Text Book for

INTERMEDIATE

First Year

Mathematics

Paper - IB

Coordinate Geometry, Calculus

Telugu and Sanskrit Akademi
Andhra Pradesh

Intermediate
First Year

Mathematics
Paper - IB
Text Book

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Y.S. JAGAN MOHAN REDDY

CHIEF MINISTER
ANDHRA PRADESH

AMARAVATI

MESSAGE

I congratulate Akademi for starting its activities with printing of textbooks from the academic year 2021 – 22.

Education is a real asset which cannot be stolen by anyone and it is the foundation on which children build their future. As the world has become a global village, children will have to compete with the world as they grow up. For this there is every need for good books and good education.

Our government has brought in many changes in the education system and more are to come. The government has been taking care to provide education to the poor and needy through various measures, like developing infrastructure, upgrading the skills of teachers, providing incentives to the children and parents to pursue education. Nutritious mid-day meal and converting Anganwadis into pre-primary schools with English as medium of instruction are the steps taken to initiate children into education from a young age. Besides introducing CBSE syllabus and Telugu as a compulsory subject, the government has taken up numerous innovative programmes.

The revival of the Akademi also took place during the tenure of our government as it was neglected after the State was bifurcated. The Akademi, which was started on August 6, 1968 in the undivided state of Andhra Pradesh, was printing text books, works of popular writers and books for competitive exams and personality development.

Our government has decided to make available all kinds of books required for students and employees through Akademi, with headquarters at Tirupati.

I extend my best wishes to the Akademi and hope it will regain its past glory.

Y.S. JAGAN MOHAN REDDY

Dr. Nandamuri Lakshmiparvathi
M.A., M.Phil., Ph.D.
Chairperson, (Cabinet Minister Rank)
Telugu and Sanskrit Akademi, A.P.

Message of Chairperson, Telugu and Sanskrit Akademi, A.P.

In accordance with the syllabus developed by the Board of Intermediate, State Council for Higher Education, SCERT etc., we design high quality Text books by recruiting efficient Professors, department heads and faculty members from various Universities and Colleges as writers and editors. We are taking steps to print the required number of these books in a timely manner and distribute through the Akademi's Regional Centers present across the Andhra Pradesh.

In addition to text books, we strive to keep monographs, dictionaries, dialect texts, question banks, contact texts, popular texts, essays, linguistics texts, school level dictionaries, glossaries, etc., updated and printed and made available to students from time to time.

For competitive examinations conducted by the Andhra Pradesh Public Service Commission and for Entrance examinations conducted by various Universities, the contents of the Akademi publications are taken as standard. So, I want all the students and Employees to make use of Akademi books of high standards for their golden future.

Congratulations and best wishes to all of you.

N. Lakshmiparvathi

Nandamuri Lakshmiparvathi
Chairperson, Telugu and Sanskrit Akademi, A.P.

J. SYAMALA RAO, I.A.S.,
Principal Secretary to Government

Higher Education Department
Government of Andhra Pradesh

MESSAGE

I Congratulate Telugu and Sanskrit Akademi for taking up the initiative of printing and distributing textbooks in both Telugu and English media within a short span of establishing Telugu and Sanskrit Akademi.

Number of students of Andhra Pradesh are competing of National Level for admissions into Medicine and Engineering courses. In order to help these students Telugu and Sanskrit Akademi consultation with NCERT redesigned their Textbooks to suit the requirement of National Level Examinations in a lucid language.

As the content in Telugu and Sanskrit Akademi books is highly informative and authentic, printed in multi-color on high quality paper and will be made available to the students in a time bound manner. I hope all the students in Andhra Pradesh will utilize the Akademi textbooks for better understanding of the subjects to compete of state and national levels.

(J. SYAMALA RAO)

THE CONSTITUTION OF INDIA

PREAMBLE

WE, THE PEOPLE OF INDIA, having solemnly resolved to constitute India into a [SOVEREIGN SOCIALIST SECULAR DEMOCRATIC REPUBLIC] and to secure to all its citizens:

JUSTICE, social, economic and political;

LIBERTY of thought, expression, belief, faith and worship;

EQUALITY of status and of opportunity; and to promote among them all

FRATERNITY assuring the dignity of the individual and the [unity and integrity of the Nation];

IN OUR CONSTITUENT ASSEMBLY this twenty-sixth day of November, 1949 do HEREBY ADOPT, ENACT AND GIVE TO OURSELVES THIS CONSTITUTION.

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Foreword

The Government of India vowed to remove the educational disparities and adopt a common core curriculum across the country especially at the Intermediate level. Ever since the Government of Andhra Pradesh and the Board of Intermediate Education (BIE) swung into action with the task of evolving a revised syllabus in all the Science subjects on par with that of CBSE, approved by NCERT, its chief intention being enabling the students from Andhra Pradesh to prepare for the National Level Common Entrance tests like NEET, ISEET etc for admission into Institutions of professional courses in our Country.

For the first time BIE AP has decided to prepare the Science textbooks. Accordingly an Academic Review Committee was constituted with the Commissioner of Intermediate Education, AP as Chairman and the Secretary, BIE AP; the Director SCERT and the Director Telugu Akademi as members. The National and State Level Educational luminaries were involved in the textbook preparation, who did it with meticulous care. The textbooks are printed on the lines of NCERT maintaining National Level Standards.

The Education Department of Government of Andhra Pradesh has taken a decision to publish and to supply all the text books with free of cost for the students of all Government and Aided Junior Colleges of newly formed state of Andhra Pradesh.

We express our sincere gratitude to the Director, NCERT for according permission to adopt its syllabi and curriculum of Science textbooks. We have been permitted to make use of their textbooks which will be of great advantage to our student community. I also express my gratitude to the Chairman, BIE and the honorable Minister for HRD and Vice Chairman, BIE and Secretary (SE) for their dedicated sincere guidance and help.

I sincerely hope that the assorted methods of innovation that are adopted in the preparation of these textbooks will be of great help and guidance to the students.

I wholeheartedly appreciate the sincere endeavors of the Textbook Development Committee which has accomplished this noble task.

Constructive suggestions are solicited for the improvement of this textbook from the students, teachers and general public in the subjects concerned so that next edition will be revised duly incorporating these suggestions.

It is very much commendable that Intermediate text books are being printed for the first time by the Akademi from the 2021-22 academic year.

Sri. V. Ramakrishna I.R.S.
Director
Telugu and Sanskrit Akademi,
Andhra Pradesh

Preface

The Board of Intermediate Education (A.P.) has recently revised the syllabus in Mathematics for the Intermediate Course, with effect from the academic year 2012-13. Accordingly, Telugu Akadcmi has prepared the necessary Text Books in Mathematics.

In accordance with the current syllabus, the topics relating to paper-IB; **Coordinate Geometry** and **Calculus** are dealt with in this Book. They are presented in the chapters. Coordinate Geometry consists of eight chapters: **Prerequisties Locus, Transformation of Axes, The Straight Line, Pair of Straight Lines, Three Dimensional Coordinates, Direction of Cosines Direction Ratios** and **The Plane.** Calculus is presented in three Chapters : **Limits and Continuity, Differentiation** and **Applications of Derivatives.**

Every chapter herein is divided into various sections and subsections, depending on the contents discussed. These contents are strictly in accordance with the prescribed syllabus and they reflect faithfully the scope and spirit of the same. Necessary definitions, theorems, corollaries. proofs and notes are given in detail. Key concepts are given at the end of each chapter, Illustrative examples and solved problems are in plenty. and these shall help the students in understanding the subject matter.

Every chapter contains exercises in a graded manner which enable the students to solve them by applying the knowledge acquired. All these problems are classified according to the nature of their answers as **I - very short, II - short and III - long.** Answers are provided for all the exercises at the end of each chapter.

Keeping in view the National level competitives examinations, some concepts and notions are highlighted for the benefit of the students. Care has been taken regarding rigor and logical consistency in the presentation of concepts and in proving theorems. Alt the end of the text Book, a list of some **Reference Books** in the subject matter is furnished.

The Members of the Mathematics Subject Committee, constituted by Board of Intermediate Education, were invited to interact with the team of the Authors and Editors. They pursued the contents chapter wise. and gave some useful suggestions and comments which are duly incorporated.

The special feature of this Book, brought out in a new format is that each chapter begins with a thought mostly on Mathematics through a quotation from a famous thinker. It carries a portrait of a noted mathematician with a brief write-up.

In the concluding part of each chapter some relevant historical notes are appended. Wherever found appropriate, references are also made of the contributions of ancient Indian scientists to the advancement of Mathematics. The purpose is to enable the students to have a glimpse into the history of Mathematics in general and the contributions of Indian mathematicians in particular.

Inspite of enough care taken in the scrutiny at various stages in the preparation of the book, errors might have crept in. The readers are therefore, requested to identify and bring them to the notice of the Akademi. We will appreciate if suggetions to enhance the quality of the book are given. Efforts will be made to incorporate them in the subsequent editions.

Prof. P.V. Arunachalam
Chief Coordinator

Preface to the Reviewed Edition

Telugu Akademi is publishing Text books for Two year Intermediate in English and Telugu medium since its inception, periodical review and revision of these publications has been undertaken as and when there was an updation of Intermediate syllabus.

In this reviewed Edition, now being undertaken by the Telugu Akademi, Andhra Pradesh the basic content of its earlier Edition is considered and it is reviewed by a team of experienced teachers. Modification by way correcting errors, print mistakes, incorporating additional content where necessary to elucidate a concept and / or a definition, modification of existing content to remove obscurities for releasing the concept more easily are carried out mainly in this review.

Not withstanding the effort and time spent by the review team in this endeavour, still a few aspects that still need modification or change might have been left unnoticed.

Constructive suggestions from the academic fraternity are welcome and the Akademi will take necessary steps to incorporate them in the forth coming edition.

We appreciate the encouragement and support extended by the Academic and Administrative staff of the Telugu Akademi in fulfilling our assignment with satisfaction.

Editors
(Reviewed Edition)

Contents

Greek Alphabet

α	Α	Alpha
β	Β	Beta
γ	Γ	Gamma
δ	Δ	Delta
ε	Ε	Epsilon
ζ	Ζ	Zeta
η	Η	Eta
θ	Θ	Theta
ι	Ι	Iota
κ	Κ	Kappa
λ	Λ	Lambda
μ	Μ	Mu
ν	Ν	Nu
ξ	Ξ	Xi
ο	Ο	Omicron
π	Π	Pi
ρ	Ρ	Rho
σ	Σ	Sigma
τ	Τ	Tau
υ	Υ	Upsilon
ϕ	Φ	Phi
χ	Χ	Chi
ψ	Ψ	Psi
ω	Ω	Omega

Coordinate Geometry

Chapter 0

Prerequisites

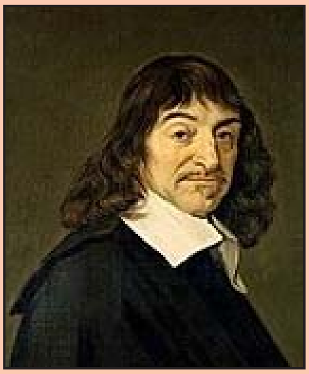
"With me everything turns into mathematics"

- Rene Descartes

Introduction

The study of geometry in coordinate method began in the third and second centuries B.C. The Greek mathematician ***Menaechmus*** (ca. 380 - 320 B.C.) used a method which had a similarity with the present methods of coordinate geometry. ***Apollonius*** (ca. 262 - ca.190 B.C.) of Perga, solved problems in a way resembling analytical geometry of one dimension. He dealt with conics wherein he developed a method which is again similar to analytical geometry, anticipating the work of ***Rene Descartes*** by eighteen centuries earlier. ***Apollonius*** came close to inventing analytic geometry, but could not do so, as he did not take into account the negative magnitudes. However, the decisive step in developing coordinate geometry as a subject was taken later by ***Descartes*** and ***Fermat***. ***Abraham de Moivre*** (1667 - 1754) also made contribution to the development of coordinate geometry.

Fermat and ***Descartes*** studied the plane curves in the seventeenth century, while ***Euler*** (1707 - 1783) took up the study of curves in space in the eighteenth century. These mathematicians had developed the coordinate geometry in a systematic way and opened the doors to a new branch of mathematics.



Rene Descartes

(1596 - 1650)

Rene Descartes *was a famous French mathematician, philosopher and writer. He was hailed as the "Father of Modern Mathematics". The Cartesian coordinate system that is used in geometry is named after him. Descartes theory provided the basis for the calculus of Newton and Leibnitz.*

The objective of coordinate geometry is to study the algebraic meaning of geometric figures and the geometric meaning of algebraic expressions. The fusion of geometric and algebraic thinking, together with the concept of function provides us an important frame for comprehension and exploration of objective reality. For example, the algebraic equation $y = mx + c$ gives rise to the geometric idea of a straight line and, on the other hand, any straight line is represented by an algebraic equation $ax + by + c = 0$ for real numbers a, b and c, with $a^2 + b^2 \neq 0$.

In 1637, ***Descartes***, in his famous book 'La Geometre', proposed that, a point in the plane can be represented as an ordered pair of real numbers which are usually the distances of that point from two mutually perpendicular lines (called coordinate axes). In honour of ***Descartes***, coordinate geometry is called '***Cartesian Geometry***'. However, after *Descartes*, the present form of coordinate geometry was developed, particularly by ***Euler***.

0.1 Prerequisites

In the present section we shall review some of the topics covered in lower classes.

0.1.1 Coordinate axes

Consider a pair of mutually perpendicular lines of reference X′ X and Y′Y in a plane (Fig. 0.1). These straight lines are called **coordinate axes** and their point of intersection is called the **origin**, and it is denoted by O.

The horizontal line is called the X–axis and the vertical line is called the **Y–axis**.

On each axis a positive direction is chosen (indicated by an arrow). Naturally, the opposite direction is referred to as the negative direction. A unit of length (scale unit) is chosen, which may be arbitrary but is the same for both the axes. The coordinate axes X′X, Y′Y (with established positive and negative directions and an appropriate scale unit) form a rectangular coordinate system (Cartesian coordinate system). The coordinate axes divide the plane into four equal parts called **quadrants**.

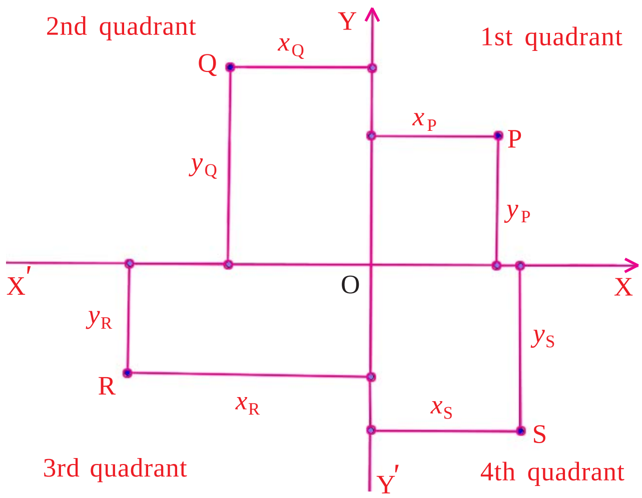


Fig. 0.1 Cartesian coordinate system

0.1.2 Coordinates

Consider a point P in the plane. Let x_P denote the perpendicular distance of P from Y-axis and y_P denote the perpendicular distance of P from X–axis. Then P is represented by the ordered pair of real numbers as shown in the following table.

Quadrant in which P lies	**Ordered pair which denotes P**
1st quadrant	(x_P, y_P)
2nd quadrant	$(-x_P, y_P)$
3rd quadrant	$(-x_P, -y_P)$
4th quadrant	$(x_P, -y_P)$

The first element of the ordered pair is called the **x – coordinate (abscissa)** and the second element is called the **y – coordinate (ordinate)**. The point P is generally represented by P (x, y). Conversely, every ordered pair of real numbers represents a unique point in the plane.

It can be easily seen that the y – coordinate y_P is zero for any point P on the X – axis and the x – coordinate x_P is zero for any point P on the Y – axis. Thus the coordinates of the origin O are $(0, 0)$ as it belongs to both the axes.

0.1.3 Distance between two points

(i) The distance between two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ in the plane is denoted by PQ and is given by

$$PQ = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

$$= \sqrt{\left[(\text{difference of } x \text{ coordinates})^2 + (\text{difference of } y \text{ coordinates})^2\right]}$$

Note that $PQ = QP$.

(ii) The distance of $P(x_1, y_1)$ from the origin $O(0, 0)$ is

$$OP = \sqrt{x_1^2 + y_1^2}\,.$$

(iii) The distance between two points $A(x_1, 0)$ and $B(x_2, 0)$ is

$$AB = \sqrt{(x_1 - x_2)^2 + (0 - 0)^2}$$

$$= \sqrt{(x_1 - x_2)^2} = |x_1 - x_2|\,.$$

(iv) The distance between two points $C(0, y_1)$ and $D(0, y_2)$ is $CD = |y_1 - y_2|$.

If A and B are two distinct points, then we denote

(i) the line segment joining A and B by $\overline{AB}$,

(ii) the ray from A through B by $\overrightarrow{AB}$, and

(iii) the straight line passing through A and B by $\overleftrightarrow{AB}$.

0.1.4 Section formula

(i) If P lies on the line segment joining two points A and B then P is said to divide the line segment $\overline{AB}$ in the ratio AP : PB internally (see Fig. 0.2).

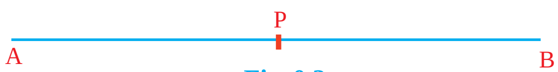


Fig. 0.2

(ii) If P lies on the extended line of the segment joining A and B, then P is said to divide the line segment $\overline{AB}$ externally (see Fig. 0.3).

Fig. 0.3

In this case – AP : PB or AP : – PB is taken as the ratio in which P divides the line segment $\overline{AB}$. The positive and negative signs of AP and PB are in accordance with the signs given to the abscissa.

(iii) The coordinates of a point P which divides the line segment joining A (x_1, y_1) and B (x_2, y_2) in the ratio $l : m$ are

$$\boxed{P = \left(\frac{lx_2 + mx_1}{l+m}, \frac{ly_2 + my_1}{l+m}\right).}$$

It can be easily seen that P divides the line segment $\overline{AB}$ internally when the ratio $l : m$ or l/m is positive and externally when the ratio l/m is negative. Note that $l + m \neq 0$, for, if $l + m = 0$, then $l = -m$ and hence P divides the line segment $\overline{AB}$ externally and AP = PB which is absurd. The point A divides the line segment $\overline{AB}$ in the ratio 0 : AB and the point B divides the line segment $\overline{AB}$ in the ratio AB : 0. All other points (on the line joining A and B) divide the line segment $\overline{AB}$ in the ratio $l : m$, where l and m are both non zero and in this case, the ratio can be taken as $\lambda : 1$, where λ is a real number ($\lambda \neq -1$).

(iv) Coordinates of the mid point of the line segment $\overline{AB}$ are

$$\boxed{\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).}$$

0.1.5 The points of trisection

The points which divide a line segment in the ratio 1 : 2 or 2 : 1 are called **the points of trisection.**

0.1.6 Area of a triangle

(i) The area of the triangle with vertices at A (x_1, y_1), B (x_2, y_2) and C (x_3, y_3) is given by

$$\Delta = \frac{1}{2}\left|x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\right|$$

$$= \frac{1}{2}\left|\Sigma x_1(y_2 - y_3)\right|.$$

(ii) The area of the triangle OAB with vertices O(0, 0), A (x_1, y_1) and B(x_2, y_2) is given by

$$\boxed{\Delta = \frac{1}{2}\left|x_1 y_2 - x_2 y_1\right|.}$$

0.1.7 Area of a Quadrilateral

The area of the quadrilateral with vertices A (x_1, y_1), B(x_2, y_2), C(x_3, y_3) and D(x_4, y_4)

$$= \frac{1}{2}\left|x_1(y_2 - y_4) + x_2(y_3 - y_1) + x_3(y_4 - y_2) + x_4(y_1 - y_3)\right|$$

$$= \frac{1}{2}\left|\Sigma x_1(y_2 - y_4)\right|$$

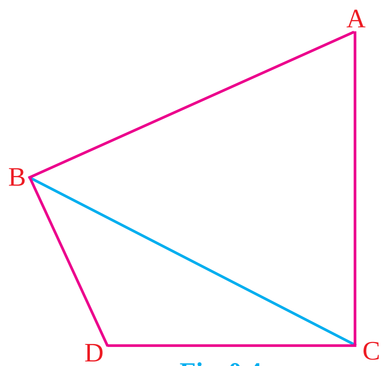


Fig. 0.4

Note that the area of the quadrilateral ABCD is the sum of the areas of triangles ABC and BCD (see Fig. 0.4).

0.1.8 Concurrent lines, Point of Concurrency

Three or more straight lines are said to be **Concurrent**, if all the straight lines have exactly one point in common and this common point is called the **Point of Concurrency**. (see Fig. 0.5)

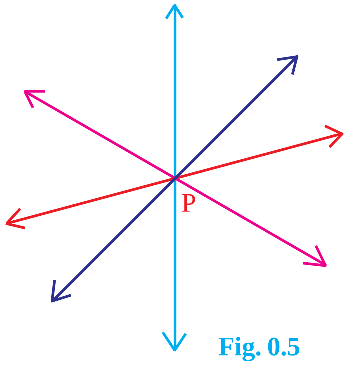


Fig. 0.5

0.1.9 Medians-Centroid

In a triangle the line segment joining a vertex to the midpoint of the opposite side is called a **Median** of the triangle. Clearly a triangle has three medians. We will prove later that the medians of a triangle are concurrent. The point of concurrency is called the **Centroid** of the triangle and is usually denoted by G. The centroid G divides each median in the ratio 2 : 1 internally (see Fig. 0.6).

The centroid G of the triangle with vertices at $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ is given by

$$\boxed{G = \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right).}$$

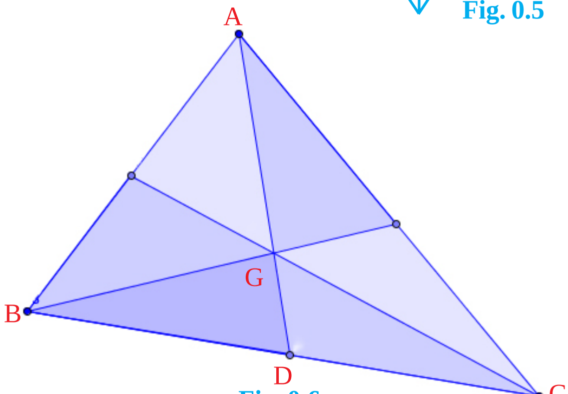


Fig. 0.6

0.1.10 Angular bisectors – In-centre

The bisectors of the internal angles of a triangle are concurrent. The point of concurrency is called the **In-centre** of the triangle and is denoted by I. It is equidistant from the three sides of the triangle and the distance is called the **In-radius** of the triangle and is denoted by r. The circle drawn with I as centre and r as radius, touches all the three sides of the triangle internally. This circle is called the **In-circle** of the triangle. (see Fig. 0.7).

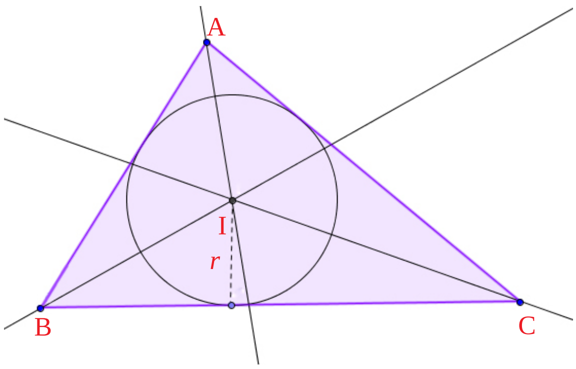


Fig. 0.7

0.1.11 Theorem : *If $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are the vertices and a, b and c are respectively the sides BC, CA and AB of the triangle ABC, then the coordinates of the In-centre are*

$$I = \left(\frac{ax_1 + bx_2 + cx_3}{a + b + c}, \frac{ay_1 + by_2 + cy_3}{a + b + c}\right).$$

Proof : Let $\overline{AD}$, $\overline{BE}$ and $\overline{CF}$ be the bisectors of $\angle A$, $\angle B$ and $\angle C$ respectively. Then by the vertical angle theorem, D divides $\overline{BC}$ in the ratio $c : b$ and hence the coordinates of D are

$$\left(\frac{cx_3 + bx_2}{c + b}, \frac{cy_3 + by_2}{c + b}\right).$$

Again by the same theorem, the point I on $\overline{AD}$ divides $\overline{AD}$ in the ratio $b + c : a$ (AI : ID).

Therefore, the coordinates of I are

$$\left[\frac{(b+c)\left(\frac{cx_3+bx_2}{c+b}\right)+ax_1}{(b+c)+a}, \frac{(b+c)\left(\frac{cy_3+by_2}{c+b}\right)+ay_1}{(b+c)+a}\right]$$

i.e., $$\boxed{I = \left(\frac{ax_1+bx_2+cx_3}{a+b+c}, \frac{ay_1+by_2+cy_3}{a+b+c}\right).}$$

0.1.12 Angular bisectors – Ex-centre

We will prove later that in a triangle ABC the bisector of internal angle A and bisectors of external angles B and C are concurrent. The point of concurrency is called the **Ex-centre** of the triangle and is denoted by I_1. It is equidistant from the side BC and from the extensions of the sides AB and AC and this distance is called the **Ex-radius** of the triangle and is denoted by r_1. The circle drawn with I_1 as centre and r_1 as radius touches these sides. This circle is called the **Ex-circle** of the triangle opposite to the vertex A (Fig. 0.8). Similarly, we get two other Ex-circles of the triangle opposite to the vertices B and C respectively. The Ex-centres of these Ex-circles are denoted by I_2 and I_3 and the corresponding Ex-radii by r_2 and r_3 respectively.

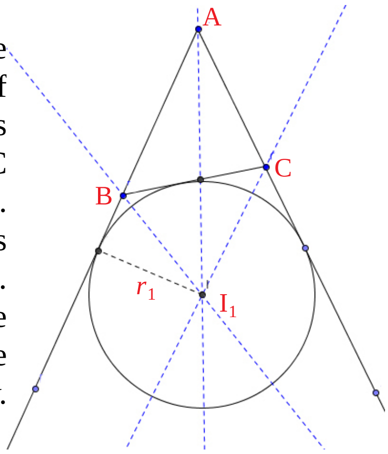


Fig. 0.8

The coordinates of the Ex-centres are given by

$$I_1 = \left(\frac{-ax_1+bx_2+cx_3}{-a+b+c}, \frac{-ay_1+by_2+cy_3}{-a+b+c}\right),$$

$$I_2 = \left(\frac{ax_1-bx_2+cx_3}{a-b+c}, \frac{ay_1-by_2+cy_3}{a-b+c}\right),$$

$$I_3 = \left(\frac{ax_1+bx_2-cx_3}{a+b-c}, \frac{ay_1+by_2-cy_3}{a+b-c}\right).$$

0.1.13 Altitudes - Orthocentre

In a triangle the perpendicular from a vertex to the opposite side is called an **Altitude** of the triangle. We will prove later by analytical methods that the three altitudes of a triangle are concurrent. The point of concurrency is called the **Orthocentre** of the triangle and is usually denoted by 'O' (see Fig. 0.9).

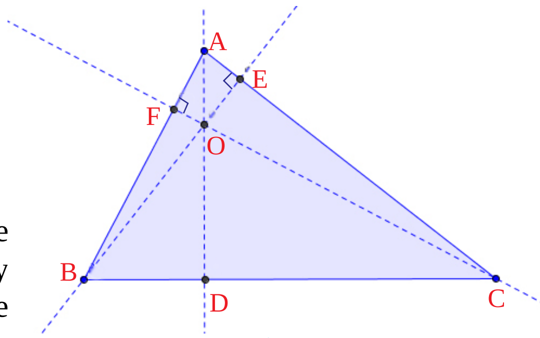


Fig. 0.9

0.1.14 Perpendicular bisectors of sides - Circumcentre

We will prove later that the perpendicular bisectors of the sides by analytical methods of a triangle are concurrent. The point of concurrency is called the **Circumcentre** of the triangle and is denoted by S. The circumcentre S is equidistant from the vertices of the triangle. Therefore the circle drawn with S as centre and the distance from S to any vertex as radius, passes through all the three vertices of the triangle. This circle is called the **Circumcircle** of the triangle (see Fig. 0.10).

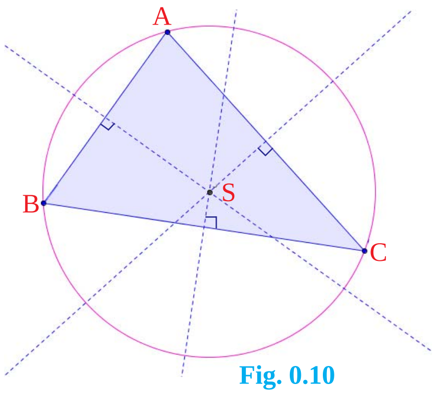


Fig. 0.10

Key Concepts

- Let O(0, 0) be the origin and P (x_1, y_1), Q (x_2, y_2) be any two points in a plane. Then
  PQ = $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$ = QP, OP = $\sqrt{x_1^2 + y_1^2}$.
- Let A (x_1, y_1) and B (x_2, y_2) be any two points in the plane. Then
  (i) the coordinates of the point which divides $\overline{AB}$ in the ratio $l : m$ are
  $$\left(\frac{lx_2 + mx_1}{l + m}, \frac{ly_2 + my_1}{l + m}\right).$$
  The division is called internal if $l : m$ or l/m is positive and external if l/m is negative.

  (ii) The coordinates of the mid point of the line segment $\overline{AB}$ are $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.
- Let A(x_1, y_1), B(x_2, y_2), C(x_3, y_3) and D(x_4, y_4) be any four points in a plane, then

  (i) the area of ΔABC = $\frac{1}{2} \left| \Sigma x_1 (y_2 - y_3) \right|$.

  (ii) The area of ΔOAB = $\frac{1}{2} \left| x_1 y_2 - x_2 y_1 \right|$.

  (iii) the area of quadrilateral ABCD = $\frac{1}{2} \left| \Sigma x_1 (y_2 - y_4) \right|$.

  (iv) the coordinates of the centroid G of ΔABC = $\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$.

  (v) the coordinates of the In-centre I of
  $$\Delta ABC = \left(\frac{ax_1 + bx_2 + cx_3}{a + b + c}, \frac{ay_1 + by_2 + cy_3}{a + b + c}\right),$$
  where a, b and c are respectively the sides BC, CA and AB.

  (vi) the coordinates of the Ex-centres of ΔABC are
  $$I_1 = \left(\frac{-ax_1 + bx_2 + cx_3}{-a + b + c}, \frac{-ay_1 + by_2 + cy_3}{-a + b + c}\right),$$
  $$I_2 = \left(\frac{ax_1 - bx_2 + cx_3}{a - b + c}, \frac{ay_1 - by_2 + cy_3}{a - b + c}\right),$$
  $$I_3 = \left(\frac{ax_1 + bx_2 - cx_3}{a + b - c}, \frac{ay_1 + by_2 - cy_3}{a + b - c}\right).$$
  where I_1, I_2 and I_3 are the Ex-centres of the Ex-circles opposite to the vertices A, B and C of the triangle ABC.

Historical Note

The earliest use of geometry that is known was that of the *Egyptians*, *Babylonians* and *Indians*. The word 'Geometry' is from the *Greek* for 'Earth-measurement', which was the primary concern of the ancient geometers. The knowledge of geometry of these early people was obtained by observation and trial-and-error methods and they made no attempt to generalize or to prove their assumptions.

During the Vedic times and later during Sulvasutra period (1500 B.C.) we find geometrical concepts like orbits of planets, the properties of right angled triangle, squaring of the circle and areas and volumes of simple geometrical figures.

It was not until about 600 B.C. that any concern was felt about why certain relationships existed and whether these relationships could be proved by logical reasoning. To the *Greeks* of the Golden Age the philosophy of geometry was more important than its applications. Interest centered upon ideas rather than physical conditions.

Thales of Miletus (ca. 624 - ca. 547 B.C.), the father of Greek mathematics, is famous for the study of geometry based on logical reasoning. His pupil *Pythagoras* (ca. 572-496 B.C.) is remembered chiefly for his theorem on a right triangle. *Apollonius* of Perga (times) studied conic sections. But it is to *Euclid*, who wrote the 'Elements' in about 300 B.C., that we owe the greatest debt.

A long time after the Golden Age of Greece, in the 17th century two great mathematicians, *Rene Descartes* (1596 - 1650) and *Pierre de Fermat* (1601 - 1665) were the codiscoverers of analytic geometry, a new method of geometry.

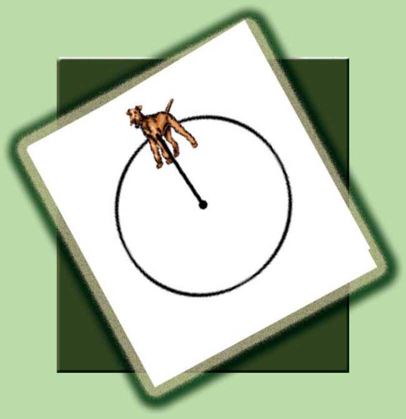
The Renaissance painters employed projective geometry to depict three dimensional real world on two dimensional canvas. *Gerard Desargues* (17th century) is a notable mathematician in this area.

The important geometric discoveries of 18th and 19th centuries were the non-Euclidean geometries of *Nicholas I. Lobachevsky* (1793 - 1856), *Janos Bolyai* (1802 - 1860), *Carl Friedrich Gauss* (1777 - 1855) and *George F.B. Riemann* (1826 - 1866).

During 18th, 19th and 20th centuries, with the advent of calculus, the geometry further developed into branches like differential geometry and topology. *David Hilbert* (1862 - 1943) gave a logical treatment to the abstract concepts of geometry. In recent times, *Mandelbrot* (1924) introduced fractal geometry, which describes the nature factually.

Chapter 1

Locus



"Where there is matter, there is geometry"

- Johannes Kepler

Introduction

In school mathematics, we learnt about the Euclidean geometry, its axioms, some theorems, constructions, proofs, deductive method, historical background and had a glimpse into Indian contribution to geometry. We have also studied some elementary concepts of **analytical geometry** also known as **coordinate geometry** which unifies the ideas of algebra and geometry. This unification is very well illustrated through the concept of locus about which we learnt a little in the 9th class. There, we studied some patterns formed by points in a plane satisfying certain condition or conditions. We recall that such patterns lead to the concept of locus.

The mathematicians knew about the conic sections as geometric curves several centuries before Kepler came to the scene. But it was **Kepler** who showed the application of these curves at the cosmic level. His laws of planetary motion magnificiently display the loci of planets as elliptic curves.

Now in this chapter, we discuss about locus in detail.

Johannes Kepler

(1571 - 1630)

Kepler was a German mathematician and astronomer who discovered that Earth and Planets move about the sun, in elliptical orbits. He gave three fundamental laws of planetary motion. He also did important work in optics and geometry. He was a key figure in the 17th century astronomical revolution.

1.1 Definition of Locus - Illustrations

Now we shall discuss the patterns formed by the points, which satisfy certain geometrical conditions, which are consistent. A set of geometric conditions is said to be 'consistent', if there is atleast one point satisfying the set of conditions. For example, when A = (1, 0) and B = (3, 0), the condition 'the sum of distances of a point P from A and B is equal to 2' is consistent, whereas the condition 'the sum of distances of a point Q from A and B is equal to 1' is not consistent, because there is no point Q such that QA + QB = 1 (since AB = 2).

1.1.1 Definition

Locus *is the set of points (and only those points) that satisfy the given consistent geometric condition(s).*

From the above definition , it follows that :

(i) Every point satisfying the given condition(s) is a point on the locus.

(ii) Every point on the locus satisfies the given condition(s).

1.1.2 Examples

1. Example : *In a plane the locus of a point whose distance from a given point A is 4.*

Any point which is at a distance 4 from A lies on the circle of radius 4 with centre A. Conversely, every point on the circle is at a distance of 4 from A. Hence, the set of all points on the circle is the locus in this example (see Fig. 1.1)

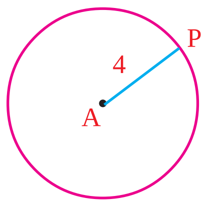


Fig. 1.1

i.e., locus = the circle, in the given plane, with centre at A and radius 4.

2. Example : *The locus of a point equidistant from two given points* A, B.

Cleary, the mid point D of $\overline{AB}$ is a point on the locus. A point P lies on the locus $\Leftrightarrow$ PA = PB

$\Leftrightarrow \Delta PAD \cong \Delta PBD$ (SAS axiom)

$\Leftrightarrow \overleftrightarrow{PD} \perp \overleftrightarrow{AB}$.

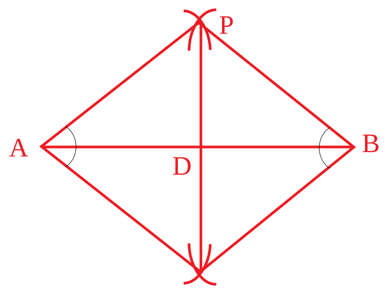


Fig. 1.2

Therefore, the required locus is the perpendicular bisector of $\overline{AB}$. (see Fig. 1.2)

3. Example: *Locus of a point above the* X*-axis whose distance from the* X*-axis is* 2.

Let M be the point on the positive direction of the Y - axis with OM = 2. P is a point above the X-axis with PD = 2, where PD is the distance of P from the X-axis (Fig. 1.3). ODPM is a rectangle so that $\overleftrightarrow{MP} \parallel \overleftrightarrow{OX}$. Conversely, we can prove that any point P on the locus lies on the line parallel to the X-axis (above the X-axis) which passes through M.

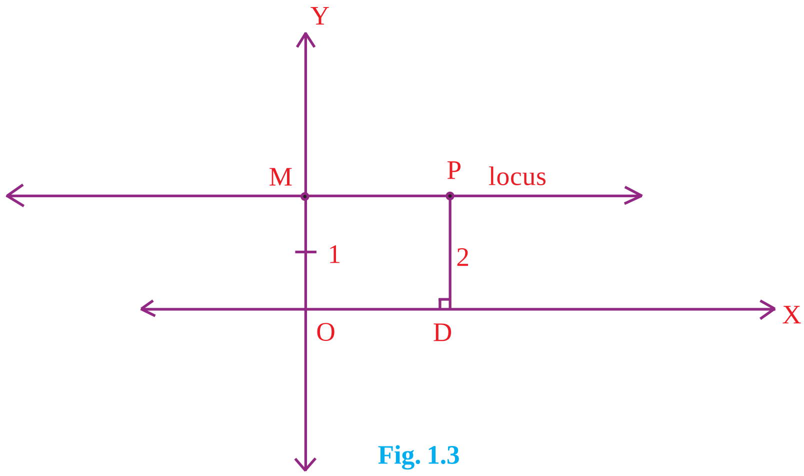


Fig. 1.3

In view of the above examples, locus of a point in the plane is generally a curve in the plane. For simplicity, we call that curve itself as locus. The locus in example 1 is a circle, whereas it is a straight line in examples 2 and 3.

