Hill Education (India) Ltd., New Delhi; 2007.
- Differential Equations; Deo; McGraw - Hill Education (India) Ltd., New Delhi; 2007.

***

BOARD OF INTERMEDIATE EDUCATION
Syllabus in Mathematics Paper - IIB
To be effective from the academic year 2013-14

Name of Topic and Sub Topics	No. of Periods
COORDINATE GEOMETRY	
01. Circle	
1.1 Equation of a circle - standard form - centre and radius - equation of a circle with a given line segment as diameter & equation of a circle through three non collinear points - parametric equations of a circle.	08
1.2 Position of a point in the plane of a circle - power of a point- definition of tangent-length of tangent.	06
1.3 Position of a straight line in the plane of a circle- conditions for a line to be tangent- chord joining two points on a circle- equation of the tangent at a point on the circle- point of contact- equation of normal.	06
1.4 Chord of contact - pole and polar-conjugate points and conjugate lines - equation of chord in terms of its mid point.	06
1.5 Relative position of two circles- circles touching each other externally, internally common tangents -centers of similitude- equation of pair of tangents from an external point.	08
	34
02. System of Circles	
2.1 Angle between two intersecting circles.	06
2.2 Radical axis of two circles- properties-common chord and common tangent of two circles - radical centre.	06
	12

03.	**Parabola**		
	3.1	Conic sections -Parabola- equation of parabola in standard form- different forms of parabola- parametric equations.	08
	3.2	Equations of tangent and normal at a point on the parabola (cartesian and parametric) - conditions for a straight line to be tangent.	07
			15
04.	**Ellipse**		
	4.1	Equation of ellipse in standard form-Parametric equations.	06
	4.2	Equation of tangent and normal at a point on the ellipse (cartesian and parametric)-condition for a straight line to be tangent.	07
			13
05.	**Hyperbola**		
	5.1	Equation of hyperbola in standard form-Parametric equations.	04
	5.2	Equations of tangent and normal at a point on the hyperbola (cartesian and parametric)- conditions for a straight line to be a tangent-Asymptotes.	04
			08
CALCULUS			
	6.1	Integration as the inverse process of differentiation- Standard forms -properties of integrals.	04
	6.2	Method of substitution- integration of Algebraic, exponential, logarithmic, trigonometric and inverse trigonometric functions. Integration by parts.	14

6.3	Integration by Partial fractions method.	05
6.4	Reduction formulae.	05
		28
07.	**Definite Integrals**	
7.1	Definite Integral as the limit of a sum	03
7.2	Interpretation of Definite Integral as an area.	03
7.3	Fundamental Theorem of Integral Calculus (without proof).	04
7.4	Properties.	04
7.5	Reduction formulae.	06
7.6	Application of Definite integral to areas.	04
		24
08.	**Differential Equations**	
8.1	Formation of differential equation-degree and order of an ordinary differential equation.	02
8.2	Solving differential equation by	
	a) Variables separable method.	03
	b) Homogeneous differential equation.	03
	c) Non - Homogeneous differential equation.	04
	d) Linear differential equations.	04
		16
	TOTAL	**150**

BOARD OF INTERMEDIATE EDUCATION, A.P.

Mathematics - IIB

Model Question Paper (w.e.f. 2013-14)

Time: 3 hrs **Max. Marks:75**

Note: This Question paper consists of three sections A, Band C.

SECTION - A

I. Very Short Answer type Questions

(i) Answer all Questions

(ii) Each Question carries 2 marks **$10 \times 2 = 20$**

1. If $ax^2 + bxy + 3y^2 - 5x + 2y - 3 = 0$ represents a circle, find the values of a and b. Also find its radius and centre.

2. State the necessary and sufficient condition for $lx + my + n = 0$ to be a normal to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$.

3. Find the angle between the circles $x^2 + y^2 - 12x - 6y + 41 = 0$ and $x^2 + y^2 + 4x + 6y - 59 = 0$.

4. Find the equation of the parabola whose focus is $S(1, -7)$ and vertex is $A(1, -2)$.

5. Find the angle between the asymptotes of the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$.

6. Evaluate $\displaystyle\int \frac{1}{(x+3)\sqrt{x+2}}\,dx$

7. Evaluate $\displaystyle\int \frac{\sin^4 x}{\cos^6 x}\,dx$

8. Evaluate $\displaystyle\int_0^1 \frac{x^2}{x^2+1}\,dx$

9. Evaluate $\int_0^{\frac{\pi}{2}} \frac{\sin^2 x - \cos^2 x}{\sin^3 x + \cos^3 x} dx$

10. Find the order and degree of the differential equation $\left[\frac{d^2 y}{dx^2} - \left(\frac{dy}{dx}\right)^3\right]^{6/5} = 6y$.

SECTION - B

II. Short Answer type Questions

(i) Answer any five Questions

(ii) Each Question carries 4 marks **$5 \times 4 = 20$**

11. Show that the tangent at $(-1, 2)$ of the circle $x^2 + y^2 - 4x - 8y + 7 = 0$ touches the circle $x^2 + y^2 + 4x + 6y = 0$. Also find its point of contact.

12. Find the equation of the circle passing through the points of intersection of the circles $x^2 + y^2 - 8x - 6y + 21 = 0$, $x^2 + y^2 - 2x - 15 = 0$ and $(1, 2)$.

13. Find the length of major axis, minor axis, latus rectum, eccentricity of the ellipse $9x^2 + 16y^2 = 144$.

14. Show that the point of intersection of the perpendicular tangents to an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, (a > b)$ lies on a circle.

15. Find the equations of the tangents to the hyperbola $3x^2 - 4y^2 = 12$ which are (i) Parallel to (ii) Perpendicular to the line $y = x - 7$.

16. Find the reduction formula for $\int_0^{\frac{\pi}{2}} \sin^n x \, dx$

17. Solve: $(1 + y^2)dx = (\mathrm{Tan}^{-1} y - x)dy$

SECTION - C

III. Long Answer type Questions

(i) Answer any five Questions

(ii) Each Question carries 7 marks **$5 \times 7 = 35$**

18. Show that the points $(1, 1), (-6, 0), (-2, 2)$ and $(-2, -8)$ are concyclic.

19. Find the direct common tangents to the circles

$x^2 + y^2 + 22x - 4y - 100 = 0, x^2 + y^2 - 22x + 4y + 100 = 0.$

20. If y_1, y_2, y_3 are the y-coordinates of the vertices of the triangle inscribed in the parabola $y^2 = 4ax$ then show that the area of the triangle is $\frac{1}{8a}|(y_1 - y_2)(y_2 - y_3)(y_3 - y_1)|$ square units.

21. Evaluate $\int \frac{9\cos x - \sin x}{4\sin x + 5\cos x} dx$

22. Evaluate $\int \frac{dx}{(1+x)\sqrt{3+2x-x^2}}$

23. Evaluate $\int_0^1 \frac{\log(1+x)}{1+x^2} dx$

24. Solve : $\frac{dy}{dx} = \frac{2x+y+3}{2y+x+1}$