1.2 Equation of Locus - Problems connected to it

It is clear that, every point on the locus satisfies the given conditions and every point which satisfies the given conditions lies on the locus.

Equation of the locus of a point is an ***algebraic equation*** *in x and y satisfied by the points (x, y) on the locus alone (and by no other point).*

Algebraic descriptions give rise to algebraic equations which some times contain more than what is required by the geometric conditions. Thus the required locus may be a part of the curve represented by the algebraic equation. Usually we call this algebraic equation as the equation of locus. However, to get the full description of the locus, the exact part of the curve, points of which satisfy the given geometric description, need to be specified.

1.2.1 Solved Problems

1. Problem : *Find the equation of the locus of a point which is at a distance 5 from* $(-2, 3)$*, in the xoy plane.*

Solution : Let the given point be $A = (-2, 3)$ and $P(x, y)$ be a point on the plane.

The geometric condition to be satisfied by P to be on the locus is that

$$AP = 5 \qquad \text{... (1)}$$

Expressing this condition algebraically, we get

$$\sqrt{(x+2)^2 + (y-3)^2} = 5$$

i.e., $x^2 + 4x + 4 + y^2 - 6y + 9 = 25$

i.e., $x^2 + y^2 + 4x - 6y - 12 = 0 \qquad \text{... (2)}$

Let $Q(x_1, y_1)$ satisfy (2).

Then, $x_1^2 + y_1^2 + 4x_1 - 6y_1 - 12 = 0$ ... (3)

Now the distance of A from Q is $AQ = \sqrt{(x_1+2)^2 + (y_1-3)^2}$

Therefore $AQ^2 = x_1^2 + 4x_1 + 4 + y_1^2 - 6y_1 + 9$

$= (x_1^2 + y_1^2 + 4x_1 - 6y_1 - 12) + 25$

$= 25$ (by using (3))

Hence $AQ = 5$.

This means that $Q(x_1, y_1)$ satisfies the geometric condition (1).

Therefore, the required equation of locus is

$$x^2 + y^2 + 4x - 6y - 12 = 0.$$

2. Problem : *Find the equation of locus of a point P, if the distance of P from A (3, 0) is twice the distance of P from B (– 3, 0).*

Solution : Let $P(x, y)$ be a point on the locus. Then the geometric condition to be satisfied by P is PA = 2PB ... (1)

i.e., $PA^2 = 4PB^2$

i.e., $(x-3)^2 + y^2 = 4\,[(x+3)^2 + y^2]$

i.e., $x^2 - 6x + 9 + y^2 = 4\,[x^2 + 6x + 9 + y^2]$

i.e., $3x^2 + 3y^2 + 30x + 27 = 0$

i.e., $x^2 + y^2 + 10x + 9 = 0$ ... (2)

Let $Q(x_1, y_1)$ satisfy (2). Then $x_1^2 + y_1^2 + 10x_1 + 9 = 0$. ... (3)

Now $QA^2 = (x_1 - 3)^2 + y_1^2$

$= x_1^2 - 6x_1 + 9 + y_1^2$

$= 4x_1^2 + 24x_1 + 36 + 4y_1^2 - 3x_1^2 - 30x_1 - 27 - 3y_1^2$

$= 4(x_1^2 + 6x_1 + 9 + y_1^2) - 3(x_1^2 + 10x_1 + 9 + y_1^2)$

$= 4(x_1^2 + 6x_1 + 9 + y_1^2)$ (by using (3))

$= 4\,[(x_1 + 3)^2 + y_1^2]$

$= 4QB^2$.

Therefore, QA = 2QB. This means that $Q(x_1, y_1)$ satisfies (1).

Hence, the required equation of locus is $x^2 + y^2 + 10x + 9 = 0$.

3. Problem : *Find the locus of the third vertex of a right angled triangle, the ends of whose hypotenuse are (4, 0) and (0, 4).*

Solution : Let A = (4, 0) and B = (0, 4). Let $P(x, y)$ be a point such that PA and PB are perpendicular. Then $PA^2 + PB^2 = AB^2$. ... (1)

i.e., $(x-4)^2 + y^2 + x^2 + (y-4)^2 = 16 + 16$,

i.e., $2x^2 + 2y^2 - 8x - 8y = 0$,

or $x^2 + y^2 - 4x - 4y = 0$. ... (2)

Let $Q(x_1, y_1)$ satisfy (2) and Q be different from A and B.

Then $\quad x_1^2 + y_1^2 - 4x_1 - 4y_1 = 0$, $(x_1, y_1) \neq (4, 0)$ and $(x_1, y_1) \neq (0, 4)$ ... (3)

$$
\begin{aligned}
\text{Now} \quad QA^2 + QB^2 &= (x_1 - 4)^2 + y_1^2 + x_1^2 + (y_1 - 4)^2 \\
&= x_1^2 - 8x_1 + 16 + y_1^2 + x_1^2 + y_1^2 - 8y_1 + 16 \\
&= 2(x_1^2 + y_1^2 - 4x_1 - 4y_1) + 32 \\
&= 32 \quad \text{(by using (3))} \\
&= AB^2
\end{aligned}
$$

Hence $\quad QA^2 + QB^2 = AB^2$, $Q \neq A$ and $Q \neq B$.

This means that $Q(x_1, y_1)$ satisfies (1). Therefore, the required equation of locus is (2), which is the circle with $\overline{AB}$ as diameter, deleting the points A and B. Though A and B satisfy equation (2), they do not satisfy the required geometric condition.

4. Problem : *Find the equation of the locus of P, if the ratio of the distances from P to A (5, – 4) and B (7, 6) is 2 : 3.*

Solution : Let P (x, y) be any point on the locus.

The geometric condition to be satisfied by P is $\dfrac{AP}{PB} = \dfrac{2}{3}$.

i.e., $\quad 3AP = 2PB$ ... (1)

i.e., $\quad 9AP^2 = 4PB^2$

i.e., $\quad 9[(x-5)^2 + (y+4)^2] = 4[(x-7)^2 + (y-6)^2]$

i.e., $\quad 9[x^2 + 25 - 10x + y^2 + 16 + 8y] = 4[x^2 + 49 - 14x + y^2 + 36 - 12y]$

i.e., $\quad 5x^2 + 5y^2 - 34x + 120y + 29 = 0$ ... (2)

Let $\quad Q(x_1, y_1)$ satisfy (2). Then $5x_1^2 + 5y_1^2 - 34x_1 + 120y_1 + 29 = 0$ ... (3)

$$
\begin{aligned}
\text{Now } 9AQ^2 &= 9[x_1^2 + 25 - 10x_1 + y_1^2 + 16 + 8y_1] \\
&= 5x_1^2 + 5y_1^2 - 34x_1 + 120y_1 + 29 + 4x_1^2 + 4y_1^2 - 56x_1 - 48y_1 + 340 \\
&= 4[x_1^2 + y_1^2 - 14x_1 - 12y_1 + 49 + 36] \quad \text{(by using (3))} \\
&= 4[(x_1 - 7)^2 + (y_1 - 6)^2] = 4\,PB^2.
\end{aligned}
$$

Thus $3AQ = 2PB$. This means that $Q(x_1, y_1)$ satisfies (1).

Hence, the required equation of locus is $5(x^2 + y^2) - 34x + 120y + 29 = 0$.

5. Problem : *A (2, 3) and B (–3, 4) are two given points. Find the equation of locus of P so that the area of the triangle PAB is 8.5*

Solution : Let P (x, y) be a point on the locus.

The geometric condition to be satisfied by P is that,

area of Δ PAB = 8.5 ... (1)

i.e., $\quad \dfrac{1}{2}\left| x(3-4) + 2(4-y) - 3(y-3) \right| = 8.5$

i.e., $\quad \left| -x + 8 - 2y - 3y + 9 \right| = 17$

i.e., $\left| -x - 5y + 17 \right| = 17$

i.e., $-x - 5y + 17 = 17$ or $-x - 5y + 17 = -17$

i.e., $x + 5y = 0$ or $x + 5y = 34$

Therefore $(x + 5y)(x + 5y - 34) = 0$

i.e., $x^2 + 10xy + 25y^2 - 34x - 170y = 0$ ... (2)

Let $Q(x_1, y_1)$ satisfy (2). Then

$x_1 + 5y_1 = 0$ or $x_1 + 5y_1 = 34$ ... (3)

Now, area of ΔQAB
$$= \frac{1}{2}\left| x_1(3-4) + 2(4 - y_1) - 3(y_1 - 3) \right|$$
$$= \frac{1}{2}\left| -x_1 + 8 - 2y_1 - 3y_1 + 9 \right|$$
$$= \frac{1}{2}\left| -x_1 - 5y_1 + 17 \right|$$
$$= \frac{17}{2} = 8.5 \qquad \text{(by using (3))}$$

This means that $Q(x_1, y_1)$ satisfies (1).

Hence, the required equation of locus is

$$(x + 5y)(x + 5y - 34) = 0 \quad \text{or} \quad x^2 + 10xy + 25y^2 - 34x - 170y = 0.$$

Exercise 1(a)

I. 1. Find the equation of locus of a point which is at a distance 5 from A (4, – 3).

2. Find the equation of locus of a point which is equidistant from the points A (– 3, 2) and B (0, 4).

3. Find the equation of locus of a point P such that the distance of P from the origin is twice the distance of P from A (1, 2).

4. Find the equation of locus of a point which is equidistant from the coordinate axes.

5. Find the equation of locus of a point equidistant from A (2, 0) and the Y-axis.

6. Find the equation of locus of a point P, the square of whose distance from the origin is 4 times its y-coordinate.

7. Find the equation of locus of a point P such that $PA^2 + PB^2 = 2c^2$, where $A = (a, 0)$, $B = (-a, 0)$ and $0 < |a| < |c|$.

II. 1. Find the equation of locus of P, if the line segment joining (2, 3) and (–1, 5) subtends a right angle at P.

2. The ends of the hypotenuse of a right angled triangle are $(0, 6)$ and $(6, 0)$. Find the equation of the locus of its third vertex.
3. Find the equation of the locus of a point, the difference of whose distances from $(-5, 0)$ and $(5, 0)$ is 8.
4. Find the equation of the locus of P, if $A = (4, 0), B = (-4, 0)$ and $|PA - PB| = 4$.
5. Find the equation of the locus of a point, the sum of whose distances from $(0, 2)$ and $(0, -2)$ is 6.
6. Find the equation of the locus of P, if $A = (2, 3), B = (2, -3)$ and $PA + PB = 8$.
7. $A(5, 3)$ and $B(3, -2)$ are two fixed points. Find the equation of the locus of P, so that the area of triangle PAB is 9.
8. Find the equation of the locus of a point, which forms a triangle of area 2 with the points $A(1, 1)$ and $B(-2, 3)$.
9. If the distance from P to the points $(2, 3)$ and $(2, -3)$ are in the ratio $2 : 3$, then find the equation of the locus of P.
10. $A(1, 2)$, $B(2, -3)$ and $C(-2, 3)$ are three points. A point P moves such that $PA^2 + PB^2 = 2PC^2$. Show that the equation to the locus of P is $7x - 7y + 4 = 0$.

Key Concepts

- Locus is the set of points (and only those points) that satisfy the given consistent geometric condition(s).
- An equation of a locus is an algebraic description of the locus. This can be obtained in the following way :

  (i) Consider a point $P(x, y)$ on the locus.

  (ii) Write the geometric condition(s) to be satisfied by P in terms of an equation or inequation in symbols.

  (iii) Apply the proper formula of coordinate geometry and translate the geometric condition(s) into an algebraic equation.

  (iv) Simplify the equation so that it is free from radicals.

  (v) Verify that if $Q(x_1, y_1)$ satisfies the equation, then Q satisfies the geometric condition.

  The equation thus obtained is the required equation of locus.

Historical Note

Analytic geometry, as we know, grew out of the need for establishing algebraic techniques for solving geometrical problems, the aim being to apply them to the study of curves, which are of particular importance in practical problems. The coordinate method was systematically developed in the first half of the 17th century in the works of ***Fermat*** and ***Descartes***.

Fermat dealt with plane loci, which according to him were collections of points lying in a plane; they are curves. He simplified the related earlier work ascribed to ***Apollonius***, by applying algebra to geometry through the use of coordinates. He noticed that an independent equation in two unknowns determines a locus of points in a plane.

John Wallis (1616-1703) described the second degree equation as representing curves for the first time in 1656. He was pioneer to describe the conic sections as the loci of the second degree equations.

Euler later extended the work from plane loci to space loci and surfaces. .

Answers

Exercise 1(a)

I. **1.** $x^2 + y^2 - 8x + 6y = 0$ **2.** $6x + 4y = 3$

3. $3x^2 + 3y^2 - 8x - 16y + 20 = 0$ **4.** $x^2 - y^2 = 0$

5. $y^2 - 4x + 4 = 0$ **6.** $x^2 + y^2 - 4y = 0$

7. $x^2 + y^2 = c^2 - a^2$

II. **1.** $x^2 + y^2 - x - 8y + 13 = 0$; $(x, y) \neq (2, 3)$ and $(x, y) \neq (-1, 5)$

2. $x^2 + y^2 - 6x - 6y = 0$, $(x, y) \neq (0, 6)$ and $(x, y) \neq (6, 0)$

3. $\frac{x^2}{16} - \frac{y^2}{9} = 1$ **4.** $\frac{x^2}{4} - \frac{y^2}{12} = 1$

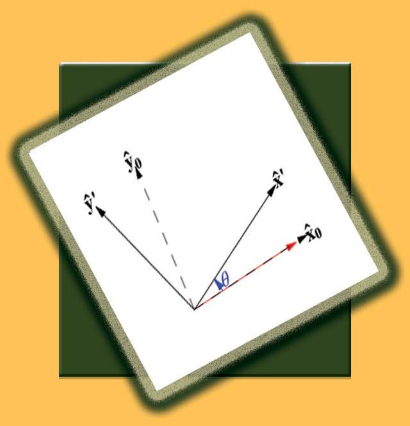
5. $\frac{x^2}{5} + \frac{y^2}{9} = 1$

6. $16x^2 + 7y^2 - 64x - 48 = 0$. i.e., $\frac{(x-2)^2}{7} + \frac{y^2}{16} = 1$

7. $(5x - 2y - 37)(5x - 2y - 1) = 0$

8. $(2x + 3y - 1)(2x + 3y - 9) = 0$

9. $5x^2 + 5y^2 - 20x - 78y + 65 = 0$



Chapter 2

Transformation of Axes

"We transform a given mathematical set up into a new one. By studying the new set up we discover some property of it. We then invert the transformation to obtain a property of the original set up"

- Howard Eves

Introduction

A plane extends infinitely in all directions. By drawing X–axis and Y–axis, and dividing the infinite plane into four quadrants, we represent any point in the plane as an ordered pair of real numbers, which are the lengths of perpendicular distances of the point from the axes chosen. It is to be noted that these axes can be chosen arbitrarily and therefore the position of these axes in the plane is not fixed. They can be changed. When the position of axes is changed, the coordinates of a point also get changed correspondingly. Consequently, equations of curves will also be changed. This process of transformation of axes will be of great advantage to solve some problems very easily.

The axes can be transformed or changed usually in the following ways : (i) Translation of axes (ii) Rotation of axes and (iii) Translation and rotation of axes.

Leonhard Paul Euler

(1707 - 1783)

Euler *was a pioneering Swiss mathematician and physicist, who spent most of his life in Russia and Germany. He published more papers than any other mathematician. Euler made important discoveries in such diverse fields as Calculus, Geometry, Trigonometry, Differential Equations, Fluid mechanics, Graph theory, Number theory and so on.*

2.1 Transformation of axes-Rules, derivations and illustrations

Transformation of axes, some times, proves to be advantageous in solving some problems. We discuss mainly two such transformations. One of these transformations, namely translation of axes is discussed in the present section.

2.1.1 Definition (Translation of axes)

The transformation obtained, by shifting the origin to a given different point in the plane, without changing the directions of coordinate axes therein is called a Translation of axes.

2.1.2 Changes in the coordinates by a translation of axes: Let $\overrightarrow{OX}$ and $\overrightarrow{OY}$ be the given coordinate axes. Suppose the origin O is shifted to $O' = (h, k)$ by the translation of the axes $\overrightarrow{OX}$ and $\overrightarrow{OY}$. Let $\overrightarrow{O'X'}$ and $\overrightarrow{O'Y'}$ be the new axes as shown in Fig. 2.1. Then with reference to $\overrightarrow{O'X'}$ and $\overrightarrow{O'Y'}$ the point O' has coordinates $(0, 0)$.

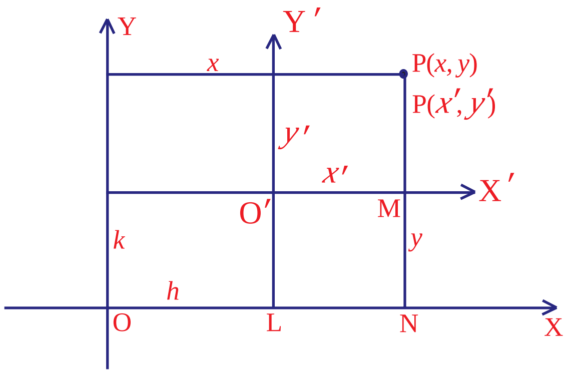


Fig. 2.1 Translation of axes

Let P be a point with coordinates (x, y) in the system $\overrightarrow{OX}$ and $\overrightarrow{OY}$; and with coordinates (x', y') in the system $\overrightarrow{O'X'}$ and $\overrightarrow{O'Y'}$.

Then $O'L = k$ and $OL = h$.

Now $x = ON = OL + LN$

$= OL + O'M$

$= h + x'$

and $y = PN = PM + MN$

$= PM + O'L$

$= y' + k.$

Thus $x = x' + h, \quad y = y' + k$; or $x' = x - h, \quad y' = y - k.$

2.1.3 Note

If the origin is shifted to (h, k) by translation of axes, then

(i) the coordinates of a point $P(x, y)$ are transformed as $P(x - h, y - k)$, and

(ii) the equation $f(x, y) = 0$ of the curve is transformed as $f(x' + h, y' + k) = 0$.

2.1.4 Note : The translation formulae always hold, irrespective of the quadrant in which the origin of the new system happens to lie. For example, if h and k are both positive, the displacement is to the right and upwards; if h and k are both negative, it is to the left and downwards. Similarly the other two cases would also be taken care of.

2.1.5 Examples

1. Example : *When the origin is shifted to (– 2, 3) by translation of axes, let us find the coordinates of (1, 2) with respect to new axes.*

Here $(h, k) = (-2, 3)$. Let $(x, y) = (1, 2)$ be shifted to (x', y') by the translation of axes.

Then $(x', y') = (x-h, y-k) = (1-(-2), 2-3) = (3, -1)$.

2. Example : *When the origin is shifted to* (3, 4) *by the translation of axes, let us find the transformed equation of* $2x^2 + 4xy + 5y^2 = 0$.

Here $(h, k) = (3, 4)$. On substituting $x = x' + 3$ and $y = y' + 4$ in the given equation.

(as per Note 2.1.3 (ii)), we get

$$2(x'+3)^2 + 4(x'+3)(y'+4) + 5(y'+4)^2 = 0.$$

Simplifying this equation, we get

$$2x'^2 + 4x'y' + 5y'^2 + 28x' + 52y' + 146 = 0.$$

This equation can be written (dropping dashes) as

$$2x^2 + 4xy + 5y^2 + 28x + 52y + 146 = 0.$$

2.2 Rotation of axes - Derivations - Illustrations

The present section is intended to discuss another transformation, namely rotation of axes.

2.2.1 Definition (Rotation of axes)

The transformation obtained, by rotating both the coordinate axes in the plane by an equal angle, without changing the position of the origin is called a Rotation of axes.

2.2.2 Changes in the coordinates when the axes are rotated through an angle 'θ'

Let $P = (x, y)$ with reference to the axes $\overrightarrow{OX}$ and $\overrightarrow{OY}$. Let the axes be rotated through an angle 'θ' in the positive direction about the origin O, to get the new system $\overrightarrow{OX'}$ and $\overrightarrow{OY'}$ as shown in Fig. 2.2. With reference to the new axes $\overrightarrow{OX'}$ and $\overrightarrow{OY'}$, let $P = (x', y')$.

Since the angle of rotation is 'θ', we have

$\angle XOX' = \angle YOY' = \theta$.

Let L, M be the feet of the perpendiculars drawn from P upon $\overrightarrow{OX}$, $\overrightarrow{OX'}$. The angle between the two straight lines is equal to the angle between their perpendiculars. Hence,

$\angle LPM = \angle XOX' = \theta$.

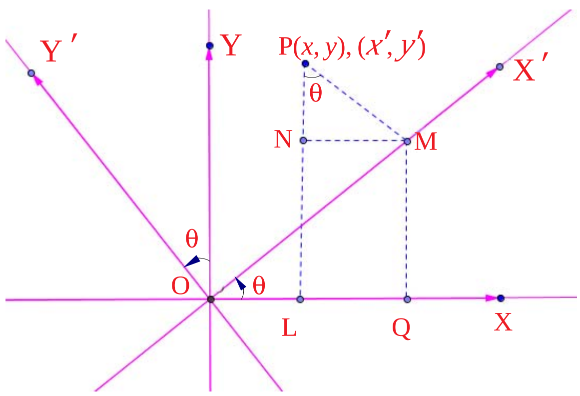


Fig. 2.2 Rotation of axes

Let N be the foot of the perpendicular from M to $\overline{PL}$.

Now $\quad x = OL = OQ - LQ$

$= OQ - NM$

$= OM\cos\theta - PM\sin\theta$

$= x'\cos\theta - y'\sin\theta$ ... (1)

Also $\quad y = PL = PN + NL$

$= PN + MQ$

$= PM\cos\theta + OM\sin\theta$

$= y'\cos\theta + x'\sin\theta$ ... (2)

Therefore $x = x'\cos\theta - y'\sin\theta, \quad y = x'\sin\theta + y'\cos\theta$. ... (3)

From the above equations, the values of x', y' can be found as

$$x' = x\cos\theta + y\sin\theta, \quad y' = -x\sin\theta + y\cos\theta \qquad \text{... (4)}$$

The results in (3) and (4) can be easily remembered by the following table.

	x'	y'
x	$\cos\theta$	$-\sin\theta$
y	$\sin\theta$	$\cos\theta$

These results can also be expressed in matrix notation as follows:

$$(x \ \ y) = (x' \ \ y')\begin{pmatrix}\cos\theta & \sin\theta \\ -\sin\theta & \cos\theta\end{pmatrix}$$

and
$$(x' \ \ y') = (x \ \ y)\begin{pmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{pmatrix}.$$

2.2.3 Note

When the axes are rotated through an angle 'θ', then

(i) the coordinates of a point $P(x, y)$ are transformed as
$P(x', y') = P(x\cos\theta + y\sin\theta, -x\sin\theta + y\cos\theta)$, and

(ii) the equation $f(x, y) = 0$ of the curve is transformed as
$f(x'\cos\theta - y'\sin\theta, x'\sin\theta + y'\cos\theta) = 0$.

2.2.4 Example: *Let us find the coordinates of $P(1, 2)$ with reference to the new axes, when the axes are rotated through an angle of 30^o.*

Let $P(x, y) = (1, 2)$ and (x', y') be the coordinates of P in the new system.

$$x' = 1(\cos 30^o) + 2(\sin 30^o) = \frac{\sqrt{3}}{2} + 2\frac{1}{2} = \frac{\sqrt{3}+2}{2}.$$

$$y' = (-1)(\sin 30^o) + 2(\cos 30^0) = \frac{-1}{2} + 2\frac{\sqrt{3}}{2} = \frac{-1+2\sqrt{3}}{2}.$$

Therefore, the new coordinates of P are $\left(\frac{\sqrt{3}+2}{2}, \frac{-1+2\sqrt{3}}{2}\right)$.

2.2.5 Note : If the origin is shifted to (h, k) and then the axes are rotated through an angle 'θ', then

(i) the coordinates of a point $P(x, y)$ are transformed as

$$P(x', y') = (x\cos\theta + y\sin\theta - h, -x\sin\theta + y\cos\theta - k), \text{ and}$$

(ii) the equation $f(x, y) = 0$ of the curve is transformed as

$$f(x'\cos\theta - y'\sin\theta + h, x'\sin\theta + y'\cos\theta + k) = 0.$$

2.2.6 Solved Problems

1. Problem : *When the origin is shifted to (2, 3) by the translation of axes, the coordinates of a point P are changed as (4, –3). Find the coordinates of P in the original system.*

Solution : Here $(h, k) = (2, 3)$; $P(x', y') = (4, -3)$

Then $x = x' + h = 4 + 2 = 6$

and $y = y' + k = -3 + 3 = 0$.

Therefore, the coordinates of P in the original system are $(6, 0)$.

2. Problem : *Find the point to which the origin is to be shifted by the translation of axes so as to remove the first degree terms from the equation*

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0, \quad \textit{where} \quad h^2 \neq ab.$$

Solution : Let the origin be shifted to (α, β) by the translation of axes. Then

$$x = x' + \alpha, \quad y = y' + \beta.$$

On substituting these in the given equation, we get

$$a(x'+\alpha)^2 + 2h(x'+\alpha)(y'+\beta) + b(y'+\beta)^2 + 2g(x'+\alpha) + 2f(y'+\beta) + c = 0.$$

On simplification, this equation gives

$$ax'^2 + 2hx'y' + by'^2 + 2x'(a\alpha + h\beta + g) + 2y'(h\alpha + b\beta + f)$$
$$+ a\alpha^2 + 2h\alpha\beta + b\beta^2 + 2g\alpha + 2f\beta + c = 0 \qquad \text{... (1)}$$

If the equation (1) has to be free from the first degree terms, then we have

$$a\alpha + h\beta + g = 0$$

and $$h\alpha + b\beta + f = 0.$$

Solving these equations for α and β, we get

$$\alpha = \frac{hf - bg}{ab - h^2}, \quad \beta = \frac{gh - af}{ab - h^2}.$$

Therefore, the origin is to be shifted to $\left(\frac{hf - bg}{ab - h^2}, \frac{gh - af}{ab - h^2}\right)$.

Note: This result can also be used as a formula.

3. Problem : *Find the point to which the origin is to be shifted by the translation of axes so as to remove the first degree terms from the equation*

$$ax^2 + by^2 + 2gx + 2fy + c = 0, \text{ where } a \neq 0, b \neq 0.$$

Solution : Here the given equation does not contain xy term. Hence writing $h = 0$ in the result of the problem 2, the required point is $\left(\frac{-g}{a}, \frac{-f}{b}\right)$.

4. Problem : *If the point P changes to $(4, -3)$ when the axes are rotated through an angle of 135^o, find the coordinates of P with respect to the original system.*

Solution : Here $(x', y') = (4, -3)$; $\theta = 135^o$.

Let (x, y) be the coordinates of P. Then

$$\begin{aligned} x &= x' \cos\theta - y' \sin\theta \\ &= 4 \cos 135^o - (-3) \sin 135^o \\ &= 4\left(\frac{-1}{\sqrt{2}}\right) + 3\left(\frac{1}{\sqrt{2}}\right) = \frac{-1}{\sqrt{2}}, \\ y &= x' \sin\theta + y' \cos\theta \\ &= 4 \sin 135^o + (-3) \cos 135^o \\ &= 4\left(\frac{1}{\sqrt{2}}\right) - 3\left(\frac{-1}{\sqrt{2}}\right) = \frac{7}{\sqrt{2}}. \end{aligned}$$

Therefore, the coordinates of P with respect to the original system are $\left(\frac{-1}{\sqrt{2}}, \frac{7}{\sqrt{2}}\right)$.

5. Problem : *Show that the axes are to be rotated through an angle of* $\frac{1}{2}\text{Tan}^{-1}\left(\frac{2h}{a-b}\right)$ *so as to remove the xy term from the equation $ax^2 + 2hxy + by^2 = 0$, if $a \neq b$ and through the angle $\frac{\pi}{4}$, if $a = b$.*

Solution : If the axes are rotated through an angle 'θ', then

$$x = x' \cos\theta - y' \sin\theta; \; y = x' \sin\theta + y' \cos\theta.$$

Therefore, the given equation transforms as

$$a(x' \cos\theta - y' \sin\theta)^2 + 2h(x' \cos\theta - y' \sin\theta)(x' \sin\theta + y' \cos\theta)$$
$$+ b(x' \sin\theta + y' \cos\theta)^2 = 0.$$

To remove $x'y'$ term from the equation, we have to equate the coefficient of $x'y'$ term to zero.

So, $(b-a)\sin\theta\cos\theta + h(\cos^2\theta - \sin^2\theta) = 0.$

$$\text{i.e.,}\qquad h\cos 2\theta = \frac{a-b}{2}\sin 2\theta$$

$$\text{i.e.,}\qquad \tan 2\theta = \frac{2h}{a-b},\quad \text{if } a \neq b$$

$$\text{and}\qquad h\cos 2\theta = 0,\quad \text{if } a = b.$$

Therefore $\theta = \dfrac{1}{2}\,\text{Tan}^{-1}\left(\dfrac{2h}{a-b}\right)$, if $a \neq b$ and $\theta = \dfrac{\pi}{4}$, if $a = b$.

Note : This result can also be used as a formula.

6. Problem : *When the origin is shifted to* $(-2, -3)$ *and the axes are rotated through an angle* 45^0*, find the transformed equation of* $2x^2 + 4xy - 5y^2 + 20x - 22y - 14 = 0$.

Solution : Here $(h, k) = (-2, -3)$, $\theta = 45^0$.

Let (x', y') be the new coordinates of any point (x, y) in the plane after the transformation.

Then $\quad x = x'\cos\theta - y'\sin\theta + h,\quad y = x'\sin\theta + y'\cos\theta + k$

i.e., $\quad x = x'\cos 45^0 - y'\sin 45^0 - 2,\quad y = x'\sin 45^0 + y'\cos\theta - 3$

$$\text{i.e.,}\qquad x = x'\left(\frac{1}{\sqrt{2}}\right) - y'\left(\frac{1}{\sqrt{2}}\right) - 2,\quad y = x'\left(\frac{1}{\sqrt{2}}\right) + y'\left(\frac{1}{\sqrt{2}}\right) - 3$$

$$\text{i.e.,}\qquad x = \left(\frac{x'-y'}{\sqrt{2}}\right) - 2,\quad y = \left(\frac{x'+y'}{\sqrt{2}}\right) - 3.$$

On substituting these values in the given equation, we get

$$2\left\{\left(\frac{x'-y'}{\sqrt{2}}\right)-2\right\}^2 + 4\left\{\left(\frac{x'-y'}{\sqrt{2}}\right)-2\right\}\left\{\left(\frac{x'+y'}{\sqrt{2}}\right)-3\right\} - 5\left\{\left(\frac{x'+y'}{\sqrt{2}}\right)-3\right\}^2$$

$$+\,20\left\{\left(\frac{x'-y'}{\sqrt{2}}\right)-2\right\} - 22\left\{\left(\frac{x'+y'}{\sqrt{2}}\right)-3\right\} - 14 = 0$$

$$\text{i.e.,}\quad 2\left[\frac{(x'-y')^2}{2} + 4 - 4\frac{(x'-y')}{\sqrt{2}}\right] + 4\left[\frac{(x'^2-y'^2)}{2} - 3\frac{(x'-y')}{\sqrt{2}} - 2\frac{(x'+y')}{\sqrt{2}} + 6\right]$$

$$-\,5\left[\frac{(x'+y')^2}{2} + 9 - 6\frac{(x'+y')}{\sqrt{2}}\right] + 10\sqrt{2}(x'-y') - 40 - 11\sqrt{2}(x'+y') + 66 - 14 = 0$$

i.e., $(x'-y')^2+8-4\sqrt{2}(x'-y')+2(x'^2-y'^2)-6\sqrt{2}(x'-y')-4\sqrt{2}(x'+y')+24$

$$-\frac{5}{2}(x'+y')^2-45+15\sqrt{2}(x'+y')+10\sqrt{2}(x'-y')-40-11\sqrt{2}(x'+y')+66-14=0.$$

i.e., $(x'-y')^2+2(x'^2-y'^2)-\frac{5}{2}(x'+y')^2-1=0$ (grouping similar terms and cancelling)

i.e., $x'^2+y'^2-2x'y'+2x'^2-2y'^2-\frac{5}{2}(x'^2+y'^2+2x'y')-1=0$

i.e., $\frac{1}{2}x'^2-\frac{7}{2}y'^2-7x'y'-1=0$

i.e., $x'^2-7y'^2-14x'y'-2=0$

Hence, the transformed equation (dropping dashes) is $x^2-7y^2-14xy-2=0$.

Exercise 2(a)

I. **1.** When the origin is shifted to (4, –5) by the translation of axes, find the coordinates of the following points with reference to new axes.

(i) (0, 3) (ii) (–2, 4) (iii) (4, –5)

2. The origin is shifted to (2, 3) by the translation of axes. If the coordinates of a point P change as follows, find the coordinates of P in the original system.

(i) (4, 5) (ii) (– 4, 3) (iii) (0, 0)

3. Find the point to which the origin is to be shifted so that the point (3, 0) may change to (2, –3).

4. When the origin is shifted to (–1, 2) by the translation of axes, find the transformed equations of the following.

(i) $x^2 + y^2 + 2x - 4y + 1 = 0$ (ii) $2x^2 + y^2 - 4x + 4y = 0$

5. The point to which the origin is shifted and the transformed equation are given below. Find the original equation.

(i) (3, –4); $x^2 + y^2 = 4$ (ii) (– 1, 2); $x^2 + 2y^2 + 16 = 0$.

6. Find the point to which the origin is to be shifted so as to remove the first degree terms from the equation

$4x^2 + 9y^2 - 8x + 36y + 4 = 0$.

7. When the axes are rotated through an angle 30°, find the new coordinates of the following points.

(i) (0, 5) (ii) (–2, 4) (iii) (0, 0)

8. When the axes are rotated through an angle 60°, the new coordinates of three points are the following

(i) (3, 4) (ii) (–7, 2) (iii) (2, 0)

Find their original coordinates.

9. Find the angle through which the axes are to be rotated so as to remove the xy term in the equation $x^2 + 4xy + y^2 - 2x + 2y - 6 = 0$.

II. 1. When the origin is shifted to the point (2, 3), the transformed equation of a curve is $x^2 + 3xy - 2y^2 + 17x - 7y - 11 = 0$. Find the original equation of the curve.

2. When the axes are rotated through an angle 45°, the transformed equation of a curve is $17x^2 - 16xy + 17y^2 = 225$. Find the original equation of the curve.

3. When the axes are rotated through an angle α, find the transformed equation of $x\cos\alpha + y\sin\alpha = p$.

4. When the axes are rotated through an angle $\frac{\pi}{6}$, find the transformed equation of $x^2 + 2\sqrt{3}\,xy - y^2 = 2a^2$.

5. When the axes are rotated through an angle $\frac{\pi}{4}$, find the transformed equation of $3x^2 + 10xy + 3y^2 = 9$.

Key Concepts

- If the origin (0, 0) is shifted to (h, k) by the translation of axes, then
  - (i) the coordinates (x, y) of a point P are transformed as $(x - h, y - k)$, and
  - (ii) the equation $f(x, y)$ of the curve is transformed as $f(x' + h, y' + k) = 0$.
- If the axes are rotated through an angle 'θ', then
  - (i) The coordinates (x, y) of a point P are transformed as $(x', y') = (x\cos\theta + y\sin\theta, -x\sin\theta + y\cos\theta)$, and
  - (ii) The equation $f(x, y) = 0$ of the curve is transformed as $f(x'\cos\theta - y'\sin\theta, x'\sin\theta + y'\cos\theta) = 0$.
- If the origin is shifted to (h, k) and then the axes are rotated through an angle 'θ', then
  - (i) The coordinates (x, y) of a point P are transformed as $(x', y') = (x\cos\theta + y\sin\theta - h, -x\sin\theta + y\cos\theta - k)$, and
  - (ii) The equation $f(x, y) = 0$ of the curve is transformed as $f(x'\cos\theta - y'\sin\theta + h, x'\sin\theta + y'\cos\theta + k) = 0$.

Historical Note

The method of transformation of axes is an ingenius method employed by great mathematicians to simplify their results. It is noted that in 1748, *Euler* used equations for rotation and translation in space to reduce the equation of general quadric surface to such a form that its principal axes coincide with coordinate axes.

Answers

Exercise 2(a)

I.

1. (i) $(-4, 8)$ (ii) $(-6, 9)$ (iii) $(0,0)$

2. (i) $(6, 8)$ (ii) $(-2, 6)$ (iii) $(2, 3)$

3. $(1, 3)$

4. (i) $x^2 + y^2 - 4 = 0$ (ii) $2x^2 + y^2 - 8x + 8y + 18 = 0$

5. (i) $x^2 + y^2 - 6x + 8y + 21 = 0$ (ii) $x^2 + 2y^2 + 2x - 8y + 25 = 0$

6. $(1, -2)$

7. (i) $\left(\frac{5}{2}, \frac{5\sqrt{3}}{2}\right)$ (ii) $\left(2-\sqrt{3},\ 1+2\sqrt{3}\right)$ (iii) $(0, 0)$

8. (i) $\left(\frac{3-4\sqrt{3}}{2}, \frac{3\sqrt{3}+4}{2}\right)$ (ii) $\left(\frac{-7-2\sqrt{3}}{2}, \frac{2-7\sqrt{3}}{2}\right)$ (iii) $\left(1, \sqrt{3}\right)$

9. 45^0

II.

1. $x^2 + 3xy - 2y^2 + 4x - y - 20 = 0$

2. $25x^2 + 9y^2 = 225$ **3.** $x = p$

4. $x^2 - y^2 = a^2$ **5.** $8x^2 - 2y^2 = 9$

Chapter 3

The Straight Line

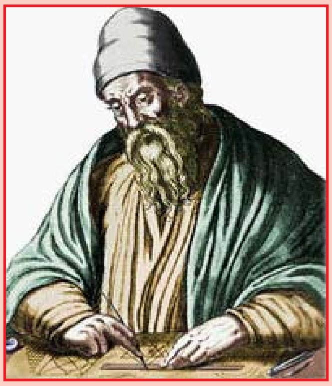
"As long as algebra and geometry have been separated, their progress has been slow and their uses are limited; but when these two sciences have been united, they have lent each other mutual forces, and have marched together towards perfection"

- Lagrange

Introduction

In the high school mathematics (in classes IX and X) the student is familiar with certain basic concepts of coordinate geometry such as representing a point in a plane by an ordered pair of real numbers called its coordinates, the distance between two points, the section formula, the area of a triangle in terms of the coordinates of its vertices, the slope of a nonvertical line when two points on the line are given, standard forms of the equation of a line and its general form, conditions for two lines to be parallel and perpendicular. Now in this chapter we study various results relating to straight lines in much more detail. Many results in this chapter can be proved by vector methods as well (see Mathematics - IA, Chapter 5).

In Chapter 1, the equation of a locus and locus of an equation were already discussed. We recall the



Euclid

(323 - 283 B.C.)

***Euclid**, also known as Euclid of Alexandria, **"The Father of Geometry"** was a Greek mathematician of the Hellenistic period who flourished in Alexandria, Egypt, almost certainly during the reign of Ptolemy. His Elements is the most successful textbook in the literature on mathematics.*

equation of a locus with particular reference to a straight line. Thus the equation of a straight line is an algebraic condition which is satisfied by every point on it and by no other point.

Hence a point lies on a straight line or not according as the coordinates of the point satisfy the equation or not. For example, the coordinates of the point (–2, 3) satisfy the equation $2x + 3y - 5 = 0$ whereas those of the point (–1, 1) do not satisfy the equation.

3.1 Revision of fundamental results

The present section is devoted to discuss some basic concepts of coordinate geometry that are covered in lower classes.

3.1.1 Horizontal lines and Vertical lines

Generally, any pair of perpendicular lines in a plane can be chosen as the axes of coordinates. But in practice, it is customary to draw the X – axis as a horizontal line and the Y– axis as a vertical line. For this reason, lines drawn parallel to the X – axis are referred to as horizontal lines and those lines drawn parallel to the Y– axis are referred to as Vertical lines. The y – coordinate of every point on the X – axis is zero. (i.e.) every point on the X – axis satisfies the equation $y = 0$. Conversely, if any point has its y-coordinate equal to 0, then the point lies on the X – axis. Therefore, the equation of the X – axis is $y = 0$. Similarly the equation of the Y – axis is $x = 0$.

Consider a straight line drawn parallel to the X – axis at a distance k from it and lying above the X – axis. The ordinate of every point on this line is k. Conversely, if a point is at a distance k from the X – axis and lies above the X – axis, then the point lies on the above horizontal line. As such, the equation $y = k$ is satisfied by only those points that lie on this line and hence, $y = k$ is the equation of the horizontal line which is at a distance k from the X – axis and lying above the X – axis.

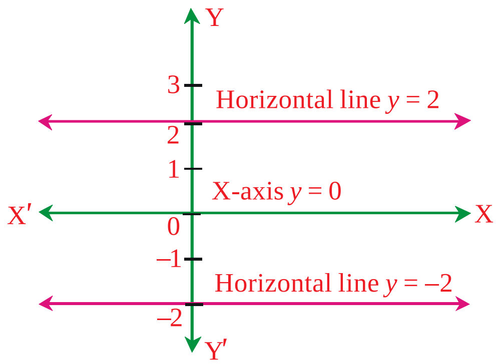


Fig. 3.1

Similarly $y = -k$ is the equation of the horizontal line which is at a distance k from the X – axis and lying below the X – axis for $k = 2$ (see Fig. 3.1).

In a similar way, it can be observed that the equation of the vertical line passing through the point $(x_0, 0)$ on the X – axis is $x = x_0$. (Here the distance of this line from the Y – axis is $|x_0|$). Also the line lies to the right of the Y – axis if $x_0 > 0$ and to the left of the Y – axis if ($x_0 < 0$) (see Fig. 3.2).

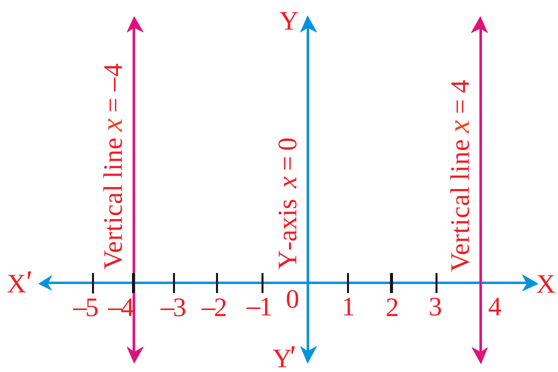


Fig. 3.2

3.1.2 The slope of a straight line

Definition

*If a non-vertical straight line L makes an angle θ with the X–axis measured counter-clock wise from the positive direction of the X–axis, then $\tan\theta$ is called the **slope** or **gradient** of the line L. The slope of a nonvertical straight line is usually denoted by m.*

3.1.3 Note

(i) A vertical line makes a right angle with the X – axis and therefore, the slope of a vertical line is not defined.

(ii) If a straight line is parallel to the X– axis, then $\theta = 0^o$ and since $\tan 0^o = 0$, the slope of a horizontal line is 0.

(iii) If θ is acute, $\tan\theta$ is positive and if θ is obtuse, $\tan\theta$ is negative (see Fig. 3.3 and 3.4)

(iv) The sign of $\tan\theta$ indicates to which side of the X – axis the straight line is inclined. Also the magnitude of $\tan\theta$ gives the amount of steepness of the line with the X–axis. Hence $\tan\theta$ is called the **slope of the line**.

(v) The variation of θ is in the interval $0 \le \theta < \pi$.

(vi) If L_1 and L_2 are two non-vertical straight lines with slopes m_1 and m_2 and if θ_1, θ_2 are the angles in $[0, \pi)$ such that $m_1 = \tan\theta_1$ and $m_2 = \tan\theta_2$, then

$$L_1 \parallel L_2 \Leftrightarrow \theta_1 = \theta_2$$

$$\Leftrightarrow \tan\theta_1 = \tan\theta_2 \; (\because 0 \le \theta_1, \; \theta_2 < \pi)$$

$$\Leftrightarrow \boxed{m_1 = m_2} \text{ (see Fig. 3.5)}$$

and
$$L_1 \perp L_2 \Leftrightarrow |\theta_1 - \theta_2| = \frac{\pi}{2}$$

$$\Leftrightarrow \cot(\theta_1 - \theta_2) = \cot\left(\pm\frac{\pi}{2}\right) = 0$$

$$(\because -\pi < (\theta_1 - \theta_2) < \pi)$$

$$\Leftrightarrow 1 + \tan\theta_1 \tan\theta_2 = 0$$

$$\Leftrightarrow \boxed{m_1 m_2 = -1} \text{ (see Fig. 3.6)}$$

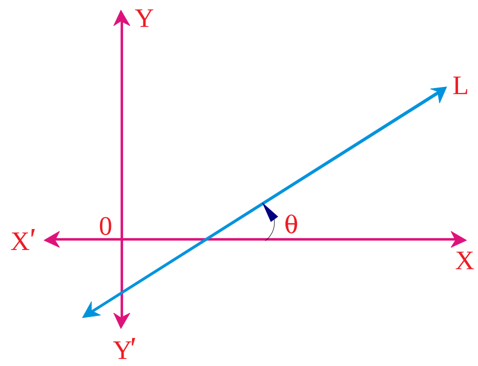


Fig. 3.3

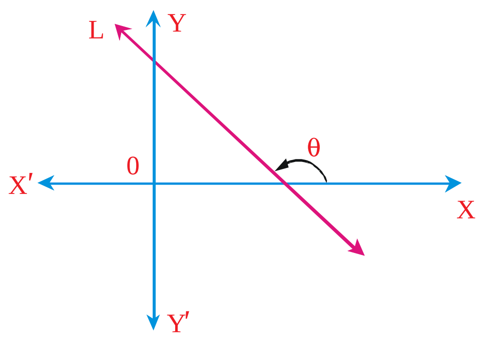


Fig. 3.4

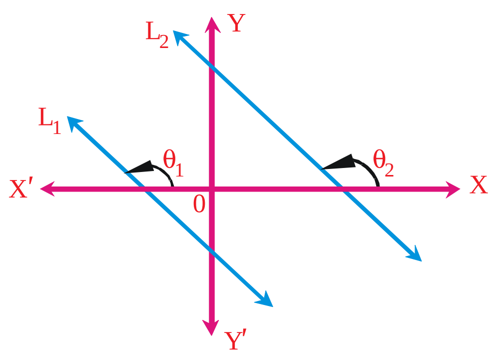


Fig. 3.5

(vii) If a non-vertical straight line passes through the points (x_1, y_1) and (x_2, y_2), then its slope is $\dfrac{y_1 - y_2}{x_1 - x_2}$. In fact, for any two points on this line, the ratio of $(y_1 - y_2)$ to $(x_1 - x_2)$ is always a constant.

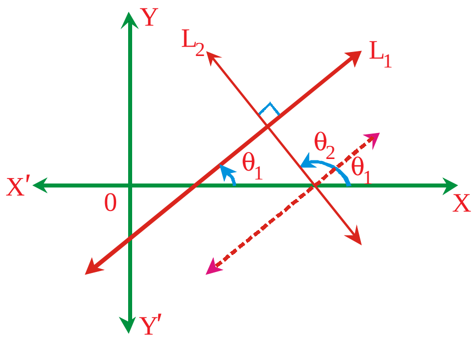


Fig. 3.6

3.1.4 Intercepts

Definition

If a straight line L intersects the X – axis at A (a, 0) and the Y – axis at B(0, b), then a and b are respectively called the X – intercept and the Y – intercept of the line L. Depending on the values of a and b, the position of the line $\overleftrightarrow{AB}$ *is as given in* Fig. 3.7.

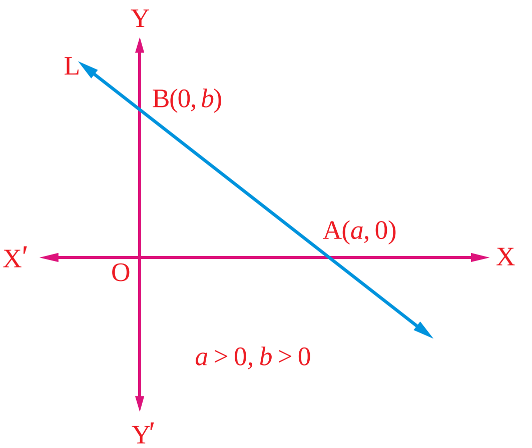


Fig. 3.7(a)

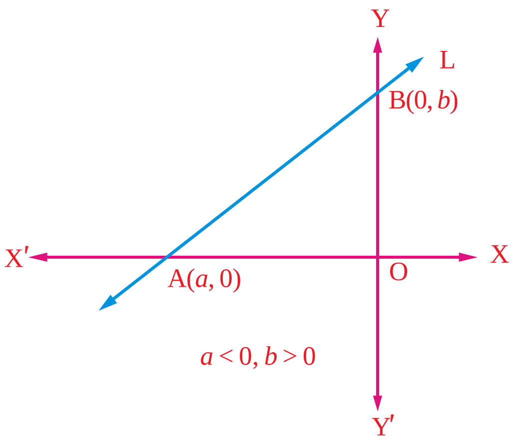


Fig. 3.7(b)

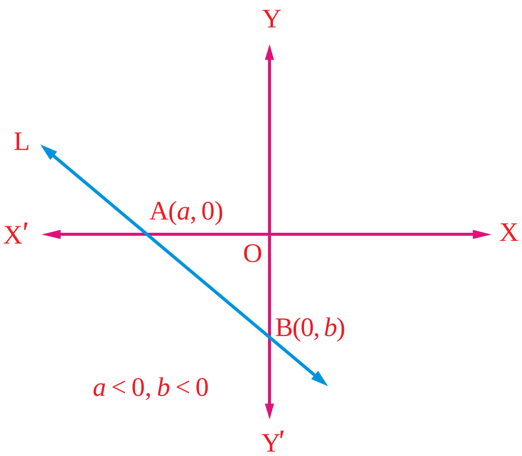


Fig. 3.7(c)

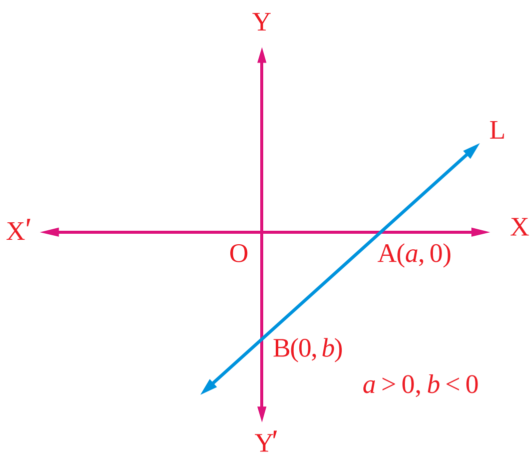


Fig. 3.7(d)

3.1.5 Note

1. A straight line passes through the origin if and only if the X – intercept and the Y – intercept of the straight line are both equal to zero.
2. The X – intercept of a horizontal line is not defined.
3. The Y – intercept of a vertical line is not defined.

3.1.6 The equation of a straight line in slope - intercept form

Theorem: *The equation of the straight line with slope 'm' and cutting off Y–intercept 'c' is* $y = mx + c$.

Proof : Let L be the straight line with slope m and cutting off an intercept 'c' on the Y-axis (see Fig. 3.8).

Clearly $B(0, c) \in L$.

Also, if a point $P(x, y)$ in the XY – plane belongs to L, then m = slope of $L = \dfrac{y-c}{x-0}$

i.e., $y = mx + c$.

Conversely, if $P(x, y)$ satisfies the equation $y = mx + c$, then $P = (0, c)$ if $x = 0$ and so, $P \in L$. For $x \neq 0$, slope of $L = m = \dfrac{y-c}{x-0}$ = slope of $\overleftrightarrow{BP}$ and in this case, $\overleftrightarrow{BP}$ and L are parallel. But L contains B, and therefore, L and $\overleftrightarrow{BP}$ are one and the same line. Accordingly $P \in L$.

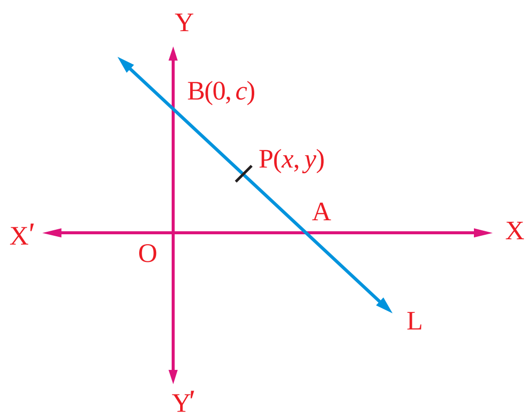


Fig. 3.8

Therefore, the equation $y = mx + c$ is satisfied by only those points which belong to L. So, the equation of L is $\boxed{y = mx + c}$.

(This is known as the equation of the line in slope-intercept form).

3.1.7 Note : The straight line $y = mx + c$ passes through the origin if $c = 0$. Thus the equation of the non-vertical straight line passing through the origin and having slope m is $y = mx$.

3.1.8 Example : Find the equation of the straight line making an angle of 120° with the positive direction of the X – axis measured counter-clockwise and passing through the point $(0, -2)$.

Solution Slope of the line $m = \tan 120^\circ = -\sqrt{3}$ and the Y-intercept of the line $c = -2$. Hence the equation of the line, using slope-intercept form, is $y = -\sqrt{3}\,x - 2$ or $\sqrt{3}\,x + y + 2 = 0$.

3.1.9 The equation of a straight line - Intercept form

Theorem : *The equation of the straight line which cuts off non-zero intercepts a and b on the X – axis and the Y – axis respectively is* $\dfrac{x}{a} + \dfrac{y}{b} = 1$.

Proof : The straight line L which cuts off intercepts a and b on the X – axis and Y – axis respectively meets the X – axis at A $(a, 0)$ and the Y – axis at B$(0, b)$. Therefore, the slope of the line L $= \dfrac{b-0}{0-a} = -\dfrac{b}{a}$.

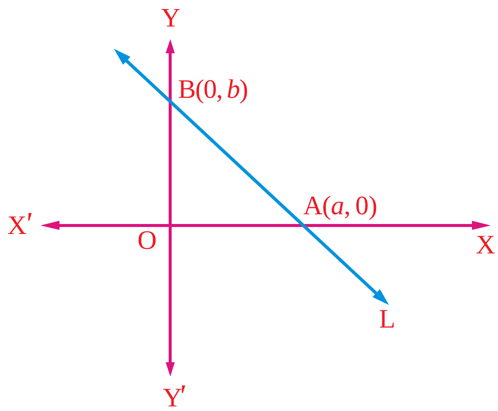


Fig. 3.9

Hence, the equation of L, by the slope-intercept form, is

$$y = \left(\frac{-b}{a}\right)x + b$$

i.e., $\boxed{\dfrac{x}{a} + \dfrac{y}{b} = 1}$.

3.1.10 Example : Find the equations of the straight line which make intercepts whose sum is 5 and product is 6.

Solution : If a and b are the intercepts of the line on the axes of coordinates, then by hypothesis, $a + b = 5$ and $ab = 6$. Solving these equations for a and b, we obtain

$$a = 3, b = 2 \quad \text{or} \quad a = 2, \ b = 3.$$

If $a = 3, b = 2$; the equation of the line is $\dfrac{x}{3} + \dfrac{y}{2} = 1$ (i.e.) $2x + 3y - 6 = 0$.

If $a = 2, b = 3$; the equation of the line is $\dfrac{x}{2} + \dfrac{y}{3} = 1$ (i.e.) $3x + 2y - 6 = 0$.

3.1.11 The equation of a straight line in point-slope form

Theorem : *The equation of the straight line with slope m and passing through the point (x_1, y_1) is $y - y_1 = m(x - x_1)$.*

Proof : Equation of any straight line with slope m is of the form $y = mx + c$.

This line passes through the point A(x_1, y_1) (see Fig. 3.10) if and only if $c = y_1 - mx_1$.

Therefore, the equation of the line with slope m and containing the point (x_1, y_1) is

$$y - mx = c = y_1 - mx_1$$

i.e., $\boxed{y - y_1 = m(x - x_1)}$.

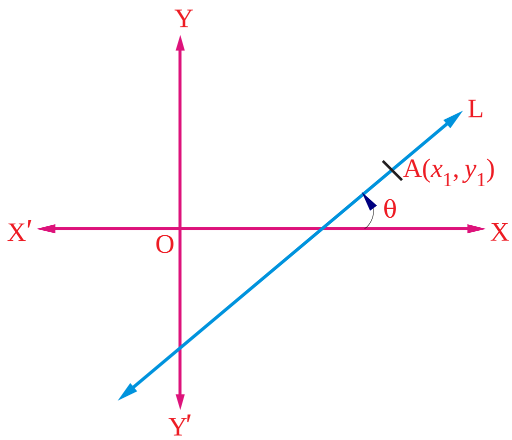


Fig. 3.10

3.1.12 Example : Find the equation of the straight line which makes an angle 135° with the positive X – axis measured counter-clockwise and passing through the point (–2, 3).

Solution : The slope of the given straight line = tan $135^o = -1$ and a point on the line is $(-2, 3)$. Hence, by point-slope form of the equation of a line, $y-3=(-1)(x+2)$ (or) $x+y-1=0$ is the equation of the given line.

3.1.13 The equation of a straight line – Two point form

Theorem : *The equation of the straight line passing through the points* $A(x_1, y_1)$ *and* $B(x_2, y_2)$ *is* $(x-x_1)(y_1-y_2)=(y-y_1)(x_1-x_2)$.

Proof : Let L be the straight line containing the points $A(x_1, y_1)$ and $B(x_2, y_2)$ (see Fig. 3.11).

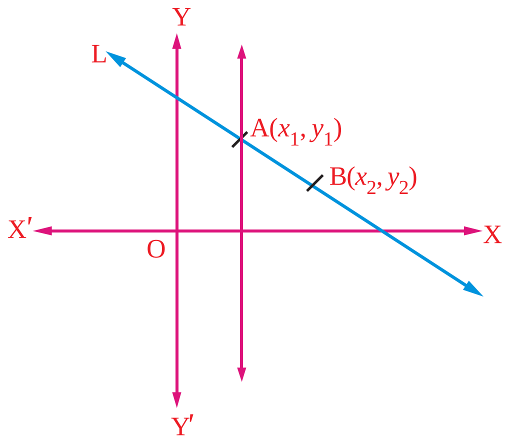


Fig. 3.11

Case 1 : Suppose L is non-vertical. Then $x_1 \neq x_2$ and

slope of the line L $= \dfrac{y_1-y_2}{x_1-x_2}$.

Therefore, the equation of L is,

$$y-y_1=\left(\frac{y_1-y_2}{x_1-x_2}\right)(x-x_1)$$

(i.e.) $(x-x_1)(y_1-y_2) = (y-y_1)(x_1-x_2)$ ...(1)

Case 2 : Suppose L is vertical. Then $x_1 = x_2$ and $y_1 \neq y_2$.

Equation of L in this case, is $x = x_1$

(see 3.1.1) which can be written in the form (1).

Therefore, equation of the straight line containing A and B is

$$\boxed{(x-x_1)(y_1-y_2) = (y-y_1)(x_1-x_2)}.$$

3.1.14 Note

1. Three points $A(x_1, y_1)$, B (x_2, y_2) and $C(x_3, y_3)$ are collinear.

$$\Leftrightarrow \quad C \in \overleftrightarrow{AB}$$

$$\Leftrightarrow \quad C(x_3, y_3) \text{ satisfies the equation of } \overleftrightarrow{AB}$$

$$\Leftrightarrow \quad (x_3-x_1)(y_1-y_2) = (y_3-y_1)(x_1-x_2)$$

$$\Leftrightarrow \quad x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)=0$$

$$\Leftrightarrow \quad \boxed{\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0}.$$

2. The expression $\Sigma x_1(y_2 - y_3)$ equals to 0 if A, B, C are collinear. If A, B, C are non-collinear, then $\Sigma x_1(y_2 - y_3) \neq 0$ and equals twice the area of Δ ABC in magnitude (disregarding the sign).

3. Equation of the straight line containing (x_1, y_1) and (x_2, y_2) can also be written as

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0 .$$

3.1.15 Example: Find the equation of the straight line passing through the points (1, –2) and (–2, 3).

Solution : The slope of the straight line containing the points (1, –2) and (–2, 3) is $\dfrac{3+2}{-2-1} = -\dfrac{5}{3}$.
Hence, by the point-slope form 3.1.11, the equation of the line containing the above two points is

$$y + 2 = -\frac{5}{3}(x - 1)$$

(i.e.) $5x + 3y + 1 = 0$.

3.1.16 Solved Problems

1. Problem : *Find the equation of the straight line passing through the point* (2, 3) *and making non-zero intercepts on the axes of coordinates whose sum is zero.*

Solution : Let the intercepts made by the straight line on the coordinate axes be $a, -a$ ($a \neq 0$). Then the equation of the straight line is $\dfrac{x}{a} + \dfrac{y}{-a} = 1$ (i.e.) $x - y = a$.

If this line passes through (2, 3), then $a = 2 - 3 = -1$.

Hence, equation of the required line is $x - y + 1 = 0$.

2. Problem : *Find the equation of the straight line passing through the points* $(at_1^2, 2at_1)$ *and* $(at_2^2, 2at_2)$.

Solution : The equation of the straight line containing the points $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ is

$$(x - at_1^2)(2at_1 - 2at_2) = (y - 2at_1)(at_1^2 - at_2^2)$$

(i.e.) $$2(x - at_1^2) = (y - 2at_1)(t_1 + t_2) \qquad (\because t_1 \neq t_2)$$

(i.e.) $2x - (t_1 + t_2)y + 2at_1t_2 = 0$.

3. Problem : *Find the equation of the straight line passing through* A (–1, 3) *and (i) parallel (ii) perpendicular to the straight line passing through* B (2, –5) *and* C (4, 6).

Solution : Slope of the straight line $\overleftrightarrow{BC} = \frac{6+5}{4-2} = \frac{11}{2}$.

(i) Slope of any straight line parallel to $\overleftrightarrow{BC}$ is also $\frac{11}{2}$ and hence, the equation of the line through A, parallel to $\overleftrightarrow{BC}$ is $y - 3 = \frac{11}{2}(x + 1)$ i.e., $11x - 2y + 17 = 0$.

(ii) Slope of any straight line perpendicular to $\overleftrightarrow{BC}$ is $\frac{-2}{11}$ and hence the equation of the line through A, perpendicular to $\overleftrightarrow{BC}$ is $y - 3 = \frac{-2}{11}(x + 1)$ (or) $2x + 11y - 31 = 0$.

4. Problem : *Prove that the points* (1, 11), (2, 15) *and* (–3, –5) *are collinear and find the equation of the straight line containing them.*

Solution: Let A = (1, 11), B = (2, 15) and C = (–3, –5) be the given points.

Then slope of $\overleftrightarrow{BC} = \frac{15+5}{2+3} = 4$.

$\therefore$ Equation of $\overleftrightarrow{BC}$ is $y + 5 = 4(x + 3)$ or $4x - y + 7 = 0$.

Also $4(1) - 11 + 7 = 0$. Hence A(1, 11) satisfies the equation of $\overleftrightarrow{BC}$. Accordingly, A, B, C are collinear and the equation of the line containing them is $4x - y + 7 = 0$.

3.1.17 Note

One of the following methods can be used for showing that three given points A, B and C are collinear.

1. Sum of two of the distances AB, BC and CA is equal to the third.
2. $\Sigma x_1(y_2 - y_3) = 0$ (or) $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$ when A, B, C are respectively (x_r, y_r); $r = 1, 2, 3$.
3. The slopes of the lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{BC}$ are equal (if the abscissae of the points are distinct).
4. Find the equation of the line $\overleftrightarrow{AB}$ and show that C satisfies the equation of $\overleftrightarrow{AB}$.

Exercise 3(a)

I. **1.** Find the slopes of the lines $x + y = 0$ and $x - y = 0$.

2. Find the equation of the line containing the points (2, –3) and (0, –3).

3. Find the equation of the line containing the points (1, 2) and (1, –2).

4. Find the angle which the straight line $y = \sqrt{3}x - 4$ makes with the Y – axis.

5. Write the equation of the reflection of the line $x = 1$ in the Y – axis.

6. Find the condition for the points $(a, 0)$, (h, k) and $(0, b)$, where $ab \neq 0$, to be collinear.

7. Write the equations of the straight lines parallel to X – axis and (i) at a distance of 3 units above the X – axis and (ii) at a distance of 4 units below the X – axis.

8. Write the equations of the straight lines parallel to Y – axis and (i) at a distance of 2 units from the Y – axis to the right of it, (ii) at a distance of 5 units from the Y – axis to the left of it.

II. 1. Find the slopes of the straight lines passing through the following pairs of points.

(i) (–3, 8), (10, 5) (ii) (3, 4), (7, –6)

(iii) (8, 1), (–1, 7) (iv) $(-p, q)$, $(q, -p)$ $(pq \neq 0)$

2. Find the value of x, if the slope of the line passing through (2, 5) and $(x, 3)$ is 2.

3. Find the value of y, if the line joining the points $(3, y)$ and (2, 7) is parallel to the line joining the points (–1, 4) and (0, 6).

4. Find the slopes of the lines (i) parallel to and (ii) perpendicular to the line passing through (6, 3) and (– 4, 5).

5. Find the equations of the straight lines which make the following angles with the positive X – axis in the positive direction and which pass through the points given below.

(i) $\frac{\pi}{4}$ and (0, 0) (ii) $\frac{\pi}{3}$ and (1, 2)

(iii) 135° and (3, – 2) (iv) 150° and (– 2, – 1).

6. Find the equations of the straight lines passing through the origin and making equal angles with the coordinate axes.

7. The angle made by a straight line with the positive X – axis in the positive direction and the Y-intercept cut off by it are given below. Find the equation of the straight line.

(i) 60°, 3 (ii) 150°, 2 (iii) 45°, – 2 (iv) $\text{Tan}^{-1}\left(\frac{2}{3}\right)$, 3

8. Find the equation of the straight line passing through $(-4, 5)$ and cutting off equal and nonzero intercepts on the coordinate axes.

9. Find the equation of the straight line passing through $(-2, 4)$ and making non-zero intercepts whose sum is zero.

III. 1. Find the equation of the straight line passing through the point $(3, -4)$ and making X and Y – intercepts which are in the ratio 2 : 3.

2. Find the equation of the straight line passing through the point $(4, -3)$ and perpendicular to the line passing through the points $(1, 1)$ and $(2, 3)$.

3. Show that the following sets of points are collinear and find the equation of the line L containing them.

   (i) $(-5, 1), (5, 5), (10, 7)$

   (ii) $(1, 3), (-2, -6), (2, 6)$

   (iii) $(a, b + c), (b, c + a), (c, a + b)$

4. A(10, 4), B(–4, 9) and C(–2, –1) are the vertices of a triangle. Find the equations of

   (i) $\overleftrightarrow{AB}$ (ii) the median through A

   (iii) the altitude through B (iv) the perpendicular bisector of the side $\overleftrightarrow{AB}$.

3.2 Straight line - Normal form - Illustrations

We now prove the theorem relating to the normal form of the equation of a straight line which was not discussed in the previous class.

3.2.1 Theorem : *The equation of the straight line, whose distance from the origin is p and the normal ray of which drawn from the origin makes an angle α with the positive direction of the X – axis measured counter clock-wise, is $x \cos \alpha + y \sin \alpha = p$.*

Proof : Let a straight line L meet the X – axis at A and the Y – axis at B.

Let N be the foot of the perpendicular drawn from the origin to the straight line L and ON = p. Also let $\overrightarrow{ON}$ make angle α with $\overrightarrow{OX}$.(see Fig. 3.12 (a), (b), (c) and (d)).

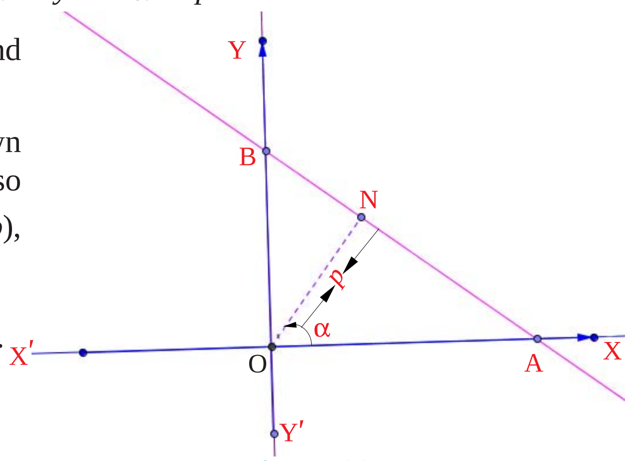


Fig. 3.12(a)

Observe that $\alpha \notin \left\{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}\right\}$ and $0 \le \alpha < 2\pi$.

From Δ AON, $\frac{ON}{OA} = |\cos \alpha|$ (or)

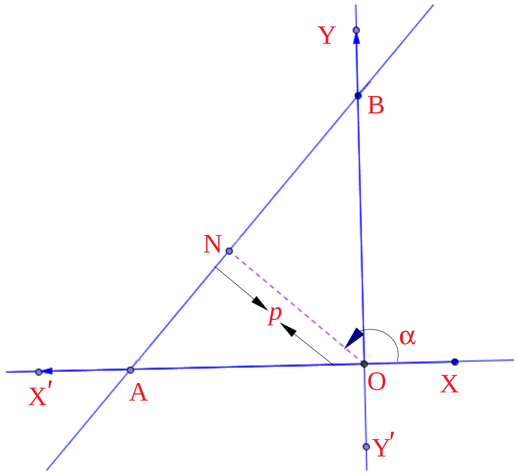


Fig. 3.12(b)

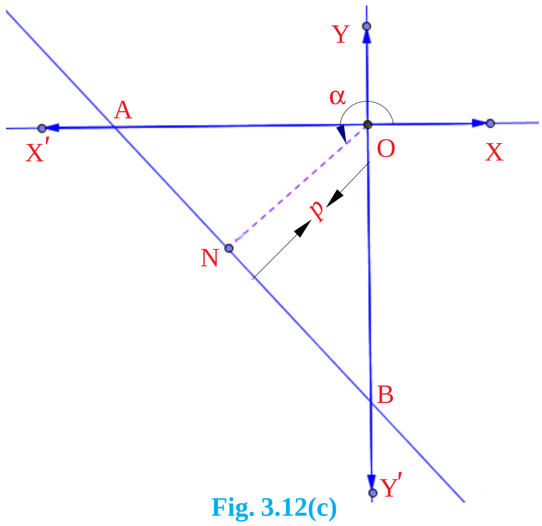


Fig. 3.12(c)

$$OA = \frac{ON}{|\cos\alpha|} = \frac{p}{|\cos\alpha|}$$

From Δ BON, $\dfrac{ON}{OB} = |\sin\alpha|$ (or)

$$OB = \frac{ON}{|\sin\alpha|} = \frac{p}{|\sin\alpha|}$$

$\therefore$ A = ($p\sec\alpha$, 0) and B = (0, $p\operatorname{cosec}\alpha$)

Hence, the equation of the straight line L (by 3.1.9)

is $\dfrac{x}{p\sec\alpha} + \dfrac{y}{p\operatorname{cosec}\alpha} = 1$

(i.e.) $x\cos\alpha + y\sin\alpha = p$.

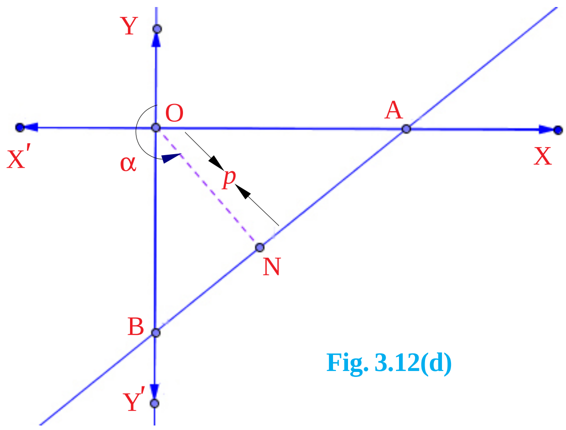


Fig. 3.12(d)

If, however, the straight line L does not intersect one of the axes of coordinates, then L is parallel to one of the axes of coordinates. If L is parallel to the X – axis, then the equation of L is either $y = p$ or $y = -p$ (see Fig. 3.12 (e) and 3.12 (f)). But then, these equations can be written as $x\cos\frac{\pi}{2} + y\sin\frac{\pi}{2} = p$ and $x\cos\frac{3\pi}{2} + y\sin\frac{3\pi}{2} = p$ respectively. Similarly, if the line L is parallel to the Y – axis at a distance p from the origin, its equation can also be expressed in the form $x\cos\alpha + y\sin\alpha = p$ where α is either 0 or π.

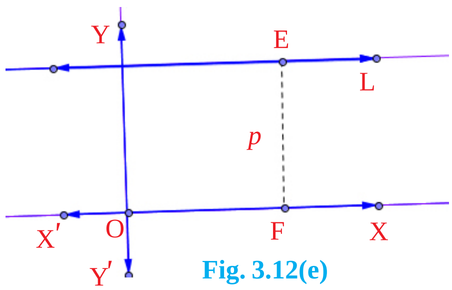


Fig. 3.12(e)

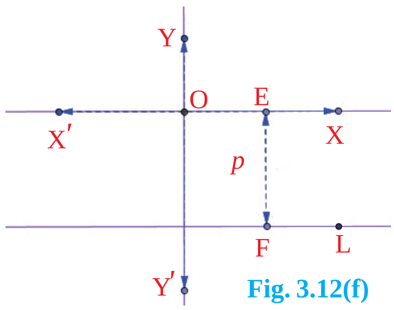


Fig. 3.12(f)

Therefore, the equation of any line L can be expressed in the form

$$\boxed{x\cos\alpha + y\sin\alpha = p \text{ where } p \geq 0 \text{ and } 0 \leq \alpha < 2\pi}\ .$$

3.2.2 Example : *Find the equation of the straight line whose distance from the origin is 4, if the normal ray from the origin to the straight line makes an angle of* 135° *with the positive direction of the* X – axis.

Solution : The equation of the given line is $x\cos\alpha + y\sin\alpha = p$ where $p = 4$ and $\alpha = 135^{\circ}$, (i.e.)

$$x\left(\frac{-1}{\sqrt{2}}\right) + y\left(\frac{1}{\sqrt{2}}\right) = 4 \text{ (or) } x - y + 4\sqrt{2} = 0.$$

3.3 Straight line - Symmetric form

Another useful form of the equation of a line is the symmetric form which is stated below.

3.3.1 Theorem : *The equation of the straight line passing through* (x_1, y_1) *and making an angle* θ *with the positive direction of the* X – *axis measured counter - clock wise is*

$$(x - x_1) : \cos\theta = (y - y_1) : \sin\theta .$$

Proof

Case 1 : Suppose the line L passing through (x_1, y_1) and making an angle θ with $\overrightarrow{OX}$ in the positive direction is non-vertical. Then $\theta \neq \frac{\pi}{2}$ and the slope of L $= \tan\theta$.

The equation of L from the point-slope form is

$$y - y_1 = \left(\frac{\sin\theta}{\cos\theta}\right)(x - x_1) \quad \text{(or)} \quad (x - x_1) : \cos\theta = (y - y_1) : \sin\theta .$$

Case 2 : If L is vertical, then $\theta = \frac{\pi}{2}$ and so, $\cos\theta = 0$ and $\sin\theta = 1$. Since L contains (x_1, y_1), equation of L is, therefore, $x = x_1$

$$\text{i.e., } (x - x_1)\sin\theta = (y - y_1)\cos\theta \text{ (or) } \boxed{(x - x_1) : \cos\theta = (y - y_1) : \sin\theta}.$$

3.3.2 Note : The symmetric form of the equation of a straight line L is

$$(x - x_1) : \cos\theta = (y - y_1) : \sin\theta$$

(i.e.) $x - x_1 = r\cos\theta, \quad y - y_1 = r\sin\theta$ where $r \in \mathbf{R}$.

Here $|r| = \sqrt{r^2} = \sqrt{(r\cos\theta)^2 + (r\sin\theta)^2} = \sqrt{(x - x_1)^2 + (y - y_1)^2}$.

That is, $|r|$ denotes the distance of the point (x_1, y_1) from the point (x, y) on the straight line. As the point (x, y) varies on the line, r also varies. This variable r which can assume any real value is called a **parameter**. Therefore, given a point P(x, y) on L, there exists a real number r such that $x = x_1 + r\cos\theta$, $y = y_1 + r\sin\theta$ are the coordinates of the point P. Conversely, the point with coordinates $(x_1 + r\cos\theta, \; y_1 + r\sin\theta)$, for different parametric values of r, always lies on the line L as it satisfies the equation $(x-x_1) : \cos\theta = (y-y_1) : \sin\theta$. For this reason, the equations $x = x_1 + r\cos\theta$, $y = y_1 + r\sin\theta$ are called the **parametric equations of the straight line** L and the parameter r varies over the interval $(-\infty, \infty)$. It can also be observed that the parameter r is positive for points which lie on one side of the point (x_1, y_1) on L and r is negative for points on the other side of the point (x_1, y_1) on L.

3.3.3 Example : *Write the parametric equations of the straight line passing through the point* (3, 2) *and making an angle of* 135^o *with the positive direction of the X – axis in the positive direction.*

Solution : In the parametric form of the equation of a straight line, $\theta = 135^o$ and $(x_1, y_1) = (3, 2)$.

So, the parametric equations of the given straight line are $x = x_1 + r\cos\theta = 3 - \frac{r}{\sqrt{2}}$ and $y = y_1 + r\sin\theta = 2 + \frac{r}{\sqrt{2}}$.

3.3.4 General form of the equation of a line : Here under, we prove that the locus of a linear equation in two variables x and y is a straight line. In view of the foregoing result, a linear equation in x and y is known as the general form of the equation of a line.

Theorem

(i) *The equation of a straight line in the XY – plane can be expressed as a first degree equation in x and y.*

(ii) *The locus of a first degree equation in x and y is a straight line.*

Proof : We recall that a first degree (or linear) equation in two variables x and y is an equation of the form $ax + by + c = 0$, where $a^2 + b^2 \neq 0$.

(i) Suppose L is a straight line in the XY – plane. Then, if L is not vertical, its equation in slope-intercept form is $y = mx + c$ (or) $mx + (-1)y + c = 0$ which is clearly linear in x and y (since $b = -1 \neq 0$).

If L is vertical, then its equation is of the form $x = k$ for some real k and this can be expressed as $1.x + 0y + (-k) = 0$ which is again linear in x and y (Here $a = 1 \neq 0$).

(ii) Consider a first degree equation $ax + by + c = 0$ in x and y.

If $b = 0$, then $a^2 + b^2 \neq 0$ gives $a \neq 0$ and in this case, the above equation can be written as $x = \frac{-c}{a}$. Clearly the locus of this equation is a vertical line. If, however, $b \neq 0$, then the equation $ax + by + c = 0$ can be expressed in the form $y = \left(-\frac{a}{b}\right)x + \left(-\frac{c}{b}\right) = mx + k$, where $m = \frac{-a}{b}$. So, the graph of this equation is a non-vertical straight line with slope $\frac{-a}{b}$.

3.3.5 Note

(i) The form $ax + by + c = 0$ of the equation of a straight line is called the **General form** of the equation of a line.

(ii) The linear equation $ax + by + c = 0$ represents a vertical line if $b = 0$ and a non-vertical line with slope $\frac{-a}{b}$ if $b \neq 0$.

3.3.6 Theorem : *Suppose* L_1 *and* L_2 *are two straight lines in the* XY – *plane with equations* $a_1x + b_1y + c_1 = 0$ *and* $a_2x + b_2y + c_2 = 0$ *respectively. Then the lines* L_1 *and* L_2 *are parallel if and only if* $a_1b_2 = a_2b_1$.

Proof : Suppose L_1 and L_2 are parallel lines.

In case both these lines are vertical, we have $b_1 = 0$ and $b_2 = 0$ so that $a_1b_2 = 0 = a_2b_1$.

If L_1 and L_2 are non vertical, then $b_1b_2 \neq 0$ and

$$L_1 \parallel L_2 \Rightarrow \text{slope of } L_1 = \text{slope of } L_2$$

$$\Rightarrow \frac{-a_1}{b_1} = \frac{-a_2}{b_2}$$

$$\Rightarrow a_1b_2 = a_2b_1.$$

Conversely, suppose L_1 and L_2 are a pair of straight lines such that

$$a_1b_2 = a_2b_1 \qquad \ldots (1)$$

We observe that $b_1 = 0 \Leftrightarrow b_2 = 0$

So, if one of the lines is vertical, then the other is also vertical and thus $L_1 \parallel L_2$.

If neither b_1 nor b_2 is zero, then from (1), it follows that $\frac{a_1}{b_1} = \frac{a_2}{b_2}$ (i.e.) slopes of L_1 and L_2 are equal. Accordingly $L_1 \parallel L_2$.

3.3.7 Theorem : *Two first degree equations* $a_1x + b_1y + c_1 = 0$ *and* $a_2x + b_2y + c_2 = 0$ *in* x *and* y *represent the same straight line if and only if* $a_1 : b_1 : c_1 = a_2 : b_2 : c_2$.

Proof : Suppose the two equations $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ represent the same straight line. Then by theorem 3.3.6, it follows that $a_1b_2 = a_2b_1$ ... (1)

If $b_1 = 0$, then $a_1 \neq 0$ and from (1), we get $b_2 = 0$. So, in this case, the equations can be written as $x = \frac{-c_1}{a_1}$ and $x = \frac{-c_2}{a_2}$. Since these equations represent the same vertical line, we must have

$$\frac{c_1}{a_1} = \frac{c_2}{a_2} \quad \text{(or)} \quad a_1c_2 = a_2c_1 \quad \text{i.e., } a_1 : c_1 = a_2 : c_2. \qquad \ldots (2)$$

Thus from (1) and (2), we obtain $a_1 : b_1 : c_1 = a_2 : b_2 : c_2$.

If $b_1 \neq 0$, then from (1), $b_2 \neq 0$. Writing $k = \frac{b_1}{b_2}$, we get , from (1), $a_1 = ka_2$.

Also, since $\left(0, \frac{-c_1}{b_1}\right)$ satisfies the equation $a_1x + b_1y + c_1 = 0$, it also satisfies $a_2x + b_2y + c_2 = 0$.

$$\text{Hence, } b_2\left(\frac{-c_1}{b_1}\right) + c_2 = 0 \text{ (i.e.) } c_1 = kc_2.$$

Thus there exists a non-zero number k such that $a_1 = ka_2$, $b_1 = kb_2$, and $c_1 = kc_2$. Hence $a_1 : b_1 : c_1 = a_2 : b_2 : c_2$.

Conversely, if $a_1 : b_1 : c_1 = a_2 : b_2 : c_2$, then for some real number $k \neq 0$, we have $a_1 = ka_2$, $b_1 = kb_2$, and $c_1 = kc_2$. Since

$$a_1x + b_1y + c_1 = 0$$

$$\Leftrightarrow k(a_2x + b_2y + c_2) = 0$$

$$\Leftrightarrow a_2x + b_2y + c_2 = 0,$$

the equations $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ represent the same line.

3.4 Straight line - Reduction into various forms

In this section, we explain how the general form of the equation of a straight line can be reduced to the intercept form and the normal form.

3.4.1 Reduction of the equation $ax + by + c = 0$; $abc \neq 0$ of a straight line to the intercept form

If $abc \neq 0$, then $ax + by + c = 0$

$$\Leftrightarrow \left(\frac{-a}{c}\right)x + \left(\frac{-b}{c}\right)y = 1$$

$$\Leftrightarrow \frac{x}{\left(\frac{-c}{a}\right)} + \frac{y}{\left(\frac{-c}{b}\right)} = 1.$$

Therefore, $\boxed{\frac{x}{\left(\frac{-c}{a}\right)} + \frac{y}{\left(\frac{-c}{b}\right)} = 1}$ is the intercept form of $ax + by + c = 0$ and $\frac{-c}{a}$ and $\frac{-c}{b}$ are respectively the x-intercept and the y-intercept of the line .

3.4.2 Reduction of the equation $ax + by + c = 0$ of a straight line to the normal form

Since $ax + by + c = 0$ is linear in x and y, we have $a^2 + b^2 \neq 0$. Therefore, $ax + by + c = 0$

$$\Leftrightarrow \boxed{\frac{a}{\sqrt{a^2+b^2}}x + \frac{b}{\sqrt{a^2+b^2}}y = \frac{-c}{\sqrt{a^2+b^2}}} \quad \text{... (A)}$$

$$\Leftrightarrow \boxed{\left(\frac{-a}{\sqrt{a^2+b^2}}\right)x + \left(\frac{-b}{\sqrt{a^2+b^2}}\right)y = \frac{c}{\sqrt{a^2+b^2}}} \quad \text{... (B)}$$

If $c < 0$, then (A) is the normal form of the equation of the line and if $c \geq 0$, then (B) is the normal form of the equation of the line. Observe that in the normal form of the equation of the line, $p = \frac{|c|}{\sqrt{a^2+b^2}}$ which is the distance of the straight line from the origin is non negative.

3.4.3 Note

1. To reduce the general form of the equation of a straight line $ax + by + c = 0$ to the normal form, rearrange the terms of the equation so that the constant term is on the right hand side and is non-negative. Then divide the equation throughout by $\sqrt{a^2+b^2}$. The resulting equation will be in the normal form as there will be only one α in $[0, 2\pi)$, satisfying

$$\cos\alpha = \frac{a}{\sqrt{a^2+b^2}}, \text{ and } \sin\alpha = \frac{b}{\sqrt{a^2+b^2}} \text{ or}$$

$$\cos\alpha = \frac{-a}{\sqrt{a^2+b^2}}, \text{ and } \sin\alpha = \frac{-b}{\sqrt{a^2+b^2}}.$$

2. The perpendicular distance of the straight line $ax + by + c = 0$ from the origin is $\boxed{\frac{|c|}{\sqrt{a^2+b^2}}}$.

3.4.4 Example : Transform the equation $x + y + 1 = 0$ into normal form

Solution :

$$x + y + 1 = 0$$

$$\Leftrightarrow \left(\frac{-1}{\sqrt{2}}\right)x + \left(\frac{-1}{\sqrt{2}}\right)y = \frac{1}{\sqrt{2}}$$

$$\Leftrightarrow x\cos\frac{5\pi}{4} + y\sin\frac{5\pi}{4} = \frac{1}{\sqrt{2}}.$$

Hence, the normal form of the equation of the given straight line is $x\cos\frac{5\pi}{4} + y\sin\frac{5\pi}{4} = \frac{1}{\sqrt{2}}$ and the distance of this line from the origin is $\frac{1}{\sqrt{2}}$.

3.4.5 Solved Problems

1. Problem : A *straight line passing through* A $(1, -2)$ *makes an angle* $\mathrm{Tan}^{-1}\frac{4}{3}$ *with the positive direction of the* X – *axis in the anti-clockwise sense. Find the points on the straight line whose distance from* A *is* 5.

Solution : The parametric equations of the line through A$(1, -2)$ and whose slope is $\frac{4}{3}$ $\left(\because \tan\theta = \frac{4}{3}\right)$ are $x = 1 + r\cos\theta = 1 + r\left(\frac{3}{5}\right)$ and $y = -2 + r\sin\theta = -2 + r\left(\frac{4}{5}\right)$.

The points on the above line at a distance of $|r| = 5$ correspond to $r = \pm 5$ in the above equations and are therefore (4, 2) and (–2, –6).

2. Problem : *A straight line parallel to the line* $y = \sqrt{3}x$ *passes through* Q(2, 3) *and cuts the line* $2x + 4y - 27 = 0$ *at* P. *Find the length of* PQ.

Solution : Since $\overleftrightarrow{PQ}$ is parallel to the straight line $y = \sqrt{3}x$, slope of $\overleftrightarrow{PQ} = \sqrt{3}$ and therefore, $\overleftrightarrow{PQ}$ makes 60^0 with $\overleftrightarrow{OX}$ in the positive direction (see Fig. 3.13). Hence, the coordinates of P are $(2 + r\cos 60^{\circ}, 3 + r\sin 60^{\circ})$.

(i.e.,) $\left(2 + \frac{r}{2}, 3 + \frac{\sqrt{3}}{2}r\right)$ where $|r| = \mathrm{PQ}$.

Since the point P lies on the straight line $2x + 4y - 27 = 0$, we must have

$$2\left(2 + \frac{r}{2}\right) + 4\left(3 + \frac{\sqrt{3}}{2}r\right) = 27.$$

This gives $r = 2\sqrt{3} - 1$ and therefore,

$$\mathrm{PQ} = |r| = 2\sqrt{3} - 1.$$

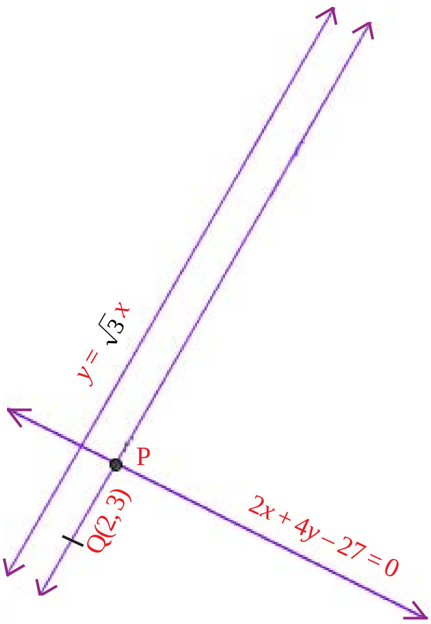


Fig. 3.13

3. Problem : *Transform the equation* $3x + 4y + 12 = 0$ *into* (i) *slope-intercept form* (ii) *intercept form and* (iii) *normal form.*

Solution

(i) Slope-intercept form :

$$3x + 4y + 12 = 0$$
$$\Leftrightarrow 4y = -3x - 12$$
$$\Leftrightarrow y = \left(-\frac{3}{4}\right)x + (-3)$$

$\therefore$ Slope $= -\frac{3}{4}$ and Y – intercept $= -3$.

(ii) Intercept form :

$$3x + 4y + 12 = 0$$
$$\Leftrightarrow \frac{-3x}{12} - \frac{4y}{12} = 1$$
$$\Leftrightarrow \frac{x}{(-4)} + \frac{y}{(-3)} = 1$$

$\therefore$ X – intercept of the line is – 4 and the Y – intercept is – 3.

(iii) Normal form :

$$3x + 4y + 12 = 0$$
$$\Leftrightarrow -3x - 4y - 12 = 0$$
$$\Leftrightarrow \left(-\frac{3}{5}\right)x + \left(-\frac{4}{5}\right)y = \frac{12}{5}$$
$$\Leftrightarrow x\cos\alpha + y\sin\alpha = p \quad \text{where} \quad p = \frac{12}{5}$$

and $\cos\alpha = -\frac{3}{5}$, $\sin\alpha = -\frac{4}{5}$ determine the angle α in $(0, 2\pi)$.

(Note that α lies in the third quadrant and $\alpha = \pi + \mathrm{Tan}^{-1}(4/3)$).

4. Problem : *If the area of the triangle formed by the straight lines* $x = 0$, $y = 0$ *and* $3x + 4y = a$ $(a > 0)$ *is* 6, *find the value of* a.

Solution: The straight line $3x + 4y = a$ (or) $\frac{x}{\left(\frac{a}{3}\right)} + \frac{y}{\left(\frac{a}{4}\right)} = 1$ has intercepts $\frac{a}{3}$ and $\frac{a}{4}$ on the axes of coordinates. Therefore, the area of the triangle formed by this line and the axes of coordinates

$= \frac{1}{2} \times \frac{a}{3} \times \frac{a}{4} = \frac{a^2}{24}$.

But $\frac{a^2}{24} = 6 \Rightarrow a^2 = 144 \Rightarrow a = 12. \quad (\because a > 0)$

Exercise 3(b)

I. **1.** Find the sum of the squares of the intercepts of the line $4x - 3y = 12$ on the axes of coordinates.

2. If the portion of a straight line intercepted between the axes of coordinates is bisected at $(2p, 2q)$, write the equation of the straight line.

3. If the linear equations $ax + by + c = 0$ $(abc \neq 0)$ and $lx + my + n = 0$ represent the same line and $r = \frac{l}{a} = \frac{n}{c}$, write the value of r in terms of m and b.

4. Find the angle made by the straight line $y = -\sqrt{3}x + 3$ with the positive direction of the X – axis measured in the counter-clockwise direction.

5. The intercepts of a straight line on the axes of coordinates are a and b. If p is the length of the perpendicular drawn from the origin to this line, write the value of p in terms of a and b.

II. 1. In what follows, p denotes the distance of the straight line from the origin and α denotes the angle made by the normal ray drawn from the origin to the straight line with $\overleftrightarrow{OX}$ measured in the anti-clockwise sense. Find the equations of the straight lines with the following values of p and α.

(i) $p = 5, \ \alpha = 60^0$ **(ii)** $p = 6, \ \alpha = 150^0$

(iii) $p = 1, \ \alpha = \frac{7\pi}{4}$ **(iv)** $p = 4, \ \alpha = 90^0$

(v) $p = 0, \ \alpha = 0$ **(vi)** $p = 2\sqrt{2}, \ \alpha = \frac{5\pi}{4}$

2. Find the equations of the straight lines in the symmetric form, given the slope and a point on the line in each part of the question.

(i) $\sqrt{3}, (2,3)$ **(ii)** $-\frac{1}{\sqrt{3}}, (-2,0)$ **(iii)** $-1, (1,1)$

3. Transform the following equations into (a) slope-intercept form (b) intercept form and (c) normal form.

(i) $3x + 4y = 5$ **(ii)** $4x - 3y + 12 = 0$

(iii) $\sqrt{3}x + y = 4$ **(iv)** $x + y + 2 = 0$

(v) $x + y - 2 = 0$ **(vi)** $\sqrt{3}x + y + 10 = 0$

4. If the product of the intercepts made by the straight line $x \tan\alpha + y \sec\alpha = 1$ $\left(0 \leq \alpha < \frac{\pi}{2}\right)$ on the coordinate axes is equal to $\sin\alpha$, find α.

5. If the sum of the reciprocals of the intercepts made by a variable straight line on the axes of coordinates is a constant, then prove that the line always passes through a fixed point.

6. Line L has intercepts a and b on the axes of coordinates. When the axes are rotated through a given angle, keeping the origin fixed, the same line L has intercepts p and q on the transformed axes. Prove that $\frac{1}{a^2}+\frac{1}{b^2}=\frac{1}{p^2}+\frac{1}{q^2}$.

7. Transform the equation $\frac{x}{a}+\frac{y}{b}=1$ into the normal form when $a>0$ and $b>0$. If the perpendicular distance of the straight line from the origin is p, deduce that $\frac{1}{p^2}=\frac{1}{a^2}+\frac{1}{b^2}$.

III. 1. A straight line passing through $A(-2,1)$ makes an angle of 30^0 with $\overleftrightarrow{OX}$ in the positive direction. Find the points on the straight line whose distance from A is 4 units.

2. Find the points on the line $3x-4y-1=0$ which are at a distance of 5 units from the point (3, 2).

3. A straight line whose inclination with the positive direction of the X – axis measured in the anti-clockwise sense is $\frac{\pi}{3}$ makes positive intercept on the Y – axis. If the straight line is at a distance of 4 from the origin, find its equation.

4. A straight line L is drawn through the point A(2, 1) such that its point of intersection with the straight line $x+y=9$ is at a distance of $3\sqrt{2}$ from A. Find the angle which the line L makes with the positive direction of the X – axis.

5. A straight line L with negative slope passes through the point (8, 2) and cuts positive coordinate axes at the points P and Q. Find the minimum value of OP + OQ as L varies, where O is the origin.

3.5 Intersection of two straight lines

In this section, we find the point of intersection of two intersecting lines and we also discuss the two half-planes partitioned by a straight line in the coordinate plane.

3.5.1 Theorem : *If* $L_1 \equiv a_1x+b_1y+c_1=0$ *and* $L_2 \equiv a_2x+b_2y+c_2=0$ *represent two intersecting lines, then their point of intersection is* $\left(\frac{b_1c_2-b_2c_1}{a_1b_2-a_2b_1}, \frac{c_1a_2-c_2a_1}{a_1b_2-a_2b_1}\right)$.

Proof : Consider the straight lines

$$L_1 \equiv a_1x+b_1y+c_1=0 \qquad \text{... (1)}$$

and

$$L_2 \equiv a_2x+b_2y+c_2=0 \qquad \text{... (2)}$$

Since the lines intersect, we must have $a_1b_2 \neq a_2b_1$.

If $P(x_0, y_0)$ is the point of intersection of lines (1) and (2), then P satisfies both the equations (1) and (2) and so,

$$a_1x_0 + b_1y_0 + c_1 = 0 \qquad \dots (3)$$

and

$$a_2x_0 + b_2y_0 + c_2 = 0 \qquad \dots (4)$$

By applying the rule of cross-multiplication to (3) and (4), we obtain

$$x_0 : y_0 : 1 = (b_1c_2 - b_2c_1) : (c_1a_2 - c_2a_1) : (a_1b_2 - a_2b_1).$$

Therefore, $x_0 = \dfrac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}, \quad y_0 = \dfrac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}$

and the point of intersection of the lines (1) and (2) is $\boxed{P = \left(\dfrac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}, \dfrac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}\right)}$.

Note that, if the straight lines (1) and (2) are parallel, then $a_1b_2 = a_2b_1$ and in this case, the equations (3) and (4) cannot be solved for x_0 and y_0. As such, the point of intersection of the lines doesn't exist.

3.5.2 Example : Find the point of intersection of the straight lines $\dfrac{x}{a} + \dfrac{y}{b} = 1$ and $\dfrac{x}{b} + \dfrac{y}{a} = 1$ $(a \neq \pm b)$.

Solution : Let $P(x_0, y_0)$ be the point of intersection of the straight lines $\dfrac{x}{a} + \dfrac{y}{b} = 1$ and $\dfrac{x}{b} + \dfrac{y}{a} = 1$.

Then, $\dfrac{x_0}{a} + \dfrac{y_0}{b} = 1$ and $\dfrac{x_0}{b} + \dfrac{y_0}{a} = 1$. From this,

we obtain $\left(\dfrac{1}{a} - \dfrac{1}{b}\right)x_0 + \left(\dfrac{1}{b} - \dfrac{1}{a}\right)y_0 = 0$ (i.e.) $x_0 = y_0$.

But $\dfrac{x_0}{a} + \dfrac{y_0}{b} = 1$ and $x_0 = y_0 \Rightarrow x_0 = \dfrac{ab}{a+b} = y_0$.

$\therefore$ P $\left(\dfrac{ab}{a+b}, \dfrac{ab}{a+b}\right)$ is the point of intersection of the given lines.

3.5.3 Half-planes : A straight line divides the coordinate plane into three mutually disjoint sets of points, namely **(i)** the set of points on the straight line, **(ii)** the set of points on one side of the straight line (shaded portion as in the Fig. 3.14) and **(iii)** the set of points on the other side of the straight line.

Notation : (i) The linear expression $ax + by + c$ is denoted by L. Then the general form of the equation of a straight line is $ax + by + c = 0$ or, briefly, L = 0.

(ii) We denote $ax_1 + by_1 + c$ by L_{11} and $ax_2 + by_2 + c$ by L_{22}. If the point $A(x_1, y_1)$ lies on the straight line $L = 0$, then the expression L_{11} equals zero. If the point A does not lie on the line $L = 0$, then $L_{11} \neq 0$ and hence, L_{11} is either positive or negative. As such, the points of the plane are divided into three parts as

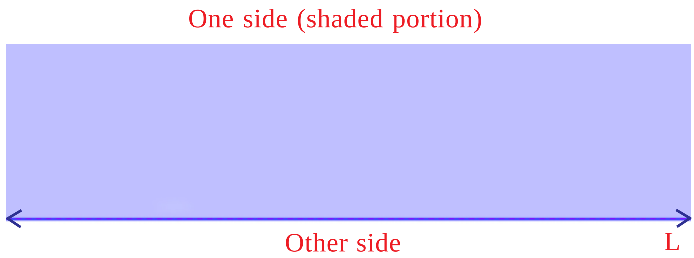


Fig. 3.14

(i) the set of points for which $L = 0$,

(ii) the set of points for which $L > 0$,

and **(iii)** the set of points for which $L < 0$.

In what follows, we find that the classification of points (x_1, y_1) on either side of a given straight line is based on whether L_{11} is positive or negative.

3.5.4 Theorem : *The ratio in which the straight line* $L \equiv ax + by + c = 0$ *divides the line segment joining the points* $A(x_1, y_1)$ *and* $B(x_2, y_2)$ *is* $-L_{11} : L_{22}$.

Proof : Let the straight line divide the line segment $\overline{AB}$ in the ratio $l : m$ at P. Then

$$P = \left(\frac{lx_2 + mx_1}{l + m}, \frac{ly_2 + my_1}{l + m}\right)$$ is a point on the straight line $L = 0$ and therefore,

$$a\left(\frac{lx_2 + mx_1}{l + m}\right) + b\left(\frac{ly_2 + my_1}{l + m}\right) + c = 0$$

(i.e.) $\quad a(lx_2 + mx_1) + b(ly_2 + my_1) + c(l + m) = 0.$

(or) $\quad l(ax_2 + by_2 + c) + m(ax_1 + by_1 + c) = 0.$

Hence, $\boxed{l : m = -(ax_1 + by_1 + c) : (ax_2 + by_2 + c) = -L_{11} : L_{22}}$.

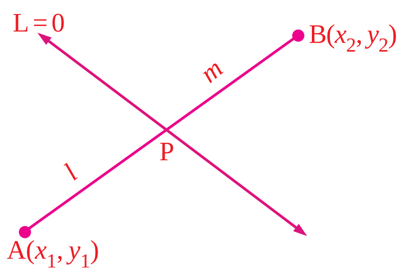


Fig. 3.15(a)

3.5.5 Note

(i) The points A, B are on opposite sides of the line $L = 0$

$\Leftrightarrow$ P divides $\overline{AB}$ internally (see Fig. 3.15 (a))

$\Leftrightarrow$ $l : m = -L_{11} : L_{22} > 0$

$\Leftrightarrow$ $\boxed{L_{11} \text{ and } L_{22} \text{ have opposite signs}}$.

(ii) The points A, B lie on the same side of the line $L = 0$

$\Leftrightarrow$ P divides $\overline{AB}$ externally (see Fig. 3.15 (b))

$\Leftrightarrow$ $l : m = -L_{11} : L_{22} < 0$

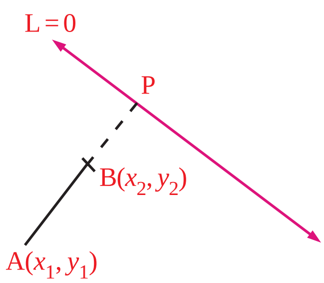


Fig. 3.15(b)

$\Leftrightarrow$ $\boxed{L_{11} \text{ and } L_{22} \text{ have the same sign.}}$

(iii) X – axis divides $\overline{AB}$ in the ratio $-y_1 : y_2$ (since the equation of the X – axis is $y = 0$ and $L_{11} = y_1$, $L_{22} = y_2$). Similarly, the Y – axis divides $\overline{AB}$ in the ratio $-x_1 : x_2$.

3.5.6 Examples

1. Example : Find the ratio in which the straight line $2x + 3y - 20 = 0$ divides the join of the points (2, 3) and (2, 10).

Solution : Here $L \equiv 2x + 3y - 20$, $L_{11} = 2(2) + 3(3) - 20 = -7$

and $L_{22} = 2(2) + 3(10) - 20 = 14$.

So, the straight line L = 0 divides the given line segment in the ratio $-L_{11} : L_{22} = 7 : 14 = 1 : 2$ and the division is internal.

2. Example : State whether (3, 2) and (– 4, – 3) are on the same side or on opposite sides of the straight line $2x - 3y + 4 = 0$.

Solution : If $L \equiv 2x - 3y + 4$, then $L_{11} = 2(3) - 3(2) + 4 = 4$

and $L_{22} = 2(-4) - 3(-3) + 4 = 5$.

As L_{11} and L_{22} have the same sign, the two points lie on the same side of the given line L = 0.

3. Example : Find the ratios in which (i) the X – axis and (ii) the Y – axis divide the line segment $\overline{AB}$ joining A (2, – 3) and B(3, –6).

Solution

(i) X – axis divides $\overline{AB}$ in the ratio $-y_1 : y_2 = -3 : 6 = -1 : 2$.

(ii) Y – axis divides $\overline{AB}$ in the ratio $-x_1 : x_2 = -2 : 3$.

So, both the axes of coordinates divide the line segment $\overline{AB}$ externally.

3.6 Family of straight lines - Concurrent lines

A set of straight lines having a common property is also known as a family of straight lines. In this section, we discuss (i) the family of straight lines parallel to a given line and (ii) the family of straight lines concurrent with two given intersecting lines.

3.6.1 Theorem : *Let $L_1 \equiv a_1x + b_1y + c_1 = 0$ and $L_2 \equiv a_2x + b_2y + c_2 = 0$ represent a pair of parallel straight lines. Then the straight line represented by $\lambda_1 L_1 + \lambda_2 L_2 = 0$ is parallel to each of the straight lines $L_1 = 0$ and $L_2 = 0$.*

Proof : The straight lines $L_1 = 0$ and $L_2 = 0$ are parallel only if $a_1b_2 = a_2b_1$.

But then, $\lambda_1 L_1 + \lambda_2 L_2 \equiv \lambda_1 (a_1x + b_1y + c_1) + \lambda_2 (a_2x + b_2y + c_2)$

$$\equiv (\lambda_1 a_1 + \lambda_2 a_2)x + (\lambda_1 b_1 + \lambda_2 b_2)y + (\lambda_1 c_1 + \lambda_2 c_2)$$

and $a_1(\lambda_1 b_1 + \lambda_2 b_2) - b_1(\lambda_1 a_1 + \lambda_2 a_2) = \lambda_2(a_1b_2 - a_2b_1) = 0$. So, the straight line represented by $\lambda_1 L_1 + \lambda_2 L_2 = 0$ is parallel to the straight line $L_1 = 0$ and hence, also to the line $L_2 = 0$.

3.6.2 Theorem : *Let* $L_1 \equiv a_1x + b_1y + c_1 = 0$ *and* $L_2 \equiv a_2x + b_2y + c_2 = 0$ *represent two intersecting lines. Then*

(i) *The equation* $\lambda_1 L_1 + \lambda_2 L_2 = 0$, *for parametric values of* λ_1 *and* λ_2 *with* $\lambda_1^2 + \lambda_2^2 \neq 0$, *represents a family of straight lines passing through the point of intersection of the lines* $L_1 = 0$ *and* $L_2 = 0$.

(ii) *Conversely, the equation of any straight line passing through the point of intersection of the given straight lines is of the form* $\lambda_1 L_1 + \lambda_2 L_2 = 0$ *for some real* λ_1, λ_2 *such that* $\lambda_1^2 + \lambda_2^2 \neq 0$.

Proof: Let $P(x_1, y_1)$ be the point of intersection of the given pair of intersecting lines $L_1 = 0$ and $L_2 = 0$. Then $a_1x_1 + b_1y_1 + c_1 = 0$ and $a_2x_1 + b_2y_1 + c_2 = 0$. Observe that $a_1b_2 \neq a_2b_1$, since L_1 and L_2 intersect.

(i) If $\lambda_1^2 + \lambda_2^2 \neq 0$, then atleast one of λ_1 and λ_2 is different from zero and since $a_1b_2 \neq a_2b_1$, it follows that the two numbers $\lambda_1 a_1 + \lambda_2 a_2$ and $\lambda_1 b_1 + \lambda_2 b_2$ cannot be both equal to zero.

Hence the equation $\lambda_1 L_1 + \lambda_2 L_2 \equiv \lambda_1(a_1x + b_1y + c_1) + \lambda_2(a_2x + b_2y + c_2) = 0$ (i.e.) $(\lambda_1 a_1 + \lambda_2 a_2)x + (\lambda_1 b_1 + \lambda_2 b_2)y + (\lambda_1 c_1 + \lambda_2 c_2) = 0$ represents a straight line. Also $\lambda_1(a_1x_1 + b_1y_1 + c_1) + \lambda_2(a_2x_1 + b_2y_1 + c_2) = 0$. Therefore, the above line passes through $P(x_1, y_1)$.

Hence, for parametric values of λ_1 and λ_2 with $\lambda_1^2 + \lambda_2^2 \neq 0$, the equation $\lambda_1 L_1 + \lambda_2 L_2 = 0$ represents a family of straight lines passing through $P(x_1, y_1)$ (see Fig. 3.16).

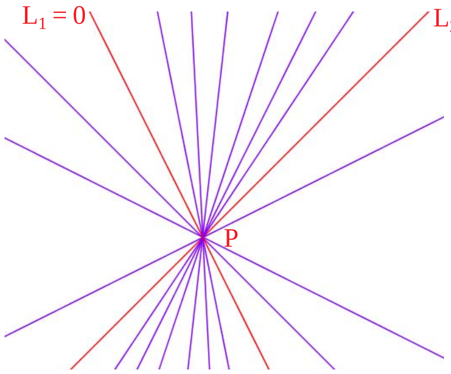


Fig. 3.16

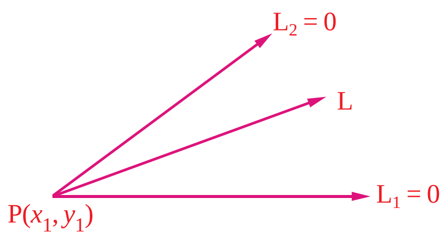


Fig. 3.17

(ii) Let $L \equiv px + qy + r = 0$ be a straight line passing through $P(x_1, y_1)$ (see Fig. 3.17).

Then $px_1 + qy_1 + r = 0$. ... (1)

Since $(p, q) \neq (0, 0)$ and $a_1b_2 \neq a_2b_1$, the equations $\lambda_1 a_1 + \lambda_2 a_2 = p$ and

$\lambda_1 b_1 + \lambda_2 b_2 = q$

have unique solution for λ_1 and λ_2 such that $(\lambda_1, \lambda_2) \neq (0, 0)$.

From (1),

$$r = -px_1 - qy_1$$
$$= -(\lambda_1 a_1 + \lambda_2 a_2)x_1 - (\lambda_1 b_1 + \lambda_2 b_2)y_1$$
$$= -\lambda_1(a_1x_1 + b_1y_1) - \lambda_2(a_2x_1 + b_2y_1) = \lambda_1 c_1 + \lambda_2 c_2.$$

$$\therefore\ px + qy + r \equiv (\lambda_1 a_1 + \lambda_2 a_2)x + (\lambda_1 b_1 + \lambda_2 b_2)y + (\lambda_1 c_1 + \lambda_2 c_2) = \lambda_1 L_1 + \lambda_2 L_2.$$

Thus, the equation of any straight line passing through the point of intersection of the lines $L_1 = 0$ and $L_2 = 0$ can be expressed in the form $\lambda_1 L_1 + \lambda_2 L_2 = 0$ for some real numbers λ_1 and λ_2 with $\lambda_1^2 + \lambda_2^2 \neq 0$.

3.6.3 Note

1. The equation $\lambda_1 L_1 + \lambda_2 L_2 = 0$ represents L_1 if $\lambda_2 = 0\ (\lambda_1 \neq 0)$ and L_2 if $\lambda_1 = 0\ (\lambda_2 \neq 0)$. The equation of any straight line different from L_1 and L_2 and passing through the point of intersection of these two lines can hence be written in the form either $L_1 + \lambda L_2 = 0$ or $L_2 + \mu L_1 = 0$ for some $\lambda \neq 0$ and $\mu \neq 0$.

2. Suppose $L_1 \equiv a_1x + b_1y + c_1 = 0$ and $L_2 \equiv a_2x + b_2y + c_2 = 0$ represent a pair of lines intersecting at P.

   If L is a straight line in the plane of $L_1 = 0$ and $L_2 = 0$ and L' is a straight line passing through P and parallel to L, then by the above theorem, the equation of L' is of the form $\lambda_1 L_1 + \lambda_2 L_2 = 0$ for $(\lambda_1, \lambda_2) \neq (0, 0)$ and hence, the equation of L is of the form $\lambda_1 L_1 + \lambda_2 L_2 = \lambda_3$, for some constant λ_3.

3.6.4 Example : Find the equation of the straight line passing through the point of intersection of the lines $x + y + 1 = 0$ and $2x - y + 5 = 0$ and containing the point $(5, -2)$.

Solution : Clearly the line $2x - y + 5 = 0$ does not contain the point $(5, -2)$. So the equation of any straight line (other than the above line) passing through the point of intersection of the given lines is of the form $(x + y + 1) + \lambda(2x - y + 5) = 0$.

This line passes through $(5, -2)$ only if $4 + \lambda(17) = 0$ (or) if $\lambda = -\frac{4}{17}$.

Therefore, the equation of the required line is $17(x + y + 1) - 4(2x - y + 5) = 0$

(i.e.) $9x + 21y - 3 = 0$ (or) $3x + 7y - 1 = 0$.

3.7 Condition for Concurrent lines

Given three straight lines in the XY – plane, we first obtain a necessary and sufficient condition for concurrency of these lines. This is followed by a sufficient condition for concurrency of three lines.

3.7.1 Theorem : *Let* $L_1 \equiv a_1x + b_1 y + c_1 = 0$, $L_2 \equiv a_2x + b_2 y + c_2 = 0$ *and* $L_3 \equiv a_3 x + b_3 y + c_3 = 0$ *be three straight lines, no two of which are parallel. Then these lines are concurrent if and only if* $a_1(b_2c_3 - b_3c_2) + b_1(c_2a_3 - c_3a_2) + c_1(a_2b_3 - a_3b_2) = 0$.

Proof: By Theorem 3.5.1, the point of intersection of the lines $L_1 = 0$ and $L_2 = 0$ is

$$P\left(\frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}, \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}\right).$$

The given straight lines are therefore concurrent

$\Leftrightarrow$ the point P lies on the line $L_3 = 0$

$$\Leftrightarrow a_3\left(\frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}\right) + b_3\left(\frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}\right) + c_3 = 0$$

$$\Leftrightarrow a_3(b_1c_2 - b_2c_1) + b_3(c_1a_2 - c_2a_1) + c_3(a_1b_2 - a_2b_1)) = 0$$

$$\Leftrightarrow \boxed{\Sigma\, a_1(b_2c_3 - b_3c_2) = 0}\,.$$

3.7.2 Note : The above necessary and sufficient condition for concurrency of three straight lines can also be expressed in the determinant form as $\boxed{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0}$.

3.7.3 Theorem (A sufficient condition for concurrency of three straight lines)

If $L_1 \equiv a_1x + b_1y + c_1 = 0$, $L_2 \equiv a_2x + b_2y + c_2 = 0$ *and* $L_3 \equiv a_3x + b_3y + c_3 = 0$ *are three straight lines, no two of which are parallel, and if non-zero real numbers* λ_1, λ_2 *and* λ_3 *exist such that* $\lambda_1 L_1 + \lambda_2 L_2 + \lambda_3 L_3 \equiv 0$, *then the straight lines* $L_1 = 0$, $L_2 = 0$ *and* $L_3 = 0$ *are concurrent.*

Proof : If $P(x_0, y_0)$ is the point of intersection of the lines $L_1 = 0$, $L_2 = 0$, then

$$a_1x_0 + b_1y_0 + c_1 = 0 \text{ and } a_2x_0 + b_2y_0 + c_2 = 0.$$

Since $L_3 \equiv \left(\frac{-\lambda_1}{\lambda_3}\right)L_1 + \left(\frac{-\lambda_2}{\lambda_3}\right)L_2$, we have

$$a_3x_0 + b_3y_0 + c_3 = \left(\frac{-\lambda_1}{\lambda_3}\right)(a_1x_0 + b_1y_0 + c_1) + \left(\frac{-\lambda_2}{\lambda_3}\right)(a_2x_0 + b_2y_0 + c_2) = 0.$$

$\therefore$ $P(x_0, y_0)$ lies on the straight line $L_3 = 0$ and accordingly, the lines $L_1 = 0$, $L_2 = 0$ and $L_3 = 0$ are concurrent at P.

3.7.4 Solved Problems

1. Problem: *Find the value of k, if the lines $2x-3y+k=0$, $3x-4y-13=0$ and $8x-11y-33=0$ are concurrent.*

Solution : Let L_1, L_2, L_3 be the straight lines whose equations are respectively

$$2x-3y+k=0 \qquad \text{... (1)}$$

$$3x-4y-13=0 \qquad \text{... (2)}$$

$$\text{and} \quad 8x-11y-33=0 \qquad \text{... (3)}$$

Solving (2) and (3) for x and y, we obtain (by applying the rule of cross-multiplication)

$$\frac{x}{132-143}=\frac{y}{-104+99}=\frac{1}{-33+32} \qquad \begin{pmatrix} -4 & -13 & 3 & -4 \\ -11 & -33 & 8 & -11 \end{pmatrix}$$

and this gives $x = 11$ and $y = 5$

$\therefore$ Point of intersection of the lines (2) and (3) is (11, 5)

Since L_1, L_2, L_3 are concurrent, L_1 contains (11, 5) and therefore, $2(11)-3(5)+k=0$ (i.e.) $k=-7$.

2. Problem : *If the straight lines $ax+by+c=0$, $bx+cy+a=0$ and $cx+ay+b=0$ are concurrent, then prove that $a^3+b^3+c^3=3abc$.*

Solution : Let L_1, L_2 and L_3 be the straight lines whose equations are respectively

$$ax+by+c=0 \qquad \text{... (1)}$$

$$bx+cy+a=0 \qquad \text{... (2)}$$

$$\text{and} \quad cx+ay+b=0 \qquad \text{... (3)}$$

Solving equations (1) and (2), we obtain

$$\frac{x}{ab-c^2}=\frac{y}{bc-a^2}=\frac{1}{ca-b^2} \qquad \begin{pmatrix} b & c & a & b \\ c & a & b & c \end{pmatrix}$$

Therefore, the point of intersection of L_1 and L_2 is $\left(\frac{ab-c^2}{ca-b^2}, \frac{bc-a^2}{ca-b^2}\right)$.

If the lines L_1, L_2, L_3 are concurrent, L_3 contains the above point of intersection of L_1 and L_2.

Hence, $c\left(\frac{ab-c^2}{ca-b^2}\right)+a\left(\frac{bc-a^2}{ca-b^2}\right)+b=0$

i.e., $c(ab-c^2)+a(bc-a^2)+b(ca-b^2)=0$

i.e., $a^3+b^3+c^3=3abc$.

3. Problem : *A variable straight line drawn through the point of intersection of the straight lines $\frac{x}{a}+\frac{y}{b}=1$ and $\frac{x}{b}+\frac{y}{a}=1$ meets the coordinate axes at A and B. Show that the locus of the mid point of $\overline{AB}$ is $2(a+b)xy=ab(x+y)$.*

Solution : The straight lines $\frac{x}{a}+\frac{y}{b}=1$ and $\frac{x}{b}+\frac{y}{a}=1$ intersect at P whose coordinates are $\left(\frac{ab}{a+b},\frac{ab}{a+b}\right)$ (see example 3.5.2)

$\therefore$ $Q(x_0, y_0)$ is a point on the given locus.

$\Leftrightarrow$ The straight line with x-intercept $2x_0$ and y-intercept $2y_0$ passes through P

$\Leftrightarrow$ P lies on the straight line $\frac{x}{2x_0}+\frac{y}{2y_0}=1$

$$\Leftrightarrow \left(\frac{1}{2x_0}+\frac{1}{2y_0}\right)\left(\frac{ab}{a+b}\right)=1$$

$\Leftrightarrow$ $2(a+b)x_0y_0 = ab(x_0+y_0)$

$\Leftrightarrow$ $Q(x_0, y_0)$ lies on the locus $2(a+b)xy = ab(x+y)$.

Hence, the locus of the midpoint Q of $\overline{AB}$ is $2(a+b)\,xy = ab\,(x+y)$.

4. Problem : *If a, b, c are in arithmetic progression, then show that the equation $ax+by+c=0$ represents a family of concurrent lines and find the point of concurrency.*

Solution : If a, b, c are in arithmetic progression, then $2b = a+c$ (or) $a-2b+c=0$. Therefore, each member of the family of straight lines given by $ax+by+c=0$ passes through the fixed point $(1,-2)$. Hence, the set of lines $ax+by+c=0$ for parametric values of a, b and c is a family of concurrent lines and the point of concurrency is $(1,-2)$.

Exercise 3(c)

I. **1.** Find the ratios in which the following straight lines divide the line segments joining the given points. State whether the points lie on the same side or on either side of the straight line.

(i) $3x-4y=7$; $(2,-7)$ and $(-1,3)$

(ii) $3x+4y=6$; $(2,-1)$ and $(1,1)$

(iii) $2x+3y=5$; $(0,0)$ and $(-2,1)$

2. Find the point of intersection of the following lines.

(i) $4x+8y-1=0,\ 2x-y+1=0$

(ii) $7x+y+3=0,\ x+y=0$

3. Show that the straight lines $(a-b)x+(b-c)y = c-a$, $(b-c)x+(c-a)y = a-b$ and $(c-a)x+(a-b)y = b-c$ are concurrent.

4. Transform the following equations into the form $L_1+\lambda L_2=0$ and find the point of concurrency of the family of straight lines represented by the equation.

(i) $(2+5k)x-3(1+2k)y+(2-k)=0$

(ii) $(k+1)x+(k+2)y+5=0$

5. Find the value of p, if the straight lines $x + p = 0$, $y + 2 = 0$ and $3x + 2y + 5 = 0$ are concurrent.

6. Find the area of the triangle formed by the following straight lines and the coordinate axes.

(i) $x - 4y + 2 = 0$ (ii) $3x - 4y + 12 = 0$.

II. 1. A straight line meets the coordinate axes in A and B. Find the equation of the straight line, when

(i) $\overline{AB}$ is divided in the ratio 2 : 3 at $(-5, 2)$

(ii) $\overline{AB}$ is divided in the ratio 1 : 2 at $(-5, 4)$

(iii) (p, q) bisects $\overline{AB}$

2. Find the equation of the straight line passing through the points $(-1, 2)$ and $(5, -1)$ and also find the area of the triangle formed by it with the axes of coordinates.

3. A triangle of area 24 sq. units is formed by a straight line and the coordinate axes in the first quadrant. Find the equation of the straight line, if it passes through (3, 4).

4. A straight line with slope 1 passes through $Q(-3, 5)$ and meets the straight line $x + y - 6 = 0$ at P. Find the distance PQ.

5. Find the set of values of 'a' if the points (1, 2) and (3, 4) lie to the same side of the straight line $3x - 5y + a = 0$.

6. Show that the lines $2x + y - 3 = 0$, $3x + 2y - 2 = 0$ and $2x - 3y - 23 = 0$ are concurrent and find the point of concurrency.

7. Find the value of p, if the following lines are concurrent.

(i) $3x + 4y = 5$, $2x + 3y = 4$, $px + 4y = 6$

(ii) $4x - 3y - 7 = 0$, $2x + py + 2 = 0$, $6x + 5y - 1 = 0$

8. Determine whether or not the four straight lines with equations $x + 2y - 3 = 0$, $3x + 4y - 7 = 0$, $2x + 3y - 4 = 0$ and $4x + 5y - 6 = 0$ are concurrent.

9. If $3a + 2b + 4c = 0$, then show that the equation $ax + by + c = 0$ represents a family of concurrent straight lines and find the point of concurrency.

10. If non-zero numbers a, b, c are in harmonic progression, then show that the equation $\frac{x}{a} + \frac{y}{b} + \frac{1}{c} = 0$ represents a family of concurrent lines and find the point of concurrency.

III. 1. Find the point on the straight line $3x + y + 4 = 0$ which is equidistant from the points $(-5, 6)$ and $(3, 2)$.

2. A straight line through P(3, 4) makes an angle of 60° with the positive direction of the X-axis. Find the coordinates of the points on the line which are 5 units away from P.

3. A straight line through Q($\sqrt{3}$, 2) makes an angle $\frac{\pi}{6}$ with the positive direction of the X – axis. If the straight line intersects the line $\sqrt{3}x - 4y + 8 = 0$ at P, find the distance PQ.

4. Show that the origin is with in the triangle whose angular points are (2, 1), (3, – 2) and (– 4, – 1).

5. A straight line through Q(2, 3) makes an angle $\frac{3\pi}{4}$ with the negative direction of the X – axis. If the straight line intersects the line $x + y - 7 = 0$ at P, find the distance PQ.

6. Show that the straight lines $x + y = 0$, $3x + y - 4 = 0$ and $x + 3y - 4 = 0$ form an isosceles triangle.

7. Find the area of the triangle formed by the straight lines $2x - y - 5 = 0$, $x - 5y + 11 = 0$ and $x + y - 1 = 0$.

3.8 Angle between two lines

In this section, we first obtain a formula for the angle between two straight lines and then deduce the conditions for two lines to be parallel and perpendicular.

3.8.1 Theorem : The angle between the straight lines

$L_1 \equiv a_1x + b_1y + c_1 = 0$ and $L_2 \equiv a_2x + b_2y + c_2 = 0$ is

$$\text{Cos}^{-1}\left(\frac{|a_1a_2 + b_1b_2|}{\sqrt{(a_1^2 + b_1^2)\,(a_2^2 + b_2^2)}}\right)$$

Proof : Let $\overleftrightarrow{OA}$ and $\overleftrightarrow{OB}$ be the straight lines passing through the origin and parallel to the given lines $L_1 = 0$ and $L_2 = 0$ (see Fig. 3.18). Then the equations of $\overleftrightarrow{OA}$ and $\overleftrightarrow{OB}$ are $a_1x + b_1y = 0$ and $a_2x + b_2y = 0$ respectively.

If $\angle XOA = \theta_1$, and $\angle XOB = \theta_2$, then the measure of $|\theta_1 - \theta_2|$ that lies in the interval $\left[0, \frac{\pi}{2}\right]$ is the angle between the lines $L_1 = 0$ and $L_2 = 0$. Clearly, P($b_1, -a_1$) and Q($b_2, -a_2$) are points on the lines $\overleftrightarrow{OA}$ and $\overleftrightarrow{OB}$ respectively. Therefore,

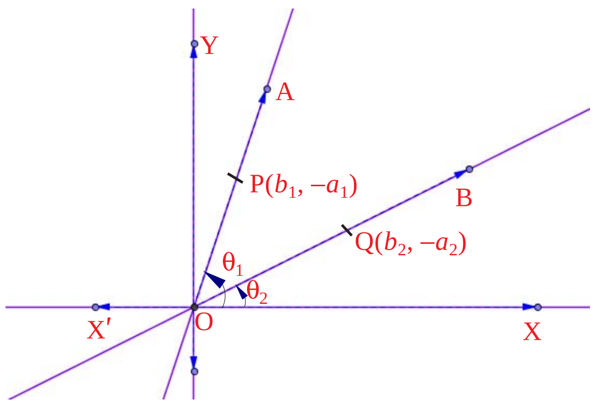


Fig. 3.18

$$\cos\theta_1 = \frac{b_1}{\sqrt{a_1^2 + b_1^2}};$$

$$\sin\theta_1 = \frac{-a_1}{\sqrt{a_1^2 + b_1^2}};$$

$$\cos\theta_2 = \frac{b_2}{\sqrt{a_2^2 + b_2^2}};$$

and $\sin\theta_2 = \dfrac{-a_2}{\sqrt{a_2^2+b_2^2}}$.

Hence, $\operatorname{Cos}(\theta_1-\theta_2) = \cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2$

$$= \frac{a_1a_2+b_1b_2}{\sqrt{\left(a_1^2+b_1^2\right)\left(a_2^2+b_2^2\right)}} = \cos(\theta_2-\theta_1).$$

Thus, if θ is the angle between $L_1=0$ and $L_2=0$, then $\theta \in \left[0, \frac{\pi}{2}\right]$ and so, $\cos\theta \geq 0$.

$$\therefore \cos\theta = \frac{|a_1a_2+b_1b_2|}{\sqrt{\left(a_1^2+b_1^2\right)\left(a_2^2+b_2^2\right)}}$$

(or) $$\boxed{\theta = \operatorname{Cos}^{-1}\frac{|a_1a_2+b_1b_2|}{\sqrt{\left(a_1^2+b_1^2\right)\left(a_2^2+b_2^2\right)}}}.$$

3.8.2 Note

1. A necessary and sufficient condition for the lines L_1 and L_2 with equations $a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$ to be perpendicular is that $a_1a_2+b_1b_2=0$ (Since $\theta = 90^0$).

Hence, the equation of a straight line perpendicular to the straight line $ax+by+c=0$ is of the form $bx-ay=k$.

2. By theorem 3.3.6, the straight lines $a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$ are parallel iff $a_1b_2=a_2b_1$. Therefore, the equation of any straight line parallel to the straight line $ax+by+c=0$ is of the form $ax+by=k$.

3. The straight line containing the points $A(x_1,y_1)$ and $B(x_2,y_2)$ is $(x-x_1)(y_1-y_2) = (y-y_1)(x_1-x_2)$. Similarly the straight line containing the points $C(x_3, y_3)$ and $D(x_4,y_4)$ is $(x-x_3)(y_3-y_4) = (y-y_3)(x_3-x_4)$.

Therefore, by the above note (1), the lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ are perpendicular if and only if

$$(x_1-x_2)(x_3-x_4) + (y_1-y_2)(y_3-y_4) = 0.$$

3.8.3 Corollary : If L_1 and L_2 are non-vertical straight lines with slopes m_1 and m_2 respectively, then the angle between them is $\operatorname{Tan}^{-1}\left|\dfrac{m_1-m_2}{1+m_1m_2}\right|$ if $m_1m_2 \neq -1$ and $\dfrac{\pi}{2}$ if $m_1m_2=-1$.

Proof : Let $a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$ be the equations of L_1 and L_2 respectively. Then $m_1 = \dfrac{-a_1}{b_1}$ and $m_2 = \dfrac{-a_2}{b_2}$.

Now $L_1 \perp L_2 \quad \Leftrightarrow \quad a_1a_2 + b_1b_2 = 0$

$$\Leftrightarrow \quad \frac{a_1a_2}{b_1b_2} + 1 = 0 \quad (\text{since} \quad b_1b_2 \neq 0)$$

$$\Leftrightarrow \quad m_1m_2 + 1 = 0$$

$$\Leftrightarrow \quad m_1m_2 = -1.$$

Therefore, angle between L_1 and L_2 is $\frac{\pi}{2}$ if $m_1m_2 = -1$. However if $m_1m_2 \neq -1$, then the angle between L_1 and L_2

$$= \text{Cos}^{-1}\left|\frac{a_1a_2 + b_1b_2}{\sqrt{\left(a_1^2 + b_1^2\right)\left(a_2^2 + b_2^2\right)}}\right|$$

$$= \text{Cos}^{-1}\frac{\left|\frac{a_1a_2}{b_1b_2} + 1\right|}{\sqrt{\left(\frac{a_1^2}{b_1^2} + 1\right)\left(\frac{a_2^2}{b_2^2} + 1\right)}}$$

$$= \text{Cos}^{-1}\left|\frac{m_1m_2 + 1}{\sqrt{\left(1 + m_1^2\right)\left(1 + m_2^2\right)}}\right|$$

$$= \text{Tan}^{-1}\left|\frac{m_1 - m_2}{1 + m_1m_2}\right|.$$

Thus, the angle between two non-perpendicular, non vertical lines with slopes m_1 and m_2 is $\text{Tan}^{-1}\left|\frac{m_1 - m_2}{1 + m_1m_2}\right|$.

3.8.4 Example : Find the angle between the lines $2x + y + 4 = 0$ and $y - 3x = 7$.

Solution : The angle between the given lines $= \text{Cos}^{-1}\frac{|-6+1|}{\sqrt{5 \times 10}}$

$$= \text{Cos}^{-1}\left(\frac{5}{5\sqrt{2}}\right) = \text{Cos}^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}.$$

3.8.5 Example : Find the angle between the lines $\sqrt{3}x + y + 1 = 0$ and $x + 1 = 0$.

Solution : The slope of the straight line $\sqrt{3}x + y + 1 = 0$ is $-\sqrt{3}$. Therefore, this line makes an angle 60^0 with the X – axis and 30^0 with the Y – axis.

But the equation $x + 1 = 0$ represents a vertical line.

Hence, the angle between the given lines = 30^0.

3.9 Length of the perpendicular from a point to a line

In this section, we obtain formulas for the perpendicular distance of a point from a given straight line.

3.9.1 Theorem : *The length of the perpendicular from the point* $P(x_0, y_0)$ *to the straight line*

$$ax + by + c = 0 \quad \text{is} \quad \left|\frac{ax_0 + by_0 + c}{\sqrt{a^2 + b^2}}\right|.$$

Proof : Let $\overleftrightarrow{AB}$ be the straight line $ax + by + c = 0$.

If the axes of coordinates are translated to the new origin $P(x_0, y_0)$, then the coordinates of a point (x, y) will be changed to (X, Y) where $x = X + x_0$ and $y = Y + y_0$ (see Fig. 3.19).

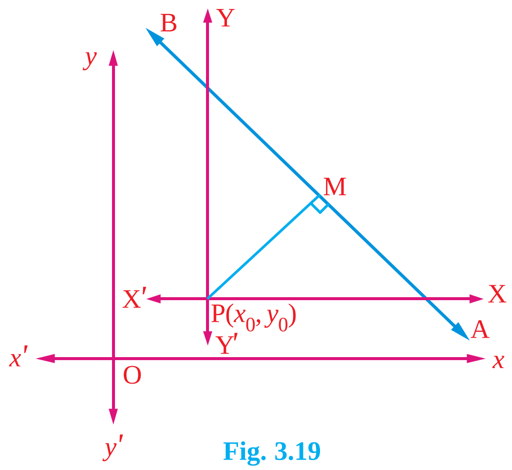


Fig. 3.19

Then the equation of $\overleftrightarrow{AB}$ w.r.t. P as the origin is

$$a(X + x_0) + b(Y + y_0) + c = 0$$

i.e., $$aX + bY + (ax_0 + by_0 + c) = 0.$$

$\therefore$ The perpendicular distance of $\overleftrightarrow{AB}$ from the origin P w.r.t. the new axes is

$$\boxed{PM = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}}$$ (see note 2 of 3.4.3).

3.9.2 Example : Find the perpendicular distance from the point $(-3, 4)$ to the straight line $5x - 12y = 2$.

Solution: The perpendicular distance of the point $(-3, 4)$ from the line $5x - 12y - 2 = 0$ is equal to

$$\frac{|5(-3) - 12(4) - 2|}{\sqrt{5^2 + 12^2}} = \frac{65}{13} = 5.$$

3.10 Distance between two parallel lines

In this section, we obtain formulas for the distance between two parallel lines.

3.10.1 Theorem : *The distance between the parallel straight lines* $ax + by + c_1 = 0$ *and*

$$ax + by + c_2 = 0 \quad \text{is} \quad \frac{|c_1 - c_2|}{\sqrt{a^2 + b^2}}.$$

Proof : Let $P(x_0, y_0)$ be a point on the straight line $L_1 : ax + by + c_1 = 0$. Let L_2 be the other line.

Then $ax_0 + by_0 = -c_1$ ... (1)

Now the distance between the parallel lines L_1 and L_2 is equal to PM where PM is the perpendicular distance of P from L_2 (see Fig. 3.20).

Therefore, $$\text{PM} = \frac{|ax_0 + by_0 + c_2|}{\sqrt{a^2+b^2}} \quad \text{(by Theorem 3.9.1)}$$

$$= \frac{|(ax_0 + by_0 + c_1) + (c_2 - c_1)|}{\sqrt{a^2+b^2}}$$

$$= \frac{|c_1 - c_2|}{\sqrt{a^2+b^2}} \quad \text{(from (1))}.$$

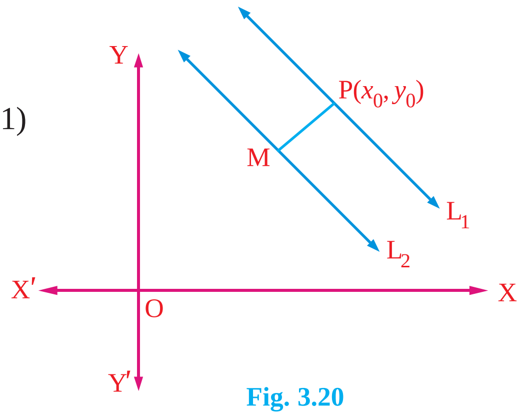


Fig. 3.20

3.10.2 Example : Find the distance between the parallel straight lines $3x + 4y - 3 = 0$ and $6x + 8y - 1 = 0$.

Solution : The equations of the given straight lines can be taken as $6x + 8y - 6 = 0$ and $6x + 8y - 1 = 0$.

Hence, by Theorem 3.10.1, the perpendicular distance between these parallel lines

$$= \frac{|-6+1|}{\sqrt{6^2+8^2}} = \frac{5}{10} = \frac{1}{2}.$$

3.10.3 Theorem : *If* $Q(h, k)$ *is the foot of the perpendicular from* $P(x_1, y_1)$ *on the straight line* $ax + by + c = 0$, *then* $(h - x_1) : a = (k - y_1) : b = -(ax_1 + by_1 + c) : (a^2 + b^2)$.

Proof: Equation of $\overleftrightarrow{PQ}$ which is normal to the given straight line $L : ax + by + c = 0$ (Fig.3.21) is

$$bx - ay = bx_1 - ay_1. \text{ Since } Q \in \overleftrightarrow{PQ}, \text{ we have}$$

$$bh - ak = bx_1 - ay_1$$

(i.e.) $$b(h - x_1) = a(k - y_1)$$

(or) $$(h - x_1) : a = (k - y_1) : b$$

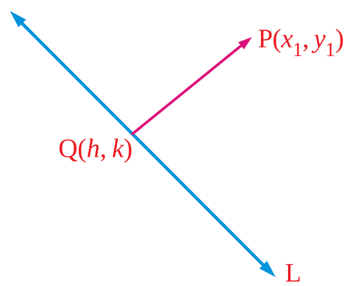


Fig. 3.21

But, this implies that $h = a\lambda + x_1$ and $k = b\lambda + y_1$ for some $\lambda \in \mathbf{R}$.

Since $Q(h, k)$ is a point on L, we have

$$a(a\lambda + x_1) + b(b\lambda + y_1) + c = 0$$

i.e., $$\lambda = -\frac{(ax_1 + by_1 + c)}{(a^2 + b^2)}.$$

Therefore, $$\boxed{(h - x_1) : a = (k - y_1) : b = -(ax_1 + by_1 + c) : (a^2 + b^2)}.$$

3.10.4 Example : Find the foot of the perpendicular from (–1, 3) on the straight line $5x - y - 18 = 0$.

Solution: (h, k) is the foot of the perpendicular from (–1, 3) on the line $5x - y - 18 = 0$

$$\Rightarrow \quad \frac{h-(-1)}{5} = \frac{k-3}{-1} = -\frac{(-5-3-18)}{5^2+1^2} = 1$$

$$\Rightarrow \quad h + 1 = 5 \quad \text{and} \quad k - 3 = -1$$

$$\Rightarrow \quad (h, k) = (4, 2).$$

3.10.5 Theorem : *If* Q (h, k) *is the image of the point* P (x_1, y_1) *w.r.t. the straight line* $ax + by + c = 0$, *then* $(h - x_1) : a = (k - y_1) : b = -2(ax_1 + by_1 + c) : (a^2 + b^2)$.

Proof : Q (h, k) is the image of the point P(x_1, y_1) w.r.t. the line L : $ax + by + c = 0$ (see Fig. 3.22)

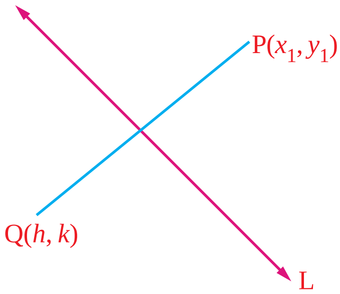


Fig. 3.22

$\Rightarrow \left(\dfrac{x_1+h}{2}, \dfrac{y_1+k}{2}\right)$ is the foot of the perpendicular from P on the line L

$$\Rightarrow \left(\frac{x_1+h}{2} - x_1\right) : a = \left(\frac{y_1+k}{2} - y_1\right) : b = -(ax_1 + by_1 + c) : (a^2 + b^2)$$

(from Theorem 3.10.3)

$$\Rightarrow (h - x_1) : 2a = (k - y_1) : 2b = -(ax_1 + by_1 + c) : (a^2 + b^2)$$

$$\Rightarrow \boxed{(h - x_1) : a = (k - y_1) : b = -2(ax_1 + by_1 + c) : (a^2 + b^2)}.$$

3.10.6 Example : Find the image of (1, – 2) w.r.t. the straight line $2x - 3y + 5 = 0$.

Solution : (h, k) is the image of (1, –2) w.r.t. the line $2x - 3y + 5 = 0$

$$\Rightarrow \quad \frac{h-1}{2} = \frac{k+2}{-3} = \frac{-2(2+6+5)}{4+9} = -2$$

$$\Rightarrow \quad h = -3, \quad k = 4.$$

$\therefore$ (– 3, 4) is the image of (1, –2) in the line $2x - 3y + 5 = 0$.

3.10.7 Solved Problems

1. Problem : *Find the value of* k, *if the angle between the straight lines* $4x - y + 7 = 0$ *and* $kx - 5y - 9 = 0$ *is* 45^o.

Solution : $$\text{Cos}^{-1}\left(\frac{|a_1a_2 + b_1b_2|}{\sqrt{(a_1^2 + b_1^2)(a_2^2 + b_2^2)}}\right) = \text{Cos}^{-1}\frac{|4k + 5|}{\sqrt{17(k^2 + 25)}} = \frac{\pi}{4}$$

$$\Leftrightarrow \quad \frac{|4k + 5|}{\sqrt{17(k^2 + 25)}} = \frac{1}{\sqrt{2}}$$

$$\Leftrightarrow \quad 2(4k+5)^2 = 17(k^2+25)$$
$$\Leftrightarrow \quad 15k^2 + 80k - 375 = 0$$
$$\Leftrightarrow \quad 3k^2 + 16k - 75 = 0$$
$$\Leftrightarrow \quad (k-3)(3k+25) = 0$$
$$\Leftrightarrow \quad k = 3 \text{ or } \frac{-25}{3}.$$

2. Problem : *Find the equations of the straight lines passing through* (x_0, y_0) *and* **(i)** *parallel* **(ii)** *perpendicular to the straight line* $ax + by + c = 0$.

Solution

(i) The equation of the straight line parallel to the line $ax + by + c = 0$ and passing through (x_0, y_0) is $ax + by = k$ where $k = ax_0 + by_0$ (i.e.) $a(x - x_0) + b(y - y_0) = 0$.

(ii) The equation of the straight line perpendicular to the line $ax + by + c = 0$ and containing the point (x_0, y_0) is $bx - ay = k$ where $k = bx_0 - ay_0$ (i.e.) $b(x - x_0) - a(y - y_0) = 0$.

3. Problem : *Find the equation of the straight line perpendicular to the line* $5x - 2y = 7$ *and passing through the point of intersection of the lines* $2x + 3y = 1$ *and* $3x + 4y = 6$.

Solution : Clearly neither of the straight lines $2x + 3y = 1$ *and* $3x + 4y = 6$ is perpendicular to the straight line $5x - 2y = 7$. Therefore, the equation of the required line is of the form $(2x + 3y - 1) + \lambda(3x + 4y - 6) = 0$ for some $\lambda (\neq 0) \in \mathbf{R}$. This line is perpendicular to the line $5x - 2y = 7$ if and only if $(2 + 3\lambda)5 + (3 + 4\lambda)(-2) = 0$

(i.e.) iff $\lambda = \dfrac{-4}{7}$.

So, the equation of the required line is $7(2x + 3y - 1) - 4(3x + 4y - 6) = 0$

(i.e.) $2x + 5y + 17 = 0$.

4. Problem : *If* $2x - 3y - 5 = 0$ *is the perpendicular bisector of the line segment joining* $(3, -4)$ *and* (α, β), *find* $\alpha + \beta$.

Solution : (α, β) is the reflection of $(3, -4)$ in the line $2x - 3y - 5 = 0$ and therefore,

$$\frac{\alpha - 3}{2} = \frac{\beta + 4}{-3} = \frac{-2(6 + 12 - 5)}{13} = -2$$

so $\alpha = -1$, $\beta = 2$ and $\alpha + \beta = 1$.

5. Problem: *If the four straight lines* $ax + by + p = 0$, $ax + by + q = 0$, $cx + dy + r = 0$ *and* $cx + dy + s = 0$ *form a parallelogram, show that the area of the parallelogram so formed is* $\left|\dfrac{(p-q)(r-s)}{bc - ad}\right|$.

Solution : Let L_1, L_2, L_3, L_4 be the straight lines represented by the equations

$$ax + by + p = 0,$$
$$ax + by + q = 0,$$
$$cx + dy + r = 0,$$
and $$cx + dy + s = 0$$ respectively.

Clearly $L_1 \parallel L_2$ and $L_3 \parallel L_4$. So L_1 and L_3 are not parallel. If θ is the angle between L_1 and L_3,

then, area of the parallelogram = $\dfrac{d_1 d_2}{\sin\theta}$ where,

$$d_1 = \text{distance between } L_1 \text{ and } L_2 = \frac{|p-q|}{\sqrt{a^2+b^2}}$$

$$d_2 = \text{distance between } L_3 \text{ and } L_4 = \frac{|r-s|}{\sqrt{c^2+d^2}}$$

and $\quad \cos\theta = \dfrac{|ac+bd|}{\sqrt{(a^2+b^2)(c^2+d^2)}}$.

So, $\quad \sin\theta = \sqrt{\dfrac{(a^2+b^2)(c^2+d^2)-(ac+bd)^2}{(a^2+b^2)(c^2+d^2)}} = \dfrac{|bc-ad|}{\sqrt{(a^2+b^2)(c^2+d^2)}}$.

$\therefore$ Area of the parallelogram = $\left|\dfrac{(p-q)(r-s)}{bc-ad}\right|$.

6. Problem : *The hypotenuse of a right angled isosceles triangle has its ends at the points (1, 3) and (–4, 1). Find the equations of the legs of the triangle.*

Solution : Let A = (1, 3) and B = (–4, 1) and ABC be a right isosceles triangle with $\overline{AB}$ as hypotenuse.

We require, therefore, the equations of $\overleftrightarrow{AC}$ and $\overleftrightarrow{BC}$

Slope of $\overleftrightarrow{AB}$ is $\dfrac{1-3}{-4-1} = \dfrac{2}{5}$.

Since the slope of $\overleftrightarrow{AB}$ is $\dfrac{2}{5}$, neither $\overleftrightarrow{AC}$ nor $\overleftrightarrow{BC}$ is vertical.

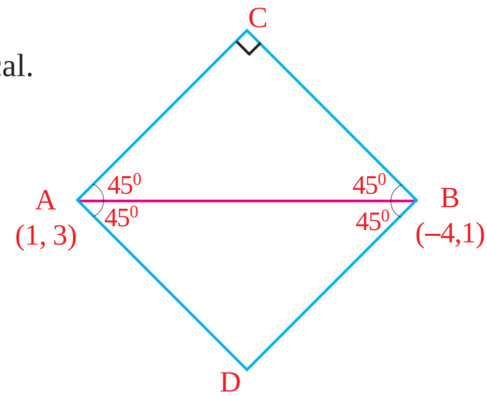


Fig. 3.23

If m be the slope of $\overleftrightarrow{AC}$, then $\tan 45^0 = \left|\dfrac{m-\frac{2}{5}}{1+\frac{2m}{5}}\right|$

$$\Rightarrow \frac{5m-2}{5+2m} = \pm 1$$

$$\Rightarrow m = \frac{7}{3} \text{ or } \frac{-3}{7}$$

Taking the slope of $\overleftrightarrow{AC}$ as $\dfrac{7}{3}$, the slope of $\overleftrightarrow{BC}$ would be $-\dfrac{3}{7}$. Therefore, the equations of $\overleftrightarrow{AC}$ and $\overleftrightarrow{BC}$ are respectively

$$y-3 = \frac{7}{3}(x-1) \text{ and } y-1 = -\frac{3}{7}(x+4),$$

which become $7x - 3y + 2 = 0$ and $3x + 7y + 5 = 0$.

If the lines drawn through A and B respectively parallel to $\overleftrightarrow{BC}$ and $\overleftrightarrow{AC}$ meet at D, then ΔABD is also right isosceles, having $\overline{AB}$ as its hypotenuse.

Therefore, the equations of $\overleftrightarrow{AD}$ and $\overleftrightarrow{BD}$ are respectively,

$$3(x-1) + 7(y-3) = 0 \text{ and } 7(x+4) - 3(y-1) = 0$$

$$\Rightarrow 3x + 7y - 24 = 0 \text{ and } 7x - 3y + 31 = 0.$$

Therefore, the two pairs of legs required are

$7x - 3y + 2 = 0$, $3x + 7y + 5 = 0$ and $3x + 7y - 24 = 0$, $7x - 3y + 31 = 0$.

Note : ADBC is a square.

7. Problem : *A line is such that its segment between the lines $5x - y + 4 = 0$ and $3x + 4y - 4 = 0$ is bisected at the point (1, 5). Obtain its equation.*

Solution : Let the required line meet $3x + 4y - 4 = 0$ at A and $5x - y + 4 = 0$ at B, so that AB is the segment between the given lines, with its mid-point at C = (1, 5).

The equation $5x - y + 4 = 0$ can be written as $y = 5x + 4$ so that any point on $\overleftrightarrow{BX}$ is $(t, 5t + 4)$ for all real t.

$\therefore$ B = $(t, 5t + 4)$ for some t. Since (1, 5) is the mid-point of $\overline{AB}$,

$$A = [2 - t, 10 - (5t + 4)]$$
$$= [2 - t, 6 - 5t]$$

Since A lies on $3x + 4y - 4 = 0$,

$$3(2 - t) + 4(6 - 5t) - 4 = 0$$

$$\Rightarrow -23t + 26 = 0$$

$$\Rightarrow t = \frac{26}{23}.$$

$$\therefore A = \left[2 - \frac{26}{23}, 6 - 5\left(\frac{26}{23}\right)\right]$$

$$= \left(\frac{20}{23}, \frac{8}{23}\right).$$

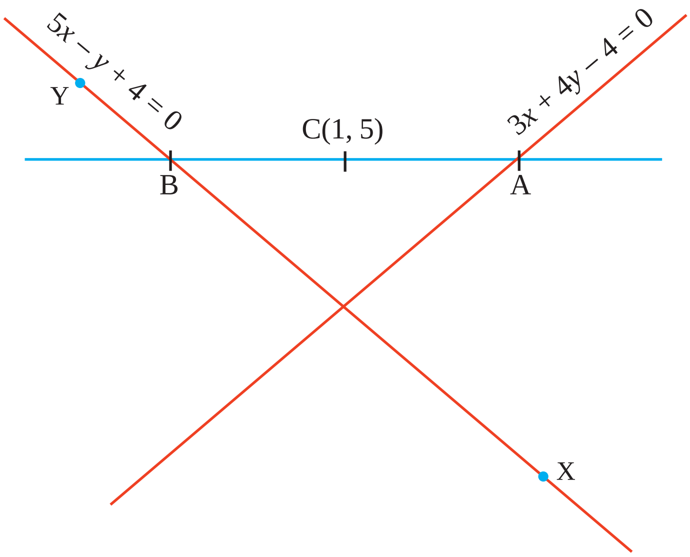


Fig. 3.24

Since slope of $\overleftrightarrow{AB}$ is $\dfrac{5 - \frac{8}{23}}{1 - \frac{20}{23}} = \dfrac{107}{3}$,

equation of $\overleftrightarrow{AB}$ is $y - 5 = \dfrac{107}{3}(x - 1)$.

$$\Rightarrow 3y - 15 = 107x - 107$$

$$\Rightarrow 107x - 3y - 92 = 0.$$

8. Problem : *An equilateral triangle has its incentre at the origin and one side as $x + y - 2 = 0$. Find the vertex opposite to $x + y - 2 = 0$.*

Solution: Let ABC be the equilateral triangle and $x + y - 2 = 0$ represent the side $\overleftrightarrow{BC}$.

Since O is the incentre of the triangle, $\overleftrightarrow{AD}$ is the bisector of $\underline{|BAC}$. Since the triangle is equilateral, $\overleftrightarrow{AD}$ is the perpendicular bisector of $\overline{BC}$.

Since O is also the centroid, AO : OD = 2 : 1. [The centroid, circumcentre incentre and orthocentre coincide]

Let D = (h, k). Since D is the foot of the perpendicular from O onto $\overleftrightarrow{BC}$, D is given by

$$\frac{h-0}{1} = \frac{k-0}{1} = \frac{-(-2)}{2}$$
$$= 1.$$

$\therefore \quad h = 1, \quad k = 1.$

$\therefore \quad D = (1, 1).$

Let A = (x_1, y_1)

$$\therefore (0,0) = \left(\frac{2+x_1}{3}, \frac{2+y_1}{3}\right).$$

$\therefore \quad x_1 = -2, \; y_1 = -2.$

$\therefore \quad A = (-2, -2)$, the required vertex.

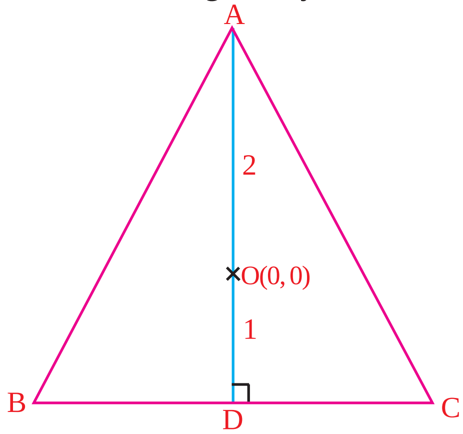


Fig. 3.25

Exercise 3(d)

I. Find the angle between the following straight lines.

1. $y = 4 - 2x, \; y = 3x + 7$
2. $3x + 5y = 7, \; 2x - y + 4 = 0$
3. $y = -\sqrt{3}x + 5, \; y = \frac{1}{\sqrt{3}}x - \frac{2}{\sqrt{3}}$
4. $ax + by = a + b, \quad a(x - y) + b(x + y) = 2b$

Find the length of the perpendicular drawn from the point given against the following straight lines.

5. $5x - 2y + 4 = 0$ .... $(-2, -3)$
6. $3x - 4y + 10 = 0$ .... $(3, 4)$
7. $x - 3y - 4 = 0$ .... $(0, 0)$

Find the distance between the following parallel lines.

8. $3x - 4y = 12, \; 3x - 4y = 7$
9. $5x - 3y - 4 = 0, \; 10x - 6y - 9 = 0$

10. Find the equation of the straight line parallel to the line $2x + 3y + 7 = 0$ and passing through the point $(5, 4)$.

11. Find the equation of the straight line perpendicular to the line $5x - 3y + 1 = 0$ and passing through the point $(4, -3)$.

12. Find the value of k, if the straight lines $6x - 10y + 3 = 0$ and $kx - 5y + 8 = 0$ are parallel.

13. Find the value of p, if the straight lines $3x + 7y - 1 = 0$ and $7x - py + 3 = 0$ are mutually perpendicular.

14. Find the value of k, if the straight lines $y - 3kx + 4 = 0$ and $(2k-1)x - (8k-1)y - 6 = 0$ are perpendicular.

15. $(-4, 5)$ is a vertex of a square and one of its diagonals is $7x - y + 8 = 0$. Find the equation of the other diagonal.

II. 1. Find the equations of the straight lines passing through $(1, 3)$ and (i) parallel to (ii) perpendicular to the line passing through the points $(3, -5)$ and $(-6, 1)$.

2. The line $\frac{x}{a} - \frac{y}{b} = 1$ meets the X – axis at P. Find the equation of the line perpendicular to this line at P.

3. Find the equation of the line perpendicular to the line $3x + 4y + 6 = 0$ and making an intercept -4 on the X – axis.

4. A(-1, 1), B(5, 3) are opposite vertices of a square in the XY-plane. Find the equation of the other diagonal (not passing through A, B) of the square.

5. Find the foot of the perpendicular drawn from $(4, 1)$ upon the straight line $3x - 4y + 12 = 0$.

6. Find the foot of the perpendicular drawn from $(3, 0)$ upon the straight line $5x + 12y - 41 = 0$.

7. $x - 3y - 5 = 0$ is the perpendicular bisector of the line segment joining the points A, B. If A = $(-1, -3)$, find the coordinates of B.

8. Find the image of the point $(1, 2)$ in the straight line $3x + 4y - 1 = 0$.

9. Show that the distance of the point $(6, -2)$ from the line $4x + 3y = 12$ is half the distance of the point $(3, 4)$ from the line $4x - 3y = 12$.

10. Find the locus of the foot of the perpendicular from the origin to a variable straight line which always passes through a fixed point (a, b).

III. 1. Show that the lines $x - 7y - 22 = 0$, $3x + 4y + 9 = 0$ and $7x + y - 54 = 0$ form a right angled isosceles triangle.

2. Find the equations of the straight lines passing through the point $(-3, 2)$ and making an angle of 45° with the straight line $3x - y + 4 = 0$.

3. Find the angles of the triangle whose sides are
$x + y - 4 = 0$, $2x + y - 6 = 0$ and $5x + 3y - 15 = 0$.

4. Prove that the feet of the perpendiculars from the origin on the lines
$x + y = 4$, $x + 5y = 26$ and $15x - 27y = 424$ are collinear.

5. Find the equations of the straight lines passing through the point of intersection of the lines $3x + 2y + 4 = 0$, $2x + 5y = 1$ and whose distance from $(2, -1)$ is 2.

6. Each side of a square is of length 4 units. The center of the square is (3, 7) and one of its diagonals is parallel to $y = x$. Find the coordinates of its vertices.

7. If $ab > 0$, find the area of the rhombus enclosed by the four straight lines $ax \pm by \pm c = 0$.

8. Find the area of the parallelogram whose sides are $3x + 4y + 5 = 0$, $3x + 4y - 2 = 0$, $2x + 3y + 1 = 0$ and $2x + 3y - 7 = 0$.

9. A person standing at the junction (crossing) of two straight paths represented by the equations $2x - 3y + 4 = 0$ and $3x + 4y - 5 = 0$ wants to reach the path whose equation is $6x - 7y + 8 = 0$ in the least time. Find the equation of the path that he should follow.

10. A ray of light passing through the point (1, 2) reflects on the X-axis at a point A and the reflected ray passes through the point (5, 3). Find the coordinates of A.

3.11 Concurrent lines – Properties related to a triangle

There are various triads of concurrent straight lines associated with a triangle, viz. medians, altitudes, angular bisectors, perpendicular bisectors of the sides etc. Geometric proofs for the concurrency of each of these triads are already learnt in lower classes. In what follows, we give the analytical proofs for the concurrency of such triads of lines. Recall the vectorial proofs of these also.

Concurrency of the medians of a triangle

3.11.1 Theorem : *The medians of a triangle are concurrent.*

Proof : Let $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be the vertices of the triangle ABC and

$L_1 \equiv a_1x + b_1y + c_1 = 0$,

$L_2 \equiv a_2x + b_2y + c_2 = 0$ and

$L_3 \equiv a_3x + b_3y + c_3 = 0$ be respectively the sides $\overleftrightarrow{BC}$, $\overleftrightarrow{CA}$ and $\overleftrightarrow{AB}$ (see Fig. 3.26).

Then $\lambda_r = a_rx_r + b_ry_r + c_r \neq 0$ for $r = 1, 2, 3$

and $a_rx_s + b_ry_s + c_r = 0$ for $r, s = 1, 2, 3$ and $r \neq s$

Suppose D, E, F are the mid points of the sides $\overline{BC}$, $\overline{CA}$ and $\overline{AB}$ respectively. Then the equation of the median $\overleftrightarrow{AD}$ is $L_3 + \lambda L_2 = 0$ where λ $(\neq 0)$ is given by ... (1)

$$\left[a_3\left(\frac{x_2+x_3}{2}\right)+b_3\left(\frac{y_2+y_3}{2}\right)+c_3\right]+$$

$$\lambda\left[a_2\left(\frac{x_2+x_3}{2}\right)+b_2\left(\frac{y_2+y_3}{2}\right)+c_2\right]=0$$

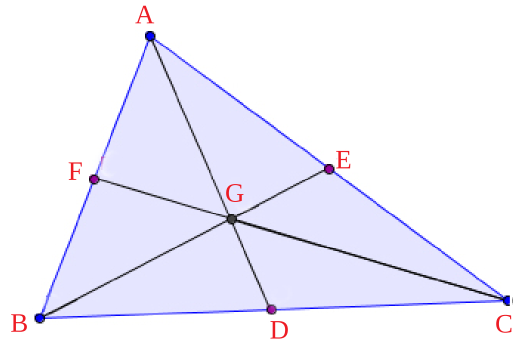


Fig. 3.26

(or) $\lambda_3 + \lambda\lambda_2 = 0$ ... (2)

Eliminating λ from (1) and (2), we obtain the equation of $\overleftrightarrow{AD}$ as

$$\lambda_2 L_3 - \lambda_3 L_2 = 0$$

Similarly the equation of $\overleftrightarrow{BE}$ is $\lambda_3 L_1 - \lambda_1 L_3 = 0$

and the equation of $\overleftrightarrow{CF}$ is $\lambda_1 L_2 - \lambda_2 L_1 = 0$.

Since $\lambda_1(\lambda_2 L_3 - \lambda_3 L_2) + \lambda_2(\lambda_3 L_1 - \lambda_1 L_3) + \lambda_3(\lambda_1 L_2 - \lambda_2 L_1) = 0$

and $\lambda_1\lambda_2\lambda_3 \neq 0$, by Theorem 3.7.3, it follows that the medians $\overleftrightarrow{AD}$, $\overleftrightarrow{BE}$ and $\overleftrightarrow{CF}$ are concurrent.

Note that G is the **Centroid** of triangle ABC.

Concurrency of the altitudes of a triangle

3.11.2 Theorem : *The altitudes of a triangle are concurrent.*

Proof : Let $\overline{AD}$, $\overline{BE}$ and $\overline{CF}$ be the altitudes of traingle ABC drawn from the vertices A, B and C respectively. Let the altitudes $\overleftrightarrow{AD}$ and $\overleftrightarrow{BE}$ intersect at 'O' (see Fig. 3.27). Choose 'O' as the origin of coordinates and a pair of perpendicular straight lines through O (not shown in Fig. 3.27) as the axes of coordinates. W.r.t. these axes, let $A = (x_1, y_1)$, $B = (x_2, y_2)$ and $C = (x_3, y_3)$.

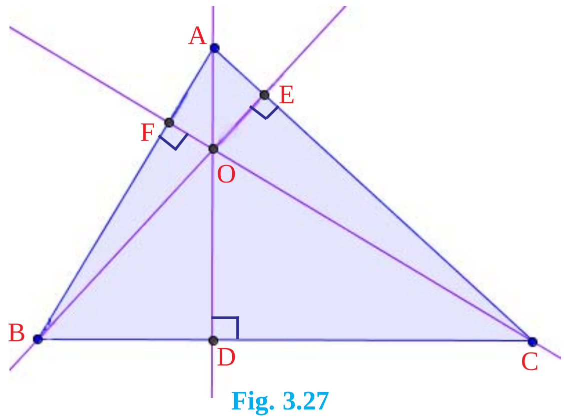


Fig. 3.27

Then $\overleftrightarrow{AD} \perp \overleftrightarrow{BC}$ $\Rightarrow (x_1 - 0)(x_2 - x_3) + (y_1 - 0)(y_2 - y_3) = 0$

(by Note 3 of 3.8.2)

$\Rightarrow x_1(x_2 - x_3) + y_1(y_2 - y_3) = 0$ ... (1)

Similarly $\overleftrightarrow{BE} \perp \overleftrightarrow{CA} \Rightarrow x_2(x_3 - x_1) + y_2(y_3 - y_1) = 0$ ... (2)

From (1) and (2), we obtain $x_3(x_2 - x_1) + y_3(y_2 - y_1) = 0$

(i.e.) $(x_3 - 0)(x_2 - x_1) + (y_3 - 0)(y_2 - y_1) = 0$.

This shows that $\overleftrightarrow{CO}$ and $\overleftrightarrow{AB}$ are perpendicular. But $\overleftrightarrow{CF}$ is the altitude drawn to $\overleftrightarrow{AB}$ from the vertex C.

Hence, $\overleftrightarrow{CF}$ passes through O. Accordingly, the altitudes $\overleftrightarrow{AD}$, $\overleftrightarrow{BE}$ and $\overleftrightarrow{CF}$ are concurrent at 'O'. Note that 'O' is the **orthocenter** of triangle ABC.

Concurrency of the internal bisectors of the angles of a triangle

3.11.3 Theorem : *The internal bisectors of the angles of a triangle are concurrent.*

Proof : Let $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be the vertices of the triangle ABC and $L_r \equiv a_r x + b_r y + c_r = 0$ $(r = 1, 2, 3)$ be respectively the sides $\overleftrightarrow{BC}$, $\overleftrightarrow{CA}$ and $\overleftrightarrow{AB}$.

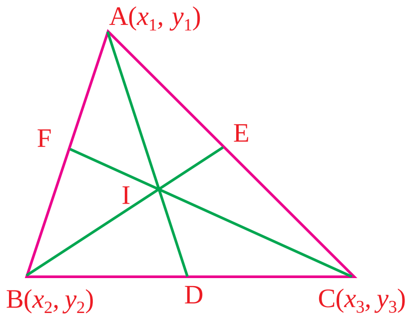


Fig. 3.28

Then $\lambda_r = a_r x_r + b_r y_r + c_r \neq 0$ $(r = 1, 2, 3)$

and $a_r x_s + b_r y_s + c_r = 0$ $(r, s = 1, 2, 3$ and $r \neq s)$.

Suppose $\overleftrightarrow{AD}$, $\overleftrightarrow{BE}$ and $\overleftrightarrow{CF}$ are the internal bisectors of the angles A, B, C respectively. With the usual notation in Δ ABC, we write $a = BC$, $b = CA$ and $c = AB$ (see Fig. 3.28). Then D divides $\overline{BC}$ internally in the ratio AB : AC = $c : b$

$$\text{and so,}\quad D = \left(\frac{bx_2 + cx_3}{b+c}, \frac{by_2 + cy_3}{b+c}\right).$$

Equation of the bisector $\overleftrightarrow{AD}$ is $L_3 + \lambda L_2 = 0$ ... (1)

where $\lambda\ (\neq 0)$ is given by

$$\left[a_3\left(\frac{bx_2 + cx_3}{b+c}\right) + b_3\left(\frac{by_2 + cy_3}{b+c}\right) + c_3\right] + \lambda\left[a_2\left(\frac{bx_2 + cx_3}{b+c}\right) + b_2\left(\frac{by_2 + cy_3}{b+c}\right) + c_2\right] = 0$$

(i.e.) $c\lambda_3 + \lambda(b\lambda_2) = 0$ ... (2)

Eliminating λ from (1) and (2), we obtain the equation of the internal bisector $\overleftrightarrow{AD}$ of angle A as $u_1 \equiv (b\lambda_2)L_3 - (c\lambda_3)L_2 = 0$.

Similarly the other bisectors $\overleftrightarrow{BE}$ and $\overleftrightarrow{CF}$ are given by

$$u_2 \equiv (c\lambda_3)L_1 - (a\lambda_1)L_3 = 0 \text{ and}$$

$$u_3 \equiv (a\lambda_1)L_2 - (b\lambda_2)L_1 = 0 \text{ respectively.}$$

Writing $k_1 = a\lambda_1$, $k_2 = b\lambda_2$ and $k_3 = c\lambda_3$, we observe that $k_1 k_2 k_3 \neq 0$ and $k_1 u_1 + k_2 u_2 + k_3 u_3 = 0$.

Hence, by theorem 3.7.3, it follows that the bisectors $\overleftrightarrow{AD}$, $\overleftrightarrow{BE}$, $\overleftrightarrow{CF}$ are concurrent. The point of concurrency is called the **Incenter** of Δ ABC, usually denoted by I. (see 4.2.3, Example 2 for alternate proof).

Concurrency of the perpendicular bisectors of the sides of a triangle

3.11.4 Theorem : *The perpendicular bisectors of the sides of a triangle are concurrent.*

Proof : Let D, E, F be the mid-points of the sides $\overline{BC}$, $\overline{CA}$ and $\overline{AB}$ respectively of triangle ABC; and let the perpendicular bisectors of the sides $\overline{BC}$, $\overline{CA}$ meet at 'O' (see Fig. 3.29). Choose 'O' as the origin of coordinates and a pair of perpendicular lines through O as the axes of coordinates (not shown in the Fig. 3.29).

Let (x_1, y_1), (x_2, y_2) and (x_3, y_3) be the coordinates of the vertices A, B, C respectively w.r.t. these axes of coordinates.

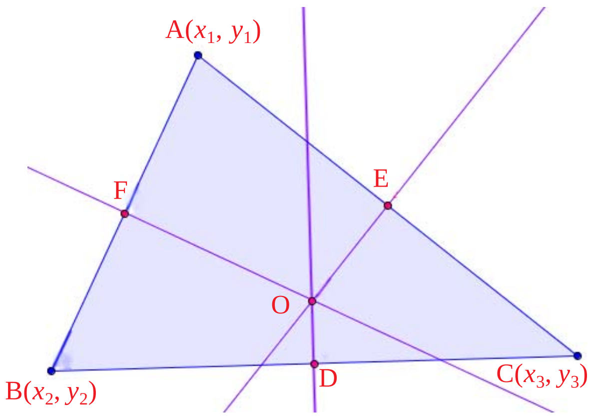


Fig. 3.29

Then $D = \left(\dfrac{x_2+x_3}{2}, \dfrac{y_2+y_3}{2}\right)$;

$E = \left(\dfrac{x_3+x_1}{2}, \dfrac{y_3+y_1}{2}\right)$

and $F = \left(\dfrac{x_1+x_2}{2}, \dfrac{y_1+y_2}{2}\right)$.

Also $\overleftrightarrow{OD} \perp \overleftrightarrow{BC} \Rightarrow \left(\dfrac{x_2+x_3}{2}\right)(x_2-x_3) + \left(\dfrac{y_2+y_3}{2}\right)(y_2-y_3) = 0$

$$\Rightarrow \left(x_2^2 - x_3^2\right) + \left(y_2^2 - y_3^2\right) = 0 \qquad \text{... (1)}$$

and $\overleftrightarrow{OE} \perp \overleftrightarrow{CA} \Rightarrow \left(\dfrac{x_3+x_1}{2}\right)(x_3-x_1) + \left(\dfrac{y_3+y_1}{2}\right)(y_3-y_1) = 0$

$$\Rightarrow \left(x_3^2 - x_1^2\right) + \left(y_3^2 - y_1^2\right) = 0 \qquad \text{... (2)}$$

From (1) and (2), we obtain $\left(x_2^2 - x_1^2\right) + \left(y_2^2 - y_1^2\right) = 0$

$$\text{(i.e.)} \quad \left(\frac{x_2+x_1}{2}\right)(x_2-x_1) + \left(\frac{y_2+y_1}{2}\right)(y_2-y_1) = 0.$$

But, this implies that $\overleftrightarrow{OF} \perp \overleftrightarrow{AB}$.

Since F is the mid point of $\overline{AB}$, $\overleftrightarrow{OF}$ is, therefore, the perpendicular bisector of $\overline{AB}$. Thus, the perpendicular bisectors of the sides are concurrent. The point of concurrence 'O' is the **circumcenter** of Δ ABC.

3.11.5 Solved Problems

1. Problem : *Find the orthocenter of the triangle whose vertices are* (–5, –7), (13, 2) *and* (–5, 6).

Solution : Let A(–5, –7), B(13, 2) and C(–5, 6) be the vertices of the given triangle. Let $\overleftrightarrow{AD}$ be the perpendicular drawn from A to $\overleftrightarrow{BC}$ and $\overleftrightarrow{BE}$ be the perpendicular drawn from B to $\overleftrightarrow{AC}$. (see Fig. 3.30)

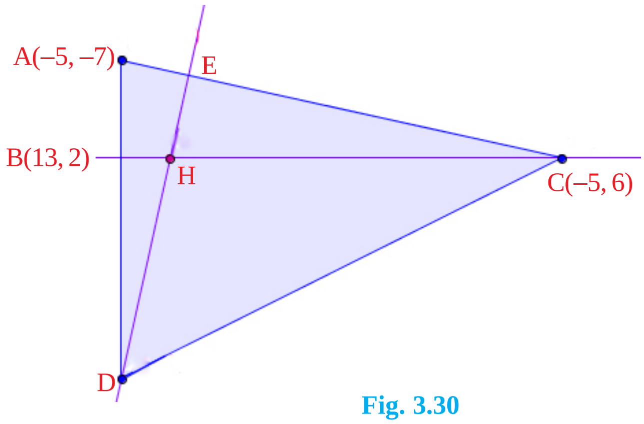


Fig. 3.30

Then, slope of $\overleftrightarrow{BC} = \dfrac{6-2}{-5-13} = \dfrac{-2}{9}$.

Since $\overleftrightarrow{AD} \perp \overleftrightarrow{BC}$, slope of $\overleftrightarrow{AD} = \dfrac{9}{2}$ and

so, the equation of $\overleftrightarrow{AD}$ is

$9x - 2y = -45 + 14 = -31$ ... (1)

Equation of $\overleftrightarrow{AC}$ is $x = -5$ and therefore, $\overleftrightarrow{AC}$ is a vertical line.

Hence, $\overleftrightarrow{BE}$ is horizontal and its equation is $y = 2$. ... (2)

The point of intersection of (1) and (2) is (–3, 2) which is the orthocenter of Δ ABC.

2. Problem : *If the equations of the sides of a triangle are* $7x + y - 10 = 0$, $x - 2y + 5 = 0$ and $x + y + 2 = 0$, *find the orthocenter of the triangle.*

Solution : Let the given triangle be ABC with the sides $\overleftrightarrow{AB}$, $\overleftrightarrow{BC}$ and $\overleftrightarrow{AC}$ represented by

$x - 2y + 5 = 0$ ... (1)

$7x + y - 10 = 0$ ... (2)

and $x + y + 2 = 0$ ... (3)

(see Fig. 3.31)

Let $\overleftrightarrow{AD}$ and $\overleftrightarrow{BE}$ be the altitudes drawn from A and B respectively to the sides $\overleftrightarrow{BC}$ and $\overleftrightarrow{CA}$.

Solving the equations (1) and (3), we obtain A = (–3, 1).

Since $\overleftrightarrow{AD} \perp \overleftrightarrow{BC}$, the equation of $\overleftrightarrow{AD}$ is

$$x - 7y = -3 - 7 = -10 \qquad \text{... (4)}$$

Solving the equations (1) and (2),

we obtain B = (1, 3).

Since $\overleftrightarrow{BE} \perp \overleftrightarrow{AC}$, the equation of $\overleftrightarrow{BE}$ is

$$x - y = 1 - 3 = -2 \qquad \text{... (5)}$$

Point of intersection of the lines (4) and (5) is $H\left(\frac{-2}{3}, \frac{4}{3}\right)$ which is the orthocenter of Δ ABC.

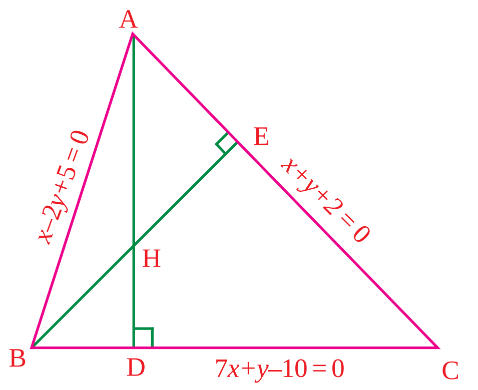


Fig. 3.31

3. Problem : *Find the circumcenter of the triangle whose vertices are* (1, 3), (–3, 5) *and* (5, –1)

Solution : Let the vertices of the triangle be A(1, 3), B(–3, 5) and C(5, –1) (see Fig. 3.32).

The mid-points of the sides $\overline{BC}$, $\overline{CA}$ are respectively D(1, 2) and E(3, 1).

Let S be the point of intersection of the perpendicular bisectors of the sides $\overline{BC}$ and $\overline{CA}$.

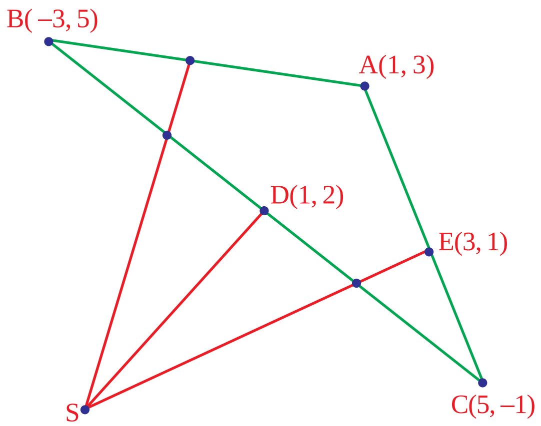


Fig. 3.32

$$\text{Slope of } \overleftrightarrow{BC} = \frac{5+1}{-3-5} = \frac{-3}{4}.$$

$\therefore$ Slope of $\overleftrightarrow{SD}$ is $\frac{4}{3}$ and so, the equation of $\overleftrightarrow{SD}$ is

$$4x - 3y = 4 - 6 = -2 \qquad \text{... (1)}$$

$$\text{Slope of } \overleftrightarrow{AC} = \frac{3+1}{1-5} = -1.$$

$\therefore$ Slope of $\overleftrightarrow{SE}$ is 1 and so, the equation of $\overleftrightarrow{SE}$ is $x - y = 3 - 1 = 2$. ... (2)

Solving the equations (1) and (2), we obtain S = (–8, –10) which is the circumcenter of Δ ABC.

4. Problem : *Find the circumcenter of the triangle whose sides are* $3x - y - 5 = 0$, $x + 2y - 4 = 0$ *and* $5x + 3y + 1 = 0$.

Solution : Let the given equations represent the sides $\overleftrightarrow{BC}$, $\overleftrightarrow{CA}$ and $\overleftrightarrow{AB}$ respectively of Δ ABC (see Fig. 3.33).

Solving the above equations by taking two at a time, we obtain the vertices A(– 2, 3), B(1, – 2) and C(2, 1) of the given triangle.

The mid-points of the sides $\overline{BC}$ and $\overline{CA}$ are respectively $D\left(\frac{3}{2}, \frac{-1}{2}\right)$ and E (0, 2).

Equation of $\overleftrightarrow{SD}$, the perpendicular bisector of $\overline{BC}$ is $x + 3y = 0$ and that of $\overleftrightarrow{SE}$, the perpendicular bisector of $\overline{CA}$ is $2x - y + 2 = 0$.

Solving these two equations, we obtain the point of intersection of the lines $\overleftrightarrow{SD}$ and $\overleftrightarrow{SE}$ which is therefore, $S = \left(\frac{-6}{7}, \frac{2}{7}\right)$, the circumcenter of Δ ABC.

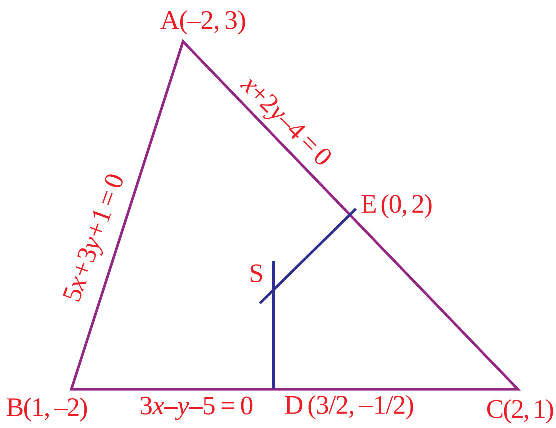


Fig. 3.33

5. Problem : *Find the incenter of the triangle formed by the straight lines $y = \sqrt{3}x$, $y = -\sqrt{3}x$ and $y = 3$.*

Solution : The straight lines $y = \sqrt{3}\,x$ and $y = -\sqrt{3}\,x$ make angles 60° and 120° respectively, with OX in the anti-clockwise sense (see Fig. 3.34). Since $y = 3$ is a horizontal line, the triangle formed by the three given lines is equilateral. So its incenter is same as the centroid which will be at a distance of 2 units from the origin (the vertex of the triangle) on the positive Y – axis (which is a median).

$\therefore$ Incentre of the triangle is I = (0, 2).

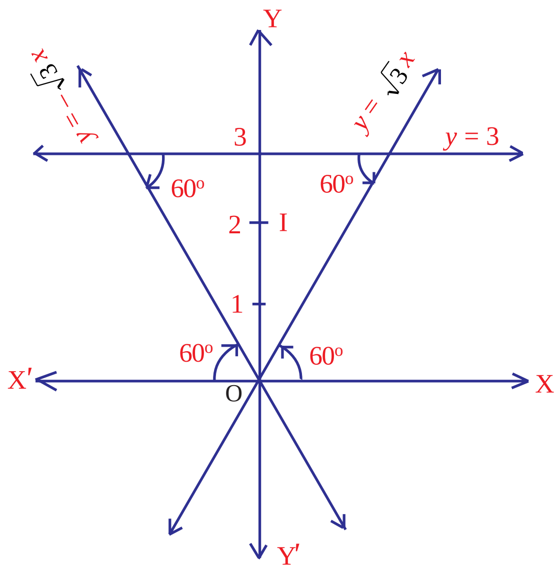


Fig. 3.34

Exercise 3(e)

I. **1.** Find the incentre of the triangle whose vertices are $(1, \sqrt{3})$, $(2, 0)$ and (0, 0).

2. Find the orthocentre of the triangle whose sides are given by $x + y + 10 = 0$, $x - y - 2 = 0$ and $2x + y - 7 = 0$.

3. Find the orthocenter of the triangle whose sides are given by $4x - 7y + 10 = 0$, $x + y = 5$ and $7x + 4y = 15$.

4. Find the circumcenter of the triangle whose sides are $x = 1$, $y = 1$ and $x + y = 1$.

5. Find the incentre of the triangle formed by the lines $x = 1, y = 1$ and $x + y = 1$.
6. Find the circumcentre of the triangle whose vertices are (1, 0), (–1, 2) and (3, 2).
7. Find the values of k, if the angle between the straight lines $kx + y + 9 = 0$ and $3x - y + 4 = 0$ is $\frac{\pi}{4}$.
8. Find the equation of the straight line passing through the origin and also through the point of intersection of the lines $2x - y + 5 = 0$ and $x + y + 1 = 0$.
9. Find the equation of the straight line parallel to the line $3x + 4y = 7$ and passing through the point of intersection of the lines $x - 2y - 3 = 0$ and $x + 3y - 6 = 0$.
10. Find the equation of the straight line perpendicular to the line $2x + 3y = 0$ and passing through the point of intersection of the lines $x + 3y - 1 = 0$ and $x - 2y + 4 = 0$.
11. Find the equation of the straight line making non-zero equal intercepts on the coordinate axes and passing through the point of intersection of the lines $2x - 5y + 1 = 0$ and $x - 3y - 4 = 0$.
12. Find the length of the perpendicular drawn from the point of intersection of the lines $3x + 2y + 4 = 0$ and $2x + 5y - 1 = 0$ to the straight line $7x + 24y - 15 = 0$.
13. Find the value of 'a' if the distances of the points (2, 3) and (–4, a) from the straight line $3x + 4y - 8 = 0$ are equal.
14. Find the circumcenter of the triangle formed by the straight lines $x + y = 0$, $2x + y + 5 = 0$ and $x - y = 2$.
15. If θ is the angle between the lines $\frac{x}{a} + \frac{y}{b} = 1$ and $\frac{x}{b} + \frac{y}{a} = 1$, find the value of $\sin\theta$ when $a > b$.

II. 1. Find the equations of the straight lines passing through the point (– 10, 4) and making an angle θ with the line $x - 2y = 10$ such that $\tan\theta = 2$.

2. Find the equations of the straight lines passing through the point (1, 2) and making an angle of 60^o with the line $\sqrt{3}x + y + 2 = 0$.
3. The base of an equilateral triangle is $x + y - 2 = 0$ and the opposite vertex is (2, –1). Find the equations of the remaining sides.
4. Find the orthocenter of the triangle with the following vertices
   (i) (–2, –1), (6, –1) and (2, 5)
   (ii) (5, –2), (–1, 2) and (1, 4)
5. Find the circumcenter of the triangle whose vertices are given below
   (i) (–2, 3), (2, –1) and (4, 0) (ii) (1, 3), (0, –2) and (–3, 1)
6. Let $\overline{PS}$ be the median of the triangle with vertices P(2, 2), Q(6, –1) and R(7, 3). Find the equation of the straight line passing through (1, –1) and parallel to the median $\overleftrightarrow{PS}$.

III. **1.** Find the orthocentre of the triangle formed by the lines $x + 2y = 0$, $4x + 3y - 5 = 0$ and $3x + y = 0$.

2. Find the circumcenter of the triangle whose sides are given by $x + y + 2 = 0$, $5x - y - 2 = 0$ and $x - 2y + 5 = 0$.

3. Find the equations of the straight lines passing through (1, 1) and which are at a distance of 3 units from (–2, 3).

4. If p and q are the lengths of the perpendiculars from the origin to the straight lines $x \sec\alpha + y \operatorname{cosec}\alpha = a$ and $x\cos\alpha - y\sin\alpha = a\cos 2\alpha$, prove that $4p^2 + q^2 = a^2$.

5. Two adjacent sides of a parallelogram are given by $4x + 5y = 0$ and $7x + 2y = 0$ and one diagonal is $11x + 7y = 9$. Find the equations of the remaining sides and the other diagonal.

6. Find the incenter of the triangle formed by the following straight lines

(i) $x + 1 = 0$, $3x - 4y = 5$ and $5x + 12y = 27$

(ii) $x + y - 7 = 0$, $x - y + 1 = 0$ and $x - 3y + 5 = 0$

7. A triangle is formed by the lines $ax + by + c = 0$, $lx + my + n = 0$ and $px + qy + r = 0$. Given that the triangle is not right-angled, show that the straight line $\dfrac{ax+by+c}{ap+bq} = \dfrac{lx+my+n}{lp+mq}$ passes through the orthocenter of the triangle.

8. The Cartesian equations of the sides $\overleftrightarrow{BC}$, $\overleftrightarrow{CA}$ and $\overleftrightarrow{AB}$ of a triangle are respectively $u_r \equiv a_r x + b_r y + c_r = 0$, $r = 1, 2, 3$. Show that the equation of the straight line passing through A and bisecting the side $\overline{BC}$ is $\dfrac{u_3}{a_3b_1 - a_1b_3} = \dfrac{u_2}{a_1b_2 - a_2b_1}$.

Key Concepts

- Slope of a non-vertical line containing the points (x_1, y_1) and (x_2, y_2) is $\dfrac{y_1 - y_2}{x_1 - x_2}$.
- Slope of a horizontal line is zero and slope of a vertical line is undefined.
- Two non-vertical lines are parallel if and only if their slopes are equal.
- Two non-vertical lines are perpendicular if and only if the product of their slopes is equal to -1.
- Equation of the horizontal line passing through (x_0, y_0) is $y = y_0$ and equation of the vertical line passing through (x_0, y_0) is $x = x_0$.

- Equation of a straight line in different forms:

  (a) **Slope - intercept form** : $y = mx + c$.

  (b) **Point - slope form** : $y - y_0 = m(x - x_0)$.

  (c) **Two - point form** : $(x - x_1)(y_1 - y_2) = (y - y_1)(x_1 - x_2)$.

  (d) **Intercept form** : $\frac{x}{a} + \frac{y}{b} = 1$.

  (e) **Normal form** : $x \cos\alpha + y \sin\alpha = p$.

  (f) **Symmetric form** : $(x - x_0) : \cos\theta = (y - y_0) : \sin\theta$.

  (g) **General form** : $ax + by + c = 0 \ (a^2 + b^2 \neq 0)$.

  (h) **Parametric form** : $x = x_0 + r\cos\theta, \quad y = y_0 \ r\sin\theta \ (r \in \mathbf{R})$.

- Lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are parallel if and only if $a_1b_2 = a_2b_1$.

- The two linear equations $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ represent the same straight line if and only if $a_1 : b_1 : c_1 = a_2 : b_2 : c_2$.

- The straight line $ax + by + c = 0$ divides the line segment joining (x_1, y_1) and (x_2, y_2) in the ratio $- \ (ax_1 + by_1 + c) : (ax_2 + by_2 + c)$.

- The points (x_1, y_1) and (x_2, y_2) lie to the same side or on either side of the straight line $ax + by + c = 0$ according as $ax_1 + by_1 + c$ and $ax_2 + by_2 + c$ have the same sign or of opposite sign.

- Point of intersection of the intersecting lines $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ is $\left(\frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}, \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}\right)$.

- If $u_1 \equiv a_1x + b_1y + c_1 = 0$ and $u_2 \equiv a_2x + b_2y + c_2 = 0$ represent two lines intersecting at P, then (i) any straight line passing through P is of the form $\lambda_1u_1 + \lambda_2u_2 = 0 \ (\lambda_1^2 + \lambda_2^2 \neq 0)$ (ii) the equation $\lambda_1u_1 + \lambda_2u_2 = 0$, for all real λ_1, λ_2 with $\lambda_1^2 + \lambda_2^2 \neq 0$, represents a straight line through P.

- If the equations $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$ and $a_3x + b_3y + c_3 = 0$ represent three lines, no two of which are parallel, then a necessary and sufficient condition for these lines to be concurrent is that $\Sigma a_1(b_2c_3 - b_3c_2) = 0$ (or) $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$.

- A sufficient condition for the above lines to be concurrent is that there exist non-zero reals $\lambda_1, \lambda_2, \lambda_3$ such that $\lambda_1(a_1x+b_1y+c_1)+\lambda_2(a_2x+b_2y+c_2)+\lambda_3(a_3x+b_3y+c_3)=0$.
- The angle between the lines $a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$ is $\text{Cos}^{-1}\dfrac{|a_1a_2+b_1b_2|}{\sqrt{(a_1^2+b_1^2)(a_2^2+b_2^2)}}$.
- The lines $a_1x+b_1y+c_1=0$ and $a_2x+b_2y+c_2=0$ are perpendicular if and only if $a_1a_2+b_1b_2=0$.
- Equation of a line parallel to the line $ax + by + c = 0$ is of the form $ax + by = k$ and that of a line perpendicular to the above line is $bx - ay = k$.
- The angle between two non-vertical, non-perpendicular lines with slopes m_1 and m_2 is $\text{Tan}^{-1}\left|\dfrac{m_1-m_2}{1+m_1m_2}\right|$.
- The length of the perpendicular from (x_0, y_0) to the straight line $ax + by + c = 0$ is $\dfrac{|ax_0+by_0+c|}{\sqrt{a^2+b^2}}$.
- The distance between the parallel lines $ax+by+c_1=0$ and $ax+by+c_2=0$ is $\dfrac{|c_1-c_2|}{\sqrt{a^2+b^2}}$.

Historical Note

Desargues. G (1593 – 1662) and *Pascal. B* (1623 – 1662) created projective geometry- a new branch of geometry. *Descartes. R* (1596–1650) and *Fermat. P* (1601 – 1665) invented analytical geometry -a method of geometry. Analytical geometry, also known as coordinate geometry, is a new and powerful method of solving geometrical problems. As applied to the plane, it establishes a one-to-one correspondence between the points in the plane and orderd pairs of real numbers - which in turn brings in a connection between plane curves and equations in two variables. "The task of proving a theorem in geometry is clearly shifted to that of proving a corresponding theorem in Algebra and Analysis".

"Perhaps analytic geometry can be regarded as the royal road to geometry that *Euclid* thought did not exist". There are several accounts about the initial flash that led Descartes to the invention of cartesian geometry. According to one account, the flash occured to him in a dream. Another episode is that the idea occured to him when watching a fly crawling about the ceiling near a corner of his room. This reminds us of the story of *Newton* and falling apple which motivated *Newton* to propound his theory of gravitation.

Answers

Exercise 3(a)

I. **1.** $-1, 1$ **2.** $y + 3 = 0$

3. $x - 1 = 0$ **4.** $\frac{\pi}{6}$

5. $x + 1 = 0$ **6.** $\frac{h}{a} + \frac{k}{b} = 1$

7. (i) $y - 3 = 0$, (ii) $y + 4 = 0$

8. (i) $x - 2 = 0$, (ii) $x + 5 = 0$

II. **1.** (i) $\frac{-3}{13}$ (ii) $\frac{-5}{2}$ (iii) $\frac{-2}{3}$ (iv) -1 **2.** $x = 1$

3. $y = 9$ **4.** (i) $\frac{-1}{5}$, (ii) 5

5. (i) $y = x$, (ii) $\sqrt{3}x - y + (2 - \sqrt{3}) = 0$, (iii) $x + y - 1 = 0$,

(iv) $x + \sqrt{3}y + (2 + \sqrt{3}) = 0$

6. $x = y$, $x = -y$

7. (i) $\sqrt{3}x - y + 3 = 0$, (ii) $x + \sqrt{3}y - 2\sqrt{3} = 0$,

(iii) $x - y - 2 = 0$, (iv) $2x - 3y + 9 = 0$

8. $x + y - 1 = 0$ **9.** $x - y + 6 = 0$

III. **1.** $3x + 2y - 1 = 0$ **2.** $x + 2y + 2 = 0$

3. (i) $2x - 5y + 15 = 0$ (ii) $3x - y = 0$ (iii) $x + y = a + b + c$

4. (i) $5x + 14y - 106 = 0$ (ii) $y = 4$ (iii) $12x + 5y + 3 = 0$

(iv) $28x - 10y - 19 = 0$.

Exercise 3(b)

I. **1.** 25 **2.** $\frac{x}{p} + \frac{y}{q} = 4$

3. $\frac{m}{b}$ **4.** $\frac{2\pi}{3}$ **5.** $\frac{|ab|}{\sqrt{a^2 + b^2}}$

II. **1.** (i) $x + \sqrt{3}y = 10$ (ii) $\sqrt{3}x - y + 12 = 0$ (iii) $x - y - \sqrt{2} = 0$

(iv) $y = 4$ (v) $x = 0$ (vi) $x + y + 4 = 0$

2. (i) $\dfrac{x-2}{\cos\frac{\pi}{3}} = \dfrac{y-3}{\sin\frac{\pi}{3}}$ (ii) $\dfrac{x+2}{\cos\frac{5\pi}{6}} = \dfrac{y}{\sin\frac{5\pi}{6}}$ (iii) $\dfrac{x-1}{\cos\frac{3\pi}{4}} = \dfrac{y-1}{\sin\frac{3\pi}{4}}$

3. (i) $y = \left(-\dfrac{3}{4}\right)x + \dfrac{5}{4}, \dfrac{x}{(5/3)} + \dfrac{y}{(5/4)} = 1, \quad x\cos\alpha + y\sin\alpha = 1\left(\alpha = \mathrm{Tan}^{-1}\dfrac{4}{3}\right)$

(ii) $y = \left(\dfrac{4}{3}\right)x + 4, \ \dfrac{x}{-3} + \dfrac{y}{4} = 1, \ x\cos\alpha + y\sin\alpha = \dfrac{12}{5}\left(\alpha = \pi - \mathrm{Tan}^{-1}\dfrac{3}{4}\right)$

(iii) $y = -\sqrt{3}x + 4, \ \dfrac{x}{(4/\sqrt{3})} + \dfrac{y}{4} = 1, \ \ x\cos\left(\dfrac{\pi}{6}\right) + y\sin\left(\dfrac{\pi}{6}\right) = 2$

(iv) $y = -x-2, \ \dfrac{x}{-2} + \dfrac{y}{-2} = 1, \ x\cos\dfrac{5\pi}{4} + y\sin\dfrac{5\pi}{4} = \sqrt{2}$

(v) $y = -x+2, \ \dfrac{x}{2} + \dfrac{y}{2} = 1, \ x\cos\dfrac{\pi}{4} + y\sin\dfrac{\pi}{4} = \sqrt{2}$

(vi) $y = -\sqrt{3}x - 10, \ \dfrac{x}{(-10/\sqrt{3})} + \dfrac{y}{-10} = 1, \ x\cos\dfrac{7\pi}{6} + y\sin\dfrac{7\pi}{6} = 5$

4. $\dfrac{\pi}{4}$ **7.** $\left(\dfrac{b}{\sqrt{a^2+b^2}}\right)x + \left(\dfrac{a}{\sqrt{a^2+b^2}}\right)y = \dfrac{ab}{\sqrt{a^2+b^2}}$

III. **1.** $(-2+2\sqrt{3}, 3)$ and $(-2-2\sqrt{3}, -1)$

2. $(7,5)$ and $(-1,-1)$ **3.** $\sqrt{3}x - y + 8 = 0$ **4.** $\dfrac{\pi}{4}$ **5.** 18

Exercise 3(c)

I. **1.** (i) 27 : 22 ; opposite sides (ii) 4 : 1, opposite sides

(iii) –5 : 6; same side **2.** (i) $\left(-\dfrac{7}{20}, \dfrac{3}{10}\right)$ (ii) $\left(-\dfrac{1}{2}, \dfrac{1}{2}\right)$

4. (i) (5, 4) (ii) (5, –5)

5. $\dfrac{1}{3}$ **6.** (i) $\dfrac{1}{2}$ (ii) 6

II. **1.** (i) $3x - 5y + 25 = 0$ (ii) $8x - 5y + 60 = 0$ (iii) $\dfrac{x}{p} + \dfrac{y}{q} = 2$

2. $x + 2y - 3 = 0, \ \dfrac{9}{4}$ **3.** $4x + 3y - 24 = 0$

4. $2\sqrt{2}$

5. $(-\infty, 7) \cup (11, +\infty)$

6. (4, –5)

7. (i) 2 (ii) 4

8. not concurrent

9. $\left(\frac{3}{4}, \frac{1}{2}\right)$

10. (1, –2)

III. 1. (–2, 2)

2. $\left(\frac{11}{2}, \frac{8+5\sqrt{3}}{2}\right), \left(\frac{1}{2}, \frac{8-5\sqrt{3}}{2}\right)$

3. 6

5. $\sqrt{2}$

7. 9

Exercise 3(d)

I. 1. $\frac{\pi}{4}$

2. $\text{Cos}^{-1}\left(\frac{1}{\sqrt{170}}\right)$

3. $\frac{\pi}{2}$

4. $\frac{\pi}{4}$

5. 0

6. $\frac{3}{5}$

7. $\frac{4}{\sqrt{10}}$

8. 1

9. $\frac{1}{2\sqrt{34}}$

10. $2x + 3y - 22 = 0$

11. $3x + 5y + 3 = 0$

12. 3

13. 3

14. -1 or $\frac{1}{6}$

15. $x + 7y - 31 = 0$

II. 1. (i) $2x + 3y - 11 = 0$ (ii) $3x - 2y + 3 = 0$

2. $\frac{x}{b} + \frac{y}{a} = \frac{a}{b}$

3. $4x - 3y + 16 = 0$

4. $3x + y - 8 = 0$

5. $\left(\frac{8}{5}, \frac{21}{5}\right)$

6. $\left(\frac{49}{13}, \frac{24}{13}\right)$

7. $\left(-\frac{8}{5}, -\frac{6}{5}\right)$

8. $\left(-\frac{7}{5}, -\frac{6}{5}\right)$

10. $x^2 + y^2 - ax - by = 0$

III. 2. $x - 2y + 7 = 0,\ 2x + y + 4 = 0$

3. $\text{Cos}^{-1}\left(\frac{4}{\sqrt{17}}\right), \text{Cos}^{-1}\left(\frac{13}{\sqrt{170}}\right)$ and $\pi - \text{Cos}^{-1}\left(\frac{3}{\sqrt{10}}\right)$

5. $y = 1,\ 4x + 3y + 5 = 0$

6. (1, 5), (1, 9), (5, 9) and (5, 5)

7. $\frac{2c^2}{ab}$

8. 56

9. $119x + 102y - 125 = 0$

10. $A = \left(\frac{13}{5}, 0\right)$

Exercise 3(e)

I.

1. $\left(1, \frac{1}{\sqrt{3}}\right)$ **2.** $(-4, -6)$ **3.** $(1, 2)$

4. $\left(\frac{1}{2}, \frac{1}{2}\right)$ **5.** $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ **6.** $(1, 2)$

7. $2, -\frac{1}{2}$ **8.** $x + 2y = 0$ **9.** $3x + 4y - 15 = 0$

10. $3x - 2y + 8 = 0$ **11.** $x + y + 32 = 0$ **12.** $\frac{1}{5}$

13. $\frac{15}{2}$ or $\frac{5}{2}$ **14.** $(-3, 1)$ **15.** $\frac{a^2 - b^2}{a^2 + b^2}$

II.

1. $3x + 4y + 14 = 0,\ x + 10 = 0$ **2.** $y = 2,\ y - 2 = \sqrt{3}\,(x - 1)$

3. $y + 1 = (2 \pm \sqrt{3})\,(x - 2)$ **4.** (i) $\left(2, \frac{5}{3}\right)$ (ii) $\left(\frac{1}{5}, \frac{14}{5}\right)$

5. (i) $\left(\frac{3}{2}, \frac{5}{2}\right)$ (ii) $\left(-\frac{1}{3}, \frac{2}{3}\right)$ **6.** $2x + 9y + 7 = 0$

III.

1. $(-4, -3)$ **2.** $\left(-\frac{1}{3}, \frac{2}{3}\right)$

3. $5x - 12y + 7 = 0,\ x = 1$

5. $7x + 2y = 9,\ 4x + 5y = 9$ and $x - y = 0$

6. (i) $\left(\frac{1}{3}, \frac{2}{3}\right)$ (iv) $(3, 1 + \sqrt{5})$

Chapter 4

Pair of Straight Lines

"One can measure the importance of a scientific work by the number of earlier publications rendered superfluous by it"

- David Hilbert

Introduction

Given the equations of two straight lines, the methods of finding their point of intersection and the angle between them were discussed in chapter 3. In this chapter, we shall find the conditions under which a second degree equation in x and y represents a pair of straight lines.

4.1 Equations of a pair of lines passing through the origin, Angle between a pair of lines

In this section, we find the nature of the combined equation of a pair of straight lines passing through the origin.

4.1.1 Combined equation of a pair of straight lines

Let L_1 and L_2 denote two straight lines and let their equations be $a_1x + b_1y + c_1 = 0$ and

David Hilbert
(1862 - 1943)

David Hilbert was a German mathematician, recognized as one of the most influential and universal mathematicians of the 19th and early 20th centuries. He led the school of formalists in the realm of mathematical philosophy.

$a_2x + b_2y + c_2 = 0$, i.e., which are linear in x and y (*i.e.*, a_1, b_1 are not both zero and a_2, b_2 are not both zero).

Consider the equation $(a_1x + b_1y + c_1)(a_2x + b_2y + c_2) = 0$. ... (1)

Now P (α, β) is a point on the locus represented by (1)

$$\Leftrightarrow (a_1\alpha + b_1\beta + c_1)(a_2\alpha + b_2\beta + c_2) = 0$$

$$\Leftrightarrow a_1\alpha + b_1\beta + c_1 = 0 \text{ or } a_2\alpha + b_2\beta + c_2 = 0$$

$\Leftrightarrow$ P lies on L_1 or P lies on L_2.

We, therefore, conclude that the locus or graph of the equation (1) is the pair of straight lines L_1 and L_2. We say that (1) is the combined equation or simply the equation of L_1 and L_2.

4.1.2 Example : The equation $6x^2 + 11xy - 10y^2 = 0$ represents the pair of straight lines $3x - 2y = 0$ and $2x + 5y = 0$, since $(3x-2y)(2x+5y) \equiv 6x^2 + 11xy - 10y^2$. ...(i)

Similarly, since $(3x+2y-1)(2x-3y+1)$

$$\equiv 6x^2 - 5xy - 6y^2 + x + 5y - 1, \qquad \text{...(ii)}$$

the equation $6x^2 - 5xy - 6y^2 + x + 5y - 1 = 0$ represents the pair of straight lines $3x + 2y - 1 = 0$ and $2x - 3y + 1 = 0$.

4.1.3 Definition

If a, b, h are real numbers, not all zero, then $H \equiv ax^2 + 2hxy + by^2 = 0$ is called a homogeneous equation of second degree in x and y; and $S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ is called a general equation of second degree in x and y.

The equation (1) in 4.1.1 and the combined equations (i) and (ii) of example 4.1.2 are second degree equations in x and y.

We shall now investigate the conditions under which the above two equations represent a pair of straight lines.

4.1.4 Theorem : *If a, b, h are not all zero and the locus of the equation $S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ contains a straight line L, then S can be written as the product of two linear factors in x and y (with real coefficients).*

Proof : Let $P(x_0, y_0)$ be a point on the straight line L. By translating the origin to the point and then rotating the axes of coordinates through a suitable angle θ about the new origin, the equation of the straight line L can be transformed into the X-axis in the new coordinate system. Let a point (x, y) have coordinates (X, Y) w.r.t. the new system of coordinate axes.

Then $x = x_0 + X\cos\theta - Y\sin\theta$

and $y = y_0 + X\sin\theta + Y\cos\theta$.

From these equations, we obtain

$$X = (x - x_0)\cos\theta + (y - y_0)\sin\theta \qquad \text{...(i)}$$

$$\text{and} \quad Y = (y - y_0)\cos\theta - (x - x_0)\sin\theta \qquad \text{...(ii)}$$

Writing the given equation $S = 0$ in the new coordinates X and Y, we find that the equation changes to the form $S \equiv AX^2 + 2HXY + BY^2 + 2GX + 2FY + C = 0$ which is a second degree equation in X and Y. (We note that A, B, H are not all zero, since $a + b = A + B$, $ab - h^2 = AB - H^2$ and a, b, h are not all zero).

Since the locus of $S = 0$ contains the straight line L whose equation is $Y = 0$ in the new coordinate system, every point on the line $Y = 0$ satisfies the equation $S = 0$ and hence,

$$AX^2 + 2GX + C = 0 \text{ for all real numbers } X.$$

Since this equation is satisfied by more than two values of X, we must have

$A = G = C = 0$. Hence

$$\begin{aligned} S &\equiv BY^2 + 2HXY + 2FY \\ &= Y(BY + 2HX + 2F) \end{aligned}$$

that is, S can be expressed as a product of two linear factors in X and Y. But X and Y are linear in x and y; and so using (i) and (ii), S can be factorised as a product of two real linear factors in x and y.

4.1.5 Note

(i) If the locus of a second degree equation in x and y contains a straight line, then the equation represents a pair of straight lines.

(ii) If the locus of a second degree equation $S = 0$ in the two variables x and y is a pair of straight lines, then we can write

$$S \equiv (l_1x + m_1y + n_1)(l_2x + m_2y + n_2),$$

where $l_1x + m_1y + n_1$ and $l_2x + m_2y + n_2$ are linear in x and y.

We now find the condition under which a homogeneous equation of second degree in x and y represents a pair of straight lines.

4.1.6 Theorem : *If a, b and h are not all zero, then the equation $H \equiv ax^2 + 2hxy + by^2 = 0$ represents a pair of straight lines if and only if $h^2 \geq ab$.*

Proof : Suppose that $H = 0$ represents a pair of lines. Then by 4.1.5. Note (ii), we can write

$$H \equiv (l_1x + m_1y + n_1)(l_2x + m_2y + n_2)$$

Here $l_1x + m_1y + n_1$ and $l_2x + m_2y + n_2$ are linear in x and y.

Since (0, 0) is a point on the locus of $ax^2 + 2hxy + by^2 = 0$, it follows that (0, 0) is a point on the line $l_1x + m_1y + n_1 = 0$ or on the line $l_2x + m_2y + n_2 = 0$.

Hence $n_1 = 0$ or $n_2 = 0$, say $n_1 = 0$. Then

$ax^2 + 2hxy + by^2 \equiv (l_1x + m_1y)(l_2x + m_2y + n_2)$ so that $l_1n_2 = 0 = m_1n_2$. Since l_1 and m_1 are not both zero, we get $n_2 = 0$. Hence

$ax^2 + 2hxy + by^2 \equiv (l_1x + m_1y)(l_2x + m_2y)$. Therefore $l_1l_2 = a$, $m_1m_2 = b$ and $l_1m_2 + l_2m_1 = 2h$.

Hence,

$$h^2 - ab = \left(\frac{l_1m_2 + l_2m_1}{2}\right)^2 - l_1l_2m_1m_2 = \left(\frac{l_1m_2 - l_2m_1}{2}\right)^2 \geq 0 \text{ so that } h^2 \geq ab.$$

Conversely, suppose that $h^2 \geq ab$.

Case (i) : Let $a \neq 0$. Then

$ax^2 + 2hxy + by^2$

$$= \frac{1}{a}(a^2x^2 + 2ahxy + aby^2)$$

$$= \frac{1}{a}\left[(ax)^2 + 2(ax)(hy) + h^2y^2 + aby^2 - h^2y^2\right]$$

$$= \frac{1}{a}\left[(ax+hy)^2 - (h^2 - ab)y^2\right]$$

$$= \frac{1}{a}\left[(ax + hy + \sqrt{h^2 - ab}\,y)(ax + hy - \sqrt{h^2 - ab})y\right], \text{ since } h^2 - ab \geq 0.$$

$$= \frac{1}{a}\left[ax + (h + \sqrt{h^2 - ab})y\right]\left[ax + (h - \sqrt{h^2 - ab})y\right].$$

Therefore, the equation $H = 0$ represents the pair of straight lines

$ax + (h + \sqrt{h^2 - ab})y = 0$ and $ax + (h - \sqrt{h^2 - ab})y = 0.$

Observe that each of these lines passes through the origin.

Case (ii) : Let $a = 0$. Then

$H \equiv 2hxy + by^2 \equiv y(2hx + by)$ and so, in this case, the equation $H = 0$ represents the straight lines $y = 0$ and $2hx + by = 0$ (since h and b are not both zero), each of which passes through the origin.

4.1.7 Note

(i) If $h^2 = ab$, we observe that the lines represented by $H = 0$ are **coincident**.

(ii) If $h^2 \geq ab$, then we can write $H \equiv (l_1x + m_1y)(l_2x + m_2y)$ so that $l_1l_2 = a$, $m_1m_2 = b$ and $l_1m_2 + l_2m_1 = 2h$. Also $l_1x + m_1y = 0$ and $l_2x + m_2y = 0$ are the straight lines represented by $H = 0$.

(iii) If $H = 0$ represents a pair of straight lines and $b \neq 0$, then these lines are non-vertical (prove). If m_1 and m_2 are the slopes of these lines, then

$$ax^2 + 2hxy + by^2 \equiv b(y - m_1x)(y - m_2x),$$

so that $m_1 + m_2 = \dfrac{-2h}{b}$ and $m_1m_2 = \dfrac{a}{b}$.

When the equations of two straight lines are given separately, finding the angle between them was discussed in 3.8. The following theorem aims at finding the angle between a pair of straight lines when their combined equation is given.

4.1.8 Theorem : *Let the equation $ax^2 + 2hxy + by^2 = 0$ represent a pair of straight lines. Then the angle θ between the lines is given by* $\cos\theta = \dfrac{|a+b|}{\sqrt{(a-b)^2 + 4h^2}}$.

Proof : It is obvious that $(a-b)^2 + 4h^2 > 0$.

Let $H \equiv ax^2 + 2hxy + by^2 = (l_1x + m_1y)(l_2x + m_2y)$. Then the lines represented by the given equation are $l_1x + m_1y = 0$ and $l_2x + m_2y = 0$. Further $l_1l_2 = a$, $m_1m_2 = b$ and $l_1m_2 + l_2m_1 = 2h$. Therefore, the angle θ between these lines is given by

$$\cos\theta = \frac{|l_1l_2 + m_1m_2|}{\sqrt{\left(l_1^2 + m_1^2\right)\left(l_2^2 + m_2^2\right)}}$$

$$= \frac{|l_1l_2 + m_1m_2|}{\sqrt{(l_1l_2 - m_1m_2)^2 + (l_1m_2 + l_2m_1)^2}}$$

Therefore $$\boxed{\cos\theta = \frac{|a+b|}{\sqrt{(a-b)^2 + 4h^2}}}.$$

4.2 Condition for perpendicular and coincident lines, bisectors of angles

It is already observed in 4.1.7 that the equation H = 0 represents a pair of coincident lines if $h^2 = ab$.

Now the lines given by $H = 0$ are perpendicular

$$\Leftrightarrow \cos\theta = 0$$

$$\Leftrightarrow a + b = 0$$

(sum of the coefficients of x^2 and y^2 in $H = 0$ is zero)

If $a + b \neq 0$, then the lines represented by $H = 0$ are not perpendicular and in such a situation, the angle θ between the lines is also given by the formula

$$\tan\theta = \frac{2\sqrt{h^2 - ab}}{|a+b|} \quad \text{because} \quad \cos\theta = \frac{|a+b|}{\sqrt{(a-b)^2 + 4h^2}} \quad \text{gives}$$

$$\sin\theta = \frac{2\sqrt{h^2 - ab}}{\sqrt{(a-b)^2 + 4h^2}}.$$

4.2.1 Example : Let us find the angle between the straight lines represented by the equation

$$2x^2 - 3xy - 6y^2 = 0.$$

Comparing this equation with $ax^2 + 2hxy + by^2 = 0$, we find $a = 2$, $b = -6$ and

$$h = \frac{-3}{2}.$$

Therefore, angle θ between the given pair of lines is given by

$$\tan\theta = \frac{2\sqrt{h^2 - ab}}{|a+b|} = \frac{2\sqrt{\frac{9}{4}+12}}{|2-6|} = \frac{\sqrt{57}}{4}.$$

Hence the angle between the lines is $\operatorname{Tan}^{-1}\left(\frac{\sqrt{57}}{4}\right)$.

4.2.2 Theorem : *Let the equations of two intersecting lines be* $L_1 \equiv a_1x + b_1y + c_1 = 0$ *and* $L_2 \equiv a_2x + b_2y + c_2 = 0$. *Then the equations of the bisectors of the angles (angle and its supplement) between* $L_1 = 0$ *and* $L_2 = 0$ *are*

$$\frac{a_1x + b_1y + c_1}{\sqrt{a_1^2 + b_1^2}} = \pm\frac{(a_2x + b_2y + c_2)}{\sqrt{a_2^2 + b_2^2}}.$$

Proof : The locus of the points equidistant from L_1 and L_2 is the pair of lines bisecting the angles between L_1 and L_2. Let PM, PN be the perpendicular distances of a point $P(x_1, y_1)$ from the lines L_1 and L_2 respectively (see Fig. 4.1).

Then P is a point on the given locus

$\Leftrightarrow$ PM = PN

$$\Leftrightarrow \left|\frac{a_1x_1 + b_1y_1 + c_1}{\sqrt{a_1^2 + b_1^2}}\right| = \left|\frac{a_2x_1 + b_2y_1 + c_2}{\sqrt{a_2^2 + b_2^2}}\right|$$

$$\Leftrightarrow \frac{(a_1x_1 + b_1y_1 + c_1)}{\sqrt{a_1^2 + b_1^2}} = \pm\frac{(a_2x_1 + b_2y_1 + c_2)}{\sqrt{a_2^2 + b_2^2}}$$

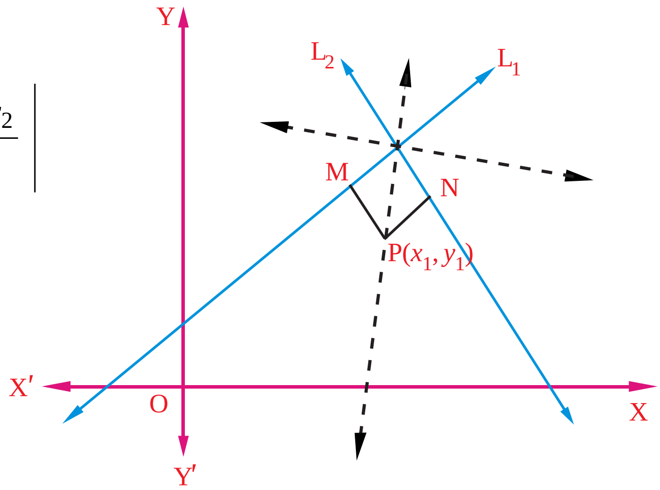


Fig. 4.1

Note : The equations of the lines bisecting the angles between L_1 and L_2 are also written as

$$\frac{(a_1x + b_1y + c_1)}{\sqrt{a_1^2 + b_1^2}} \pm \frac{(a_2x + b_2y + c_2)}{\sqrt{a_2^2 + b_2^2}} = 0.$$

4.2.3 Examples

1. Example

Let us find the equations of the straight lines bisecting the angles between the lines $7x + y + 3 = 0$ and $x - y + 1 = 0$. By Theorem 4.2.2, the equations of the straight lines bisecting the angles between the given lines are

$$\left(\frac{7x + y + 3}{\sqrt{50}}\right) \pm \left(\frac{x - y + 1}{\sqrt{2}}\right) = 0$$

that is, $(7x + y + 3) \pm 5(x - y + 1) = 0$

(or) $x + 3y - 1 = 0$ and $3x - y + 2 = 0$.

2. Example : Let us prove that the internal bisectors of the angles of a triangle are concurrent.

Let $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ be the vertices of a given triangle ABC (see Fig. 4.2) whose sides $\overleftrightarrow{BC}$, $\overleftrightarrow{CA}$ and $\overleftrightarrow{AB}$ are represented respectively by the equations

$$L_1 \equiv a_1x + b_1y + c_1 = 0,$$
$$L_2 \equiv a_2x + b_2y + c_2 = 0$$
and $$L_3 \equiv a_3x + b_3y + c_3 = 0.$$

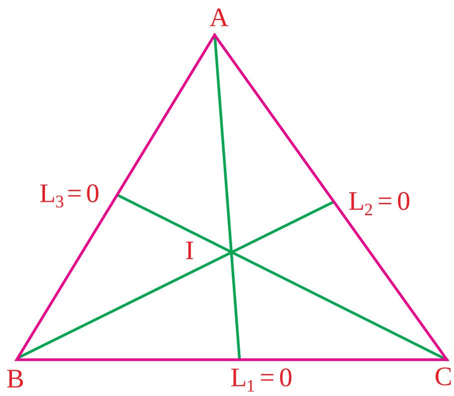


Fig. 4.2

Without loss of generality, we can assume that the non-zero numbers $a_rx_r + b_ry_r + c_r$ $(r = 1, 2, 3)$ are positive (that is, if necessary, we rewrite the equations so that these numbers are positive).

Also $a_rx_s + b_ry_s + c_r = 0$ for $r \neq s$ and $r, s = 1, 2, 3$.

Now the equation $u_1 \equiv \dfrac{a_2x + b_2y + c_2}{\sqrt{a_2^2 + b_2^2}} - \dfrac{a_3x + b_3y + c_3}{\sqrt{a_3^2 + b_3^2}} = 0$

(or) $$u_1 \equiv \frac{L_2}{\sqrt{a_2^2 + b_2^2}} - \frac{L_3}{\sqrt{a_3^2 + b_3^2}} = 0$$

represents one of the bisectors of the angle BAC.

Since $a_3x_2 + b_3y_2 + c_3 = a_2x_3 + b_2y_3 + c_2 = 0$, we have

$$\frac{a_2x_2 + b_2y_2 + c_2}{\sqrt{a_2^2 + b_2^2}} - \frac{a_3x_2 + b_3y_2 + c_3}{\sqrt{a_3^2 + b_3^2}} > 0 \text{ and}$$
$$\frac{a_2x_3 + b_2y_3 + c_2}{\sqrt{a_2^2 + b_2^2}} - \frac{a_3x_3 + b_3y_3 + c_3}{\sqrt{a_3^2 + b_3^2}} < 0.$$

Hence the vertices $B(x_2, y_2)$ and $C(x_3, y_3)$ lie on either side of the bisector $u_1 = 0$ and accordingly, it is the internal bisector of angle A of Δ ABC.

Similarly the internal bisectors of the angles B and C of the triangle are respectively

$$u_2 \equiv \frac{L_3}{\sqrt{a_3^2 + b_3^2}} - \frac{L_1}{\sqrt{a_1^2 + b_1^2}} = 0$$
and $$u_3 \equiv \frac{L_1}{\sqrt{a_1^2 + b_1^2}} - \frac{L_2}{\sqrt{a_2^2 + b_2^2}} = 0.$$

Now letting $k_1 = k_2 = k_3 = 1$, we observe that $k_1u_1 + k_2u_2 + k_3u_3 \equiv 0$ and therefore, by Theorem 3.7.3, the bisectors $u_1 = 0$, $u_2 = 0$ and $u_3 = 0$ are concurrent.

4.3 Pair of bisectors of anlges

4.3.1 Theorem : *If the equation $ax^2 + 2hxy + by^2 = 0$ represents a pair of intersecting lines, then the combined equation of the pair of bisectors of the angles between these lines is $h(x^2 - y^2) = (a - b)xy$.*

Proof : Since the straight lines represented by the given equation pass through the origin, their separate equations (in symmetric form) can be taken as $x \sin\theta - y\cos\theta = 0$ and $x\sin\phi - y\cos\phi = 0$.

Therefore,

$$\begin{aligned} H &\equiv ax^2 + 2hxy + by^2 \\ &\equiv \lambda\,(x\sin\theta - y\cos\theta)(x\sin\phi - y\cos\phi) \end{aligned}$$

for some real $\lambda \neq 0$. From this, we obtain

$$\sin\theta\sin\phi = \frac{a}{\lambda},\ \cos\theta\cos\phi = \frac{b}{\lambda} \text{ and } \sin(\theta+\phi) = \frac{-2h}{\lambda} \qquad \dots (1)$$

The equations of the straight lines bisecting the angles between the given pair of lines is therefore

$$(x\sin\theta - y\cos\theta) \pm (x\sin\phi - y\cos\phi) = 0.$$

Accordingly, their combined equation is

$$(x\sin\theta - y\cos\theta)^2 - (x\sin\phi - y\cos\phi)^2 = 0$$

that is, $(\sin^2\theta - \sin^2\phi)x^2 - 2(\sin\theta\cos\theta - \sin\phi\cos\phi)xy + (\cos^2\theta - \cos^2\phi)y^2 = 0$

that is, $(\sin^2\theta - \sin^2\phi)(x^2 - y^2) = (\sin 2\theta - \sin 2\phi)xy$

that is, $[\sin(\theta+\phi)\sin(\theta-\phi)](x^2-y^2) = [2\cos(\theta+\phi)\sin(\theta-\phi)]xy$

that is, $[\sin(\theta+\phi)](x^2-y^2) = [2\cos(\theta+\phi)]xy$ (since $\theta \neq \phi$)

(or) $\frac{-2h}{\lambda}(x^2-y^2) = \frac{2(b-a)}{\lambda}xy$ (using the relations (1)) which is the same equation as $h(x^2 - y^2) = (a-b)xy$.

4.3.2 Note

The sum of the coefficients of x^2 and y^2 in the bisector's equation is zero, which verifies their perpendicularity.

4.3.3 Example : Let us find the combined equation of the pair of bisectors of the angles between the pair of straight lines represented by $6x^2 + 11xy + 3y^2 = 0$.

Comparing the given equation with $ax^2 + 2hxy + by^2 = 0$, we observe that $a = 6$, $b = 3$ and $h = \frac{11}{2}$.

Therefore, the equation of the pair of bisectors of the angles between the given pair of lines is $h(x^2 - y^2) = (a-b)xy$

that is, $\frac{11}{2}(x^2 - y^2) = (6-3)xy$ (or) $11(x^2 - y^2) - 6xy = 0$.

4.3.4 Solved Problems

1. Problem : *Does the equation $x^2 + xy + y^2 = 0$ represent a pair of lines?*

Solution : No. For, $a = b = 1$, $h = \frac{1}{2}$ and

$$h^2 - ab = \frac{1}{4} - 1 < 0, \text{ that is, } h^2 < ab.$$

2. Problem : *Find the nature of the triangle formed by the lines $x^2 - 3y^2 = 0$ and $x = 2$.*

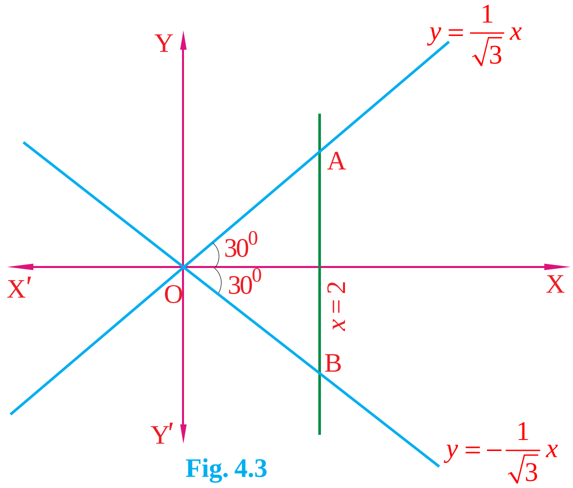


Fig. 4.3

Soltuion : The lines $x^2 - 3y^2 = 0$, that is, $y = \frac{1}{\sqrt{3}}x$, $y = -\frac{1}{\sqrt{3}}x$ are equally inclined to the x-axis, the inclination being 30^0 (see Fig.4.3).

Further $\angle OAB = \angle OBA = 60^0$.

Hence the triangle is equilateral.

3. Problem : *Find the centriod of the triangle formed by the lines* $12x^2 - 20xy + 7y^2 = 0$ *and* $2x - 3y + 4 = 0$.

Solution : The pair of straight lines $12x^2 - 20xy + 7y^2 = 0$ intersect the straight line $2x - 3y + 4 = 0$ in the points A and B whose coordinates are given by the equation $3(3y-4)^2 - 10y(3y-4) + 7y^2 = 0$ (eliminant of x from the above equations) that is, $y^2 - 8y + 12 = 0$ (or) $(y-2)(y-6) = 0$ and so, $y = 2$ or 6 and correspondingly $x = 1$ or 7.

Therefore, the points of intersection are A (1, 2) and B (7, 6). Accordingly, the triangle OAB formed by the given triad of lines has its centroid at $\left(\frac{8}{3}, \frac{8}{3}\right)$.

4. Problem : *Prove that the lines represented by the equations $x^2 - 4xy + y^2 = 0$ and $x + y = 3$ form an equilateral triangle.*

Solution : The slope of the line $L \equiv x + y - 3 = 0$ is -1 and hence it makes an angle of 45^0 with the negative direction of the x-axis. Therefore, no straight line which makes an angle of 60^0 with L is vertical. Let the equation of a line passing through the origin and making an angle of 60^0 with L be $y = mx$. Then

$$\sqrt{3} = \tan 60^0 = \left|\frac{m+1}{1-m}\right|$$

so that $(m+1)^2 = 3(m-1)^2$

(or) $m^2 - 4m + 1 = 0$

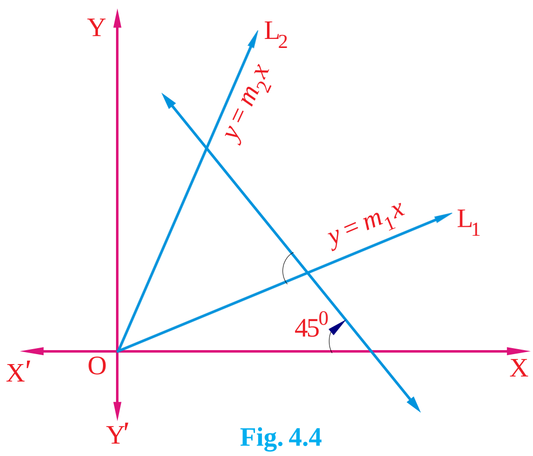


Fig. 4.4

whose roots m_1 and m_2 are real and distinct. Therefore, there are two lines L_1 and L_2 passing through the origin, each making an angle of 60^0 with L. Their equations are $y = m_1 x$ and $y = m_2 x$ where $m_1 + m_2 = 4, \quad m_1 m_2 = 1$.

The combined equation of L_1 and L_2 is

$$(y - m_1 x)(y - m_2 x) = 0$$

i.e., $y^2 - (m_1 + m_2)xy + m_1 m_2 x^2 = 0$

i.e., $y^2 - 4xy + x^2 = 0$, which is same as the given pair of lines.

Hence L, L_1, L_2 form an equilateral triangle.

5. Problem : *Show that the product of the perpendicular distances from a point* (α,β) *to the pair of straight lines* $ax^2 + 2hxy + by^2 = 0$ *is* $\dfrac{\left|a\alpha^2 + 2h\alpha\beta + b\beta^2\right|}{\sqrt{(a-b)^2 + 4h^2}}$.

Solution : Let $ax^2 + 2hxy + by^2 \equiv (l_1 x + m_1 y)(l_2 x + m_2 y)$.

Then the lines represented by the equation are $l_1 x + m_1 y = 0$ and $l_2 x + m_2 y = 0$.

Further, $l_1 l_2 = a; \quad m_1 m_2 = b$ and $l_1 m_2 + l_2 m_1 = 2h$.

d_1 = length of the perpendicular from (α,β) to $l_1 x + m_1 y = 0$

$$= \frac{\left|l_1\alpha + m_1\beta\right|}{\sqrt{l_1^2 + m_1^2}}.$$

d_2 = length of the perpendicular from (α,β) to $l_2 x + m_2 y = 0$

$$= \frac{\left|l_2\alpha + m_2\beta\right|}{\sqrt{l_2^2 + m_2^2}}.$$

Then, the product of the lengths of the perpendiculars from (α,β) to the given pair of lines

$$= d_1 d_2 = \frac{\left|(l_1\alpha + m_1\beta)(l_2\alpha + m_2\beta)\right|}{\sqrt{\left(l_1^2 + m_1^2\right)\left(l_2^2 + m_2^2\right)}}$$

$$= \frac{\left|a\alpha^2 + 2h\alpha\beta + b\beta^2\right|}{\sqrt{(a-b)^2 + 4h^2}}.$$

6. Problem : *Let* $ax^2 + 2hxy + by^2 = 0$ *represent a pair of straight lines. Then show that the equation of the pair of straight lines*

(i) *passing through* (x_0, y_0) *and parallel to the given pair of lines is*

$$a(x-x_0)^2+2h(x-x_0)(y-y_0)+b(y-y_0)^2=0 \text{ and}$$

(ii) *passing through* (x_0, y_0) *and perpendicular to the given pair of lines is*

$$b(x-x_0)^2-2h(x-x_0)(y-y_0)+a(y-y_0)^2=0.$$

Solution : Let $ax^2+2hxy+by^2 \equiv (l_1x+m_1y)(l_2x+m_2y)$.

Then the equations of the lines are $L_1 \equiv l_1x+m_1y=0$, $L_2 \equiv l_2x+m_2y=0$.

Further, $l_1l_2=a$, $m_1m_2=b$ and $l_1m_2+l_2m_1=2h$.

(i) Now the equations of the straight lines passing through (x_0, y_0) and parallel to L_1 and L_2 respectively are

$$l_1(x-x_0)+m_1(y-y_0)=0$$

and $$l_2(x-x_0)+m_2(y-y_0)=0.$$

Therefore, their combined equation is

$$[l_1(x-x_0)+m_1(y-y_0)]\,[l_2(x-x_0)+m_2(y-y_0)]=0$$

(or) $$\boxed{a(x-x_0)^2+2h(x-x_0)\,(y-y_0)+b(y-y_0)^2=0}.$$

(ii) The straight lines passing through (x_0, y_0) and perpendicular to the pair L_1 and L_2 are respectively

$$m_1x-l_1y=m_1x_0-l_1y_0$$

(or) $m_1(x-x_0)-l_1(y-y_0)=0$ and $m_2(x-x_0)-l_2(y-y_0)=0$.

Hence their combined equation is

$$[m_1(x-x_0)-l_1(y-y_0)]\,[m_2(x-x_0)-l_2(y-y_0)]=0$$

that is, $$\boxed{b(x-x_0)^2-2h(x-x_0)\,(y-y_0)+a(y-y_0)^2=0}.$$

Note: The pair of lines passing through the origin and perpendicular to the pair of lines given by $ax^2+2hxy+by^2=0$ is $bx^2-2hxy+ay^2=0.$

7. Problem : *Show that the area of the triangle formed by the lines* $ax^2+2hxy+by^2=0$ *and* $lx+my+n=0$ *is* $\left|\dfrac{n^2\sqrt{h^2-ab}}{am^2-2hlm+bl^2}\right|$.

Solution : Let $\overleftrightarrow{OA}$ and $\overleftrightarrow{OB}$ be the pair of straight lines represented by the equation $ax^2+2hxy+by^2=0$ (see Fig. 4.5) and $\overleftrightarrow{AB}$ be the line $lx+my+n=0$.

Let $ax^2+2hxy+by^2 \equiv (l_1x+m_1y)(l_2x+m_2y)$ and $\overleftrightarrow{OA}$ and $\overleftrightarrow{OB}$ be the lines $l_1x+m_1y=0$ and $l_2x+m_2y=0$ respectively.

Let $A=(x_1, y_1)$, $B=(x_2, y_2)$. Then,

since A lies on $\overleftrightarrow{OA}$ and $\overleftrightarrow{AB}$, $l_1x_1+m_1y_1=0$ and $lx_1+my_1+n=0$.

Since, by hypothesis, the given three lines form a triangle,

$$l_1m - lm_1 \neq 0 \text{ and } l_2m - lm_2 \neq 0.$$

So, by the rule of cross-multiplication, we obtain

$$\frac{x_1}{m_1n} = \frac{y_1}{-nl_1} = \frac{1}{l_1m - lm_1} \quad \text{and hence}$$

$$x_1 = \frac{m_1n}{l_1m - lm_1};\ y_1 = \frac{-nl_1}{l_1m - lm_1}.$$

Similarly, $$x_2 = \frac{m_2n}{l_2m - lm_2};\ y_2 = \frac{-nl_2}{l_2m - lm_2}.$$

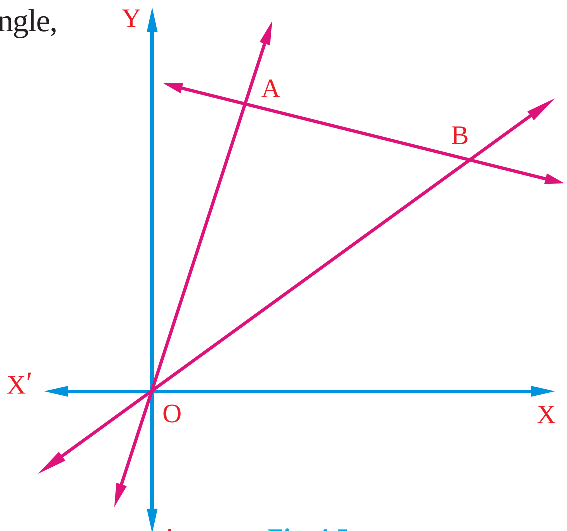


Fig. 4.5

Therefore, area of triangle OAB

$$= \frac{1}{2}\left|x_1y_2 - x_2y_1\right|$$

$$= \frac{1}{2}\left|\frac{n^2(l_1m_2 - l_2m_1)}{(l_1m - lm_1)(l_2m - lm_2)}\right|$$

$$= \frac{1}{2}\left|\frac{n^2\sqrt{(l_1m_2 + l_2m_1)^2 - 4l_1l_2m_1m_2}}{l_1l_2m^2 - lm(l_1m_2 + l_2m_1) + m_1m_2l^2}\right| = \frac{1}{2}\left|\frac{n^2\sqrt{4h^2 - 4ab}}{am^2 - 2hlm + bl^2}\right| = \left|\frac{n^2\sqrt{h^2 - ab}}{am^2 - 2hlm + bl^2}\right|.$$

(since $l_1l_2 = a,\ m_1m_2 = b$ and $l_1m_2 + l_2m_1 = 2h$)

8. Problem : *Two equal sides of an isosceles triangle are $7x - y + 3 = 0$ and $x + y - 3 = 0$ and its third side passes through the point $(1, 0)$. Find the equation of the third side.*

Solution : Let the lines $7x - y + 3 = 0$ and $x + y - 3 = 0$ intersect at A. If we draw lines (not passing through A) perpendicular to each of the bisectors of the angles at A, we get isosceles triangles, equal sides being along the given lines.

(ΔABF $\cong$ ΔAFC and ΔADG $\cong$ ΔAGE)

Of them, we require those triangles whose third sides pass through (1, 0).

The equations of the bisectors of the angles between $7x - y + 3 = 0$ and $x + y - 3 = 0$ are

$$\frac{7x - y + 3}{5\sqrt{2}} = \pm\left(\frac{x + y - 3}{\sqrt{2}}\right)$$

$$\Rightarrow 7x - y + 3 = \pm 5(x + y - 3)$$

$$\Rightarrow x - 3y + 9 = 0 \text{ and } 3x + y - 3 = 0.$$

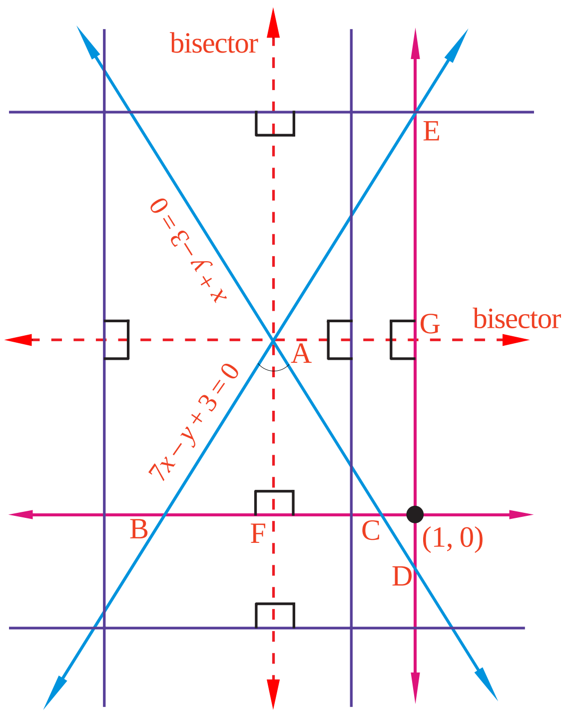


Fig. 4.6

The third sides will be those lines perpendicular to the bisectors and intersecting at (1, 0).

The side perpendicular to $x - 3y + 9 = 0$ and passing through $(1, 0)$ is $3x + y - 3 = 0$. The other one is $(x - 1) - 3(y - 0) = 0$ i.e., $x - 3y - 1 = 0$. Therefore, $3x + y - 3 = 0$ and $x - 3y - 1 = 0$ are the required ones. [In the Figure ΔABC and ΔADE are isosceles with $\overline{BC}$ and $\overline{DE}$ as third sides].

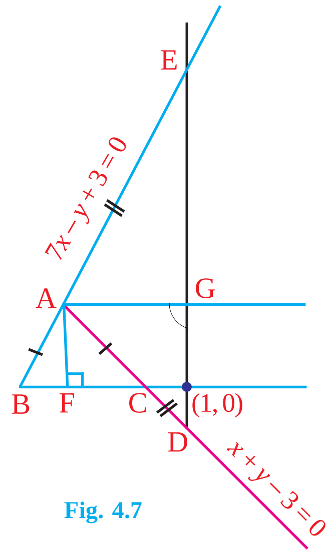


Fig. 4.7

Exercise 4(a)

I. 1. Find the acute angle between the pair of lines represented by the following equations.

(i) $x^2 - 7xy + 12y^2 = 0$

(ii) $y^2 - xy - 6x^2 = 0$

(iii) $(x \cos\alpha - y \sin\alpha)^2 = (x^2 + y^2)\sin^2\alpha$

(iv) $x^2 + 2xy \cot\alpha - y^2 = 0$

II. 1. Show that the following pairs of straight lines have the same set of angular bisectors (that is they are equally inclined to each other).

(i) $2x^2 + 6xy + y^2 = 0,\ 4x^2 + 18xy + y^2 = 0$

(ii) $a^2x^2 + 2h(a+b)xy + b^2y^2 = 0,$

$ax^2 + 2hxy + by^2 = 0;\ a + b \neq 0.$

(iii) $ax^2 + 2hxy + by^2 + \lambda(x^2 + y^2) = 0\ ;\ (\lambda \in \mathbf{R}),$

$ax^2 + 2hxy + by^2 = 0.$

2. Find the value of h, if the slopes of the lines represented by $6x^2 + 2hxy + y^2 = 0$ are in the ratio $1 : 2$.

3. If $ax^2 + 2hxy + by^2 = 0$ represents two straight lines such that the slope of one line is twice the slope of the other, prove that $8h^2 = 9ab$.

4. Show that the equation of the pair of straight lines passing through the origin and making an angle of 30^0 with the line $3x - y - 1 = 0$ is $13x^2 + 12xy - 3y^2 = 0$.

5. Find the equation to the pair of straight lines passing through the origin and making an acute angle α with the straight line $x + y + 5 = 0$.

6. Show that the straight lines represented by $(x + 2a)^2 - 3y^2 = 0$ and $x = a$ form an equilateral triangle.

7. Show that the pair of bisectors of the angles between the straight lines $(ax+by)^2 = c(bx-ay)^2, c>0$ are parallel and perpendicular to the line $ax+by+k=0$.

8. The adjacent sides of a parallelogram are $2x^2-5xy+3y^2=0$ and one diagonal is $x+y+2=0$. Find the vertices and the other diagonal.

9. Find the centroid and the area of the triangle formed by the following lines

   (i) $2y^2 - xy - 6x^2 = 0,\ x + y + 4 = 0$

   (ii) $3x^2 - 4xy + y^2 = 0,\ 2x - y = 6$

10. Find the equation of the pair of lines intersecting at $(2, -1)$ and

    (i) perpendicular to the pair $6x^2 - 13xy - 5y^2 = 0$, and

    (ii) parallel to the pair $6x^2 - 13xy - 5y^2 = 0$

11. Find the equation of the bisector of the acute angle between the lines $3x - 4y + 7 = 0$ and $12x + 5y - 2 = 0$.

12. Find the equation of the bisector of the obtuse angle between the lines $x + y - 5 = 0$ and $x - 7y + 7 = 0$.

III. 1. Show that the lines represented by $(lx+my)^2 - 3(mx-ly)^2 = 0$ and $lx+my+n=0$ form an equilateral triangle with area $\dfrac{n^2}{\sqrt{3}(l^2+m^2)}$.

2. Show that the straight lines represented by $3x^2+48xy+23y^2=0$ and $3x-2y+13=0$ form an equilateral triangle of area $\dfrac{13}{\sqrt{3}}$ sq.units.

3. Show that the equation of the pair of lines bisecting the angles between the pair of bisectors of the angles between the pair of lines $ax^2+2hxy+by^2=0$ is $(a-b)(x^2-y^2)+4hxy=0$.

4. If one line of the pair of lines $ax^2+2hxy+by^2=0$ bisects the angle between the coordinate axes, prove that $(a+b)^2=4h^2$.

5. If (α, β) is the centroid of the triangle formed by the lines $ax^2+2hxy+by^2=0$ and $lx+my=1$, prove that $\dfrac{\alpha}{bl-hm} = \dfrac{\beta}{am-hl} = \dfrac{2}{3(bl^2-2hlm+am^2)}$.

6. Prove that the distance from the origin to the orthocentre of the triangle formed by the lines $\dfrac{x}{\alpha} + \dfrac{y}{\beta} = 1$ and $ax^2 + 2hxy + by^2 = 0$ is

$$\left(\alpha^2+\beta^2\right)^{\frac{1}{2}} \left| \frac{(a+b)\alpha\beta}{a\alpha^2 - 2h\alpha\beta + b\beta^2} \right|.$$

7. The straight line $lx + my + n = 0$ bisects an angle between the pair of lines of which one is $px + qy + r = 0$. Show that the other line is
$(px + qy + r)(l^2 + m^2) - 2(lp + mq)(lx + my + n) = 0$.

4.4 Pair of lines - Second degree general equation

We now obtain conditions for a general equation of second degree in x and y to represent a pair of lines.

4.4.1 Theorem

If the second degree equation $S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ *in the two variables* x *and* y *represents a pair of straight lines, then*

(i) $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$ and

(ii) $h^2 \geq ab$, $g^2 \geq ac$ and $f^2 \geq bc$.

Proof : Since $S = 0$ represents a pair of lines, we can write

$$S \equiv (l_1x + m_1y + n_1)(l_2x + m_2y + n_2) \text{ (see Note (ii) of 4.1.5).}$$

Comparing the coefficients of like terms on either side,

we obtain
$$l_1l_2 = a \;;\; l_1m_2 + l_2m_1 = 2h$$
$$m_1m_2 = b \;;\; l_1n_2 + l_2n_1 = 2g$$
$$n_1n_2 = c \;;\; m_1n_2 + m_2n_1 = 2f.$$

(i) $8fgh = (2h)(2g)(2f) = (l_1m_2 + l_2m_1)(l_1n_2 + l_2n_1)(m_1n_2 + m_2n_1)$

$$= l_1l_2\left(m_1^2 n_2^2 + m_2^2 n_1^2\right) + m_1m_2\left(n_1^2 l_2^2 + n_2^2 l_1^2\right) + n_1n_2\left(l_1^2 m_2^2 + l_2^2 m_1^2\right) + 2l_1l_2\, m_1m_2n_1n_2$$

$$= a(4f^2 - 2bc) + b(4g^2 - 2ac) + c(4h^2 - 2ab) + 2abc$$

$$= 4(af^2 + bg^2 + ch^2 - abc).$$

Therefore, $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$.

(ii) $$h^2 - ab = \left(\frac{l_1m_2 + l_2m_1}{2}\right)^2 - l_1l_2\, m_1m_2$$

$$= \left(\frac{l_1m_2 - l_2m_1}{2}\right)^2 \geq 0.$$

Therefore $h^2 \geq ab$.

Similarly, it can be shown that $g^2 \geq ac$ and $f^2 \geq bc$.

4.4.2 Note

1. Both the above sets of conditions are necessary for the equation $S = 0$ to represent a pair of straight lines. That is, $S = 0$ cannot represent a pair of lines if any of the above conditions fails.

For example, in the equation $x^2 + y^2 + 2xy + 1 = 0$, we have $a = b = c = h = 1$; $f = g = 0$ and therefore $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$. But this equation which is the same as $(x + y)^2 + 1 = 0$ does not represent a pair of straight lines. Infact, the locus of this equation is the empty set ϕ .

Similarly, in the equation $x^2 + y^2 = 0$, we have $a = b = 1$; $c = f = g = h = 0$ and so $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$; $f^2 = bc$ and $g^2 = ac$. But $h^2 < ab$; and once again this equation does not represent a pair of straight lines. The locus of the equation $x^2 + y^2 = 0$ is the origin.

2. The equation of second degree in x and y,

$S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of straight lines if

(i) $\Delta \equiv abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

and (ii) $h^2 \geq ab,\ g^2 \geq ac,\ f^2 \geq bc$.

(The proof of this is beyond the scope of the book)

From 4.4.1 Theorem and 4.4.2 Note (2) we have

$S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of straight lines

$\Leftrightarrow \Delta \equiv abc + 2fgh - af^2 - bg^2 - ch^2 = 0,\quad h^2 \geq ab,\ g^2 \geq ac,\ f^2 \geq bc.$

4.4.3 Theorem : *If the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents two straight lines, then the equation $ax^2 + 2hxy + by^2 = 0$ represents a pair of lines passing through the origin and parallel to the former pair of lines.*

Proof: Let the equation $S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represent a pair of straight lines. Then we can write $S \equiv (l_1x + m_1y + n_1)(l_2x + m_2y + n_2)$ so that $l_1 l_2 = a, m_1m_2 = b$ and $l_1m_2 + l_2m_1 = 2h$.

Therefore, $H \equiv ax^2 + 2hxy + by^2 = l_1l_2x^2 + (l_1m_2 + l_2m_1)\,xy + m_1m_2y^2$

$= (l_1x + m_1y)(l_2x + m_2y)$.

Hence the equation $H = 0$ represents the pair of straight lines $l_1x + m_1y = 0$ and $l_2x + m_2y = 0$ which are respectively parallel to the pair of lines $l_1x + m_1y + n_1 = 0$ and $l_2x + m_2y + n_2 = 0$ represented by $S = 0$ and they pass through the origin.

4.5 Conditions for parallel lines - Distance between them, Point of intersection of pair of lines

In this section we find the condition for two lines to be parallel and to find the distance between two parallel lines. We will also find the intersection of two lines when their combined equation is given.

$S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of straight lines. The angle between this pair of lines is the same as the angle between the pair of lines represented by $H \equiv ax^2 + 2hxy + by^2 = 0$. Hence the angle between the pair of lines $S = 0$ is

$$\cos^{-1}\left(\frac{|a+b|}{\sqrt{(a-b)^2+4h^2}}\right)$$

$$= \quad \tan^{-1}\left(\frac{2\sqrt{h^2-ab}}{a+b}\right) \qquad \text{if } (a+b)>0$$

$$= \tan^{-1}\left(\frac{2\sqrt{h^2-ab}}{-(a+b)}\right) \qquad \text{if } (a+b)<0$$

$$= \frac{\pi}{2} \qquad \text{if } a+b=0.$$

Therefore, the lines represented by S = 0 are parallel if $h^2 = ab$, perpendicular if $a + b = 0$ and intersecting if $h^2 > ab$.

4.5.1 Theorem : *If the equation* $S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ *represents a pair of parallel straight lines, then (i)* $h^2 = ab$ *(ii)* $af^2 = bg^2$ *and*

(iii) the distance between the parallel lines $= 2\sqrt{\dfrac{g^2-ac}{a(a+b)}} = 2\sqrt{\dfrac{f^2-bc}{b(a+b)}}$.

Proof

Let the two parallel straight lines represented by $S=0$ be

$$lx + my + n_1 = 0 \qquad ...(1)$$

and $$lx + my + n_2 = 0. \qquad ...(2)$$

Then, $S \equiv \lambda(lx + my + n_1)(lx + my + n_2)$, for some real $\lambda \neq 0$.

From this we have

$$l^2 = \frac{a}{\lambda},\ m^2 = \frac{b}{\lambda}, n_1n_2 = \frac{c}{\lambda}, lm = \frac{h}{\lambda},$$

$$l(n_1+n_2) = \frac{2g}{\lambda} \text{ and } m(n_1+n_2) = \frac{2f}{\lambda}.$$

Now (i) $$h^2 = \lambda^2 l^2 m^2 = (\lambda l^2)(\lambda m^2) = ab.$$

(ii) $$4af^2 = (\lambda l^2)(\lambda^2 m^2)(n_1+n_2)^2$$

$$= (\lambda m^2)(\lambda^2 l^2)(n_1+n_2)^2$$

$$= b(4g^2)$$

so that $af^2 = bg^2$.

(iii) Distance between the parallel lines

$$= \frac{|n_1-n_2|}{\sqrt{l^2+m^2}} = \sqrt{\frac{(n_1+n_2)^2-4n_1n_2}{l^2+m^2}}$$

$$= \sqrt{\frac{\lambda^2(n_1+n_2)^2 - 4\lambda^2 n_1 n_2}{\lambda^2(l^2+m^2)}}$$

$$= \sqrt{\frac{\frac{4g^2}{l^2} - 4\lambda c}{\lambda(a+b)}} \quad \text{(or)} \quad \sqrt{\frac{\frac{4f^2}{m^2} - 4\lambda c}{\lambda(a+b)}}$$

$$= 2\sqrt{\frac{g^2 - ac}{a(a+b)}} \quad \text{(or)} \quad 2\sqrt{\frac{f^2 - bc}{b(a+b)}}.$$

4.5.2 Theorem

If the equation $S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ *represents a pair of straight lines intersecting at the origin, then* $g = f = c = 0$.

Proof : Let $S = 0$ represent a pair of lines. Then $S \equiv (l_1x + m_1y + n_1)(l_2x + m_2y + n_2)$ and $L_1 \equiv l_1x + m_1y + n_1 = 0$ and $L_2 \equiv l_2x + m_2y + n_2 = 0$ are the lines represented by $S = 0$. Since origin is a point on both the lines $L_1 = 0$ and $L_2 = 0$, we get $n_1 = 0 = n_2$. Hence

$$S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c$$
$$\equiv (l_1x + m_1y)(l_2x + m_2y).$$

Comparing the constant terms and the coefficients of x and y on either side,

we get $g = f = c = 0$.

4.5.3 Theorem : *If the equation* $S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ *represents a pair of intersecting straight lines, then their point of intersection is* $\left(\frac{hf - bg}{ab - h^2}, \frac{gh - af}{ab - h^2}\right)$.

Proof : Suppose that $S = 0$ represents a pair of intersecting lines. Then $h^2 > ab$. Let $P(x_0, y_0)$ be the point of intersection of the lines $S = 0$. Shifting the origin to $P(x_0, y_0)$ by the translation of axes, the equation $S = 0$ changes to:

$$a(X + x_0)^2 + 2h(X + x_0)(Y + y_0) + b(Y + y_0)^2 + 2g(X + x_0) + 2f(Y + y_0) + c = 0$$

thas is, $(aX^2 + 2hXY + bY^2) + 2(ax_0 + hy_0 + g)X + 2(hx_0 + by_0 + f)Y + S_{00} = 0$

where $S_{00} = ax_0^2 + 2hx_0y_0 + by_0^2 + 2gx_0 + 2fy_0 + c$.

W.r.t. the transformed axes, the above equation represents a pair of straight lines intersecting at the origin. Therefore, by Theorem 4.5.2, it follows that

$$ax_0 + hy_0 + g = 0 \quad \text{... (1)}$$

$$hx_0 + by_0 + f = 0 \quad \text{... (2)}$$

and $$ax_0^2 + 2hx_0y_0 + by_0^2 + 2gx_0 + 2fy_0 + c = 0$$

that is, $$(ax_0 + hy_0 + g)x_0 + (hx_0 + by_0 + f)y_0 + (gx_0 + fy_0 + c) = 0$$

(or) $\quad gx_0 + fy_0 + c = 0 \quad$ (using (1) and (2)). ... (3)

Solving the equations (1) and (2) for x_0 and y_0 by applying the rule of cross-multiplication, we get

$$\frac{x_0}{hf-bg} = \frac{y_0}{gh-af} = \frac{1}{ab-h^2} \text{ and this yields } P(x_0, y_0) = \left(\frac{hf-bg}{ab-h^2}, \frac{gh-af}{ab-h^2}\right).$$

4.5.4 Note : If $h^2 > ab$, then the point of intersection of the pair of lines $S = 0$ satisfies the three equations $ax + hy + g = 0$, $hx + by + f = 0$ and $gx + fy + c = 0$.

Observe that the eliminant of x_0, y_0 from the equations (1), (2) and (3) above is $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$

which is the same as one of the necessary conditions given in the Theorem 4.4.1 for the equation S = 0 to represent a pair of straight lines.

4.5.5 Example : Let us find the point of intersection of the pair of straight lines represented by

$x^2 + 4xy + 3y^2 - 4x - 10y + 3 = 0$.

Comparing this equation with the general equation of second degree in x and y,

we get $\quad a = 1; \quad f = -5$

$\quad b = 3; \quad g = -2$

$\quad c = 3; \quad h = 2$

Therefore, the point of intersection of the lines is

$$\left(\frac{hf-bg}{ab-h^2}, \frac{gh-af}{ab-h^2}\right) = \left(\frac{-10+6}{3-4}, \frac{-4+5}{3-4}\right) = (4, -1).$$

The point of intersection can also be obtained by solving the equations

$ax + hy + g = 0$, that is, $\quad x + 2y - 2 = 0$

and $\quad hx + by + f = 0$, that is, $\quad 2x + 3y - 5 = 0$.

4.5.6 Solved Problems

1. Problem : *Find the angle between the straight lines represented by*

$$2x^2 + 5xy + 2y^2 - 5x - 7y + 3 = 0.$$

Solution : Here $a = 2$, $2h = 5$ and $b = 2$ and

$$\theta = \cos^{-1}\frac{a+b}{\sqrt{(a-b)^2 + (2h)^2}} = \cos^{-1}\frac{4}{\sqrt{0+5^2}} = \cos^{-1}\left(\frac{4}{5}\right).$$

2. Problem : *Find the equation of the pair of lines passing through the origin and parallel to the pair of lines* $2x^2+3xy-2y^2-5x+5y-3=0$.

Solution : Equation of the pair of lines passing through the origin and parallel to the lines represented by $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ is $ax^2 + 2hxy + by^2 = 0$. Hence the required equation is $2x^2 + 3xy - 2y^2 = 0$.

3. Problem : *Find the equation of the pair of lines passing through the origin and prependicular to the pair of lines* $ax^2+2hxy+by^2+2gx+2fy+c=0$.

Solution : The equation of the lines passing through the origin and parallel to the given pair of lines is $ax^2+2hxy+by^2=0$. Hence the required lines are the lines passing through the origin and perpendicular to the lines $ax^2+2hxy+by^2=0$. Hence their equation is $bx^2-2hxy+ay^2=0$.

4. Problem: *If* $x^2+xy-2y^2+4x-y+k=0$ *represents a pair of straight lines, find k.*

Solution : Since the given equation represents a pair of lines,

$$abc+2fgh-af^2-bg^2-ch^2=0.$$

Here, $a=1, b=-2, c=k, f=-\frac{1}{2}, g=2, h=\frac{1}{2}$. Hence $k = 3$.

5. Problem : *Prove that the equation* $2x^2 + xy - 6y^2 + 7y - 2 = 0$ *represents a pair of straight lines.*

Solution : Here $a=2, b=-6, c=-2, f=\frac{7}{2}, g=0, h=\frac{1}{2}$.

Hence $abc+2fgh-af^2-bg^2-ch^2$

$$= 2(-6)(-2)+2.\frac{7}{2}.0.\frac{1}{2}-2\left(\frac{7}{2}\right)^2-(-6).0-(-2)\left(\frac{1}{2}\right)^2$$

$$= 24-\frac{49}{2}+\frac{1}{2}=\frac{1}{2}(48-49+1)=0,$$

$$h^2-ab = \frac{1}{4}+12>0,\ g^2-ac=0-(2)(-2)=4>0,$$

$$f^2-bc = \frac{49}{4}-(-6)(-2)=\frac{49}{4}-12=\frac{1}{4}>0.$$

Therefore, $h^2>ab, g^2>ac$ and $f^2>bc$. Hence the given equation represents a pair of straight lines.

6. Problem : *Prove that the equation* $2x^2 + 3xy - 2y^2 - x + 3y - 1 = 0$ *represents a pair of perpendicular straight lines.*

Solution : $a=2,\ b=-2,\ c=-1,\ h=\frac{3}{2},\ g=-\frac{1}{2},\ f=\frac{3}{2}$.

$$\text{Hence } abc+2fgh-af^2-bg^2-ch^2 = 4+2.\frac{3}{2}\left(-\frac{1}{2}\right)\frac{3}{2}-2.\frac{9}{4}-(-2)\frac{1}{4}-(-1).\frac{9}{4}$$

$$= 4-\frac{9}{4}-\frac{9}{2}+\frac{1}{2}+\frac{9}{4}=\frac{9}{2}-\frac{9}{2}=0.$$

$$h^2-ab=\frac{9}{4}+4>0,\ g^2-ac = \frac{1}{4}+2>0,\ f^2-bc=\frac{9}{4}-2=\frac{1}{4}>0,$$

$a+b=2-2=0.$ Hence the given equation represents a pair of perpendicular lines.

7. Problem : *Show that the equation* $2x^2-13xy-7y^2+x+23y-6=0$ *represents a pair of straight lines. Also find the angle between them and the coordinates of the point of intersection of the lines.*

Solution : Let $S \equiv 2x^2-13xy-7y^2+x+23y-6.$

Now $12x^2-13xy-7y^2=(x-7y)(2x+y)$.

Let us see whether we can find C_1 and C_2 such that $2C_1+C_2=1,\ C_1-7C_2=23$ and $C_1C_2=-6$. From the first two, we get $C_1=2, C_2=-3$. These values satisfy $C_1C_2=-6$. Hence there exist C_1 and C_2 such that

$$S \equiv (x-7y+C_1)(2x+y+C_2)$$

$$= (x-7y+2)(2x+y-3).$$

Therefore, the given equation represents the straight lines $2x+y-3=0$ and $x-7y+2=0$.

$$\text{Angle between the lines } = \mathrm{Tan}^{-1}\left|\frac{2+\frac{1}{7}}{1-\frac{2}{7}}\right| = \mathrm{Tan}^{-1}3.$$

Solving the equations $2x+y-3=0$ and $x-7y+2=0$, we obtain the point of intersection of the given pair of lines which is $\left(\frac{19}{15},\frac{7}{15}\right)$.

8. Problem : *Find that value of* λ *for which the equation*

$\lambda x^2-10xy+12y^2+5x-16y-3=0$ *represents a pair of straight lines.*

Solution : A necessary condition for the given equation to represent a pair of lines is

$$abc+2fgh-af^2-bg^2-ch^2=0$$

where $a=\lambda, b=12, c=-3, h=-5, g=\frac{5}{2}, f=-8$.

Therefore, $-36\lambda + 2\times -8\times \frac{5}{2}\times -5 - \lambda(-8)^2 - 12\left[\frac{5}{2}\right]^2 - (-3)(-5)^2 = 0$.

This gives $\lambda = 2 = a$.

Now $h^2 - ab = 25 - 24 = 1 > 0$,

$$g^2 - ac = \frac{25}{4} + 6 = \frac{49}{4} > 0,$$

$$f^2 - bc = 64 + 36 = 100 > 0$$

that is, $h^2 > ab, g^2 > ac$ and $f^2 > bc$.

Therefore, the given equation represents a pair of lines for $\lambda = 2$.

9. Problem : *Show that the pairs of straight lines* $6x^2 - 5xy - 6y^2 = 0$ *and*

$6x^2 - 5xy - 6y^2 + x + 5y - 1 = 0$ *form a square.*

Solution : $\mathrm{H} \equiv 6x^2 - 5xy - 6y^2 = (3x + 2y)(2x - 3y)$ and

$$\mathrm{S} \equiv 6x^2 - 5xy - 6y^2 + x + 5y - 1$$

$$= (3x + 2y - 1)(2x - 3y + 1).$$

Clearly, $\mathrm{H} = 0$ represents a pair of perpendicular lines and $\mathrm{S} = 0$ also represents a pair of perpendicular lines. Further the lines represented by $\mathrm{H} = 0$ are parallel to the lines represented by $\mathrm{S} = 0$. Therefore, the four lines form a rectangle.

But the distance of each of the lines $3x + 2y - 1 = 0$, $2x - 3y + 1 = 0$ from the origin is $\frac{1}{\sqrt{13}}$. Hence the rectangle is a square.

Aliter

Since the second degree terms in both the equations are identical, they form a parallelogram. Also, since the sum of the coefficients of x^2 and y^2 is zero, the parallelogram becomes a rectangle.

If OABC represents the rectangle,

then $6x^2 - 5xy - 6y^2 + x + 5y - 1 = 0$ ....(1)

represents the combined equation of $\overleftrightarrow{AB}$ and $\overleftrightarrow{BC}$.

Comparing the equation (1) with

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, we get

$$a = 6,\ b = -6,\ h = -\frac{5}{2},\ g = \frac{1}{2},\ f = \frac{5}{2}.$$

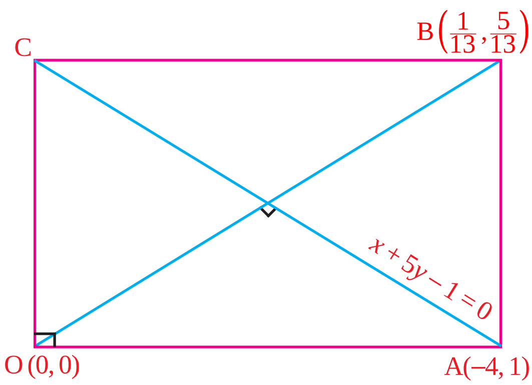


Fig. 4.8

Since $B = \left(\frac{hf-bg}{ab-h^2}, \frac{gh-af}{ab-h^2}\right) = \left(\frac{1}{13}, \frac{5}{13}\right)$.

$\therefore$ Slope of $\overleftrightarrow{OB}$ is 5.

Since the equation of $\overleftrightarrow{AC}$ is $x + 5y - 1 = 0$, its slope is $-\frac{1}{5}$.

Clearly, $\overleftrightarrow{AC}$ is perpendicular to $\overleftrightarrow{OB}$. OABC is thus a square.

10. Problem : *Show that the equation* $8x^2 - 24xy + 18y^2 - 6x + 9y - 5 = 0$ *represents a pair of parallel straight lines and find the distance between them.*

Solution : $S \equiv 8x^2 - 24xy + 18y^2 - 6x + 9y - 5$

$= 2(2x-3y)^2 - 3(2x-3y) - 5$

$= [2(2x-3y)-5][(2x-3y)+1]$

$= (4x-6y-5)(2x-3y+1)$.

Therefore, the equation $S = 0$ represents the straight lines $4x - 6y - 5 = 0$ and $2x - 3y + 1 = 0$ which are clearly a pair of parallel lines.

Distance between them $= \frac{2+5}{\sqrt{4^2+6^2}} = \frac{7}{\sqrt{52}}$.

Note : This problem can also be solved by using the result of 4.5.1 Theorem.

11. Problem : *If the pairs of lines represented by* $ax^2 + 2hxy + by^2 = 0$ *and* $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ *form a rhombus, prove that* $(a-b)fg + h(f^2 - g^2) = 0$.

Solution : Let $\overleftrightarrow{OA}$, $\overleftrightarrow{OB}$ be the pair of straight lines given by $H \equiv ax^2 + 2hxy + by^2 = 0$ and $\overleftrightarrow{AC}$, $\overleftrightarrow{BC}$ be the pair of lines given by $S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$. We know that the lines represented by $H = 0$ are parallel to the lines represented by $S = 0$ that is, Figure OACB (See Fig. 4.9) is a parallelogram. C, the point of intersection of the lines $S = 0$

$$\left(\frac{hf-bg}{ab-h^2}, \frac{gh-af}{ab-h^2}\right).$$

Since O and C are distinct points, $hf-bg$ and $gh-af$ are not both zero. Now the equation of the diagonal OC is

$$(gh - af)x - (hf - bg)\,y = 0.$$

Since A is a point on the locus H = 0 as well as on the locus S = 0, coordinates of A satisfy the equation S – H = 0. Similarly, the coordinates of B also satisfy the equation. Now $S - H \equiv 2gx + 2fy + c = 0$, being linear in x and y, represents a straight line, (note that g and f are not both zero). Hence S – H = 0 is the equation of the diagonal AB. Since, by hypothesis, figure OACB is a rhombus, the diagonals OC and AB are perpendicular to each other. Hence $(gh - af)2g - (hf - bg)2f = 0$ that is, $(a-b)fg + h(f^2 - g^2) = 0$.

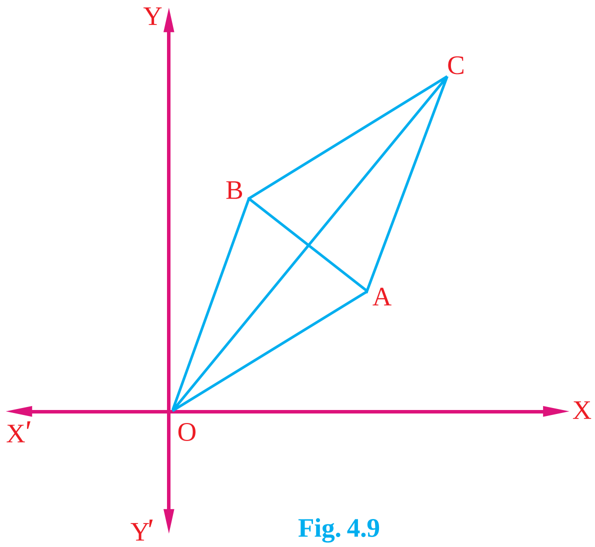


Fig. 4.9

12. Problem : *If two of the sides of a parallelogram are represented by* $ax^2 + 2hxy + by^2 = 0$ *and* $px + qy = 1$ *is one of its diagonals, prove that the other diagonal is*

$$y(bp - hq) = x(aq - hp).$$

Solution : Let OACB be the parallelogram two of whose sides $\overleftrightarrow{OA}$, $\overleftrightarrow{OB}$ are represented by the equation $H \equiv ax^2 + 2hxy + by^2 = 0$ (see Fig. 4.9). Since the other pair of sides AC and BC are respectively parallel to $\overleftrightarrow{OB}$ and $\overleftrightarrow{OA}$ their combined equation will be of the form

$$S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0.$$

Then the equation of the diagonal AB is $2gx + 2fy + c = 0$ (see solved problem 11). But this line is given to be

$$px + qy = 1 \text{ (or) } - pcx - qcy + c = 0, \quad \text{since} \quad c \neq 0.$$

Therefore, $2g = -pc,\ 2f = -qc.$ ...(1)

The vertex C of the parallelogram $= \left(\dfrac{hf - bg}{ab - h^2}, \dfrac{gh - af}{ab - h^2}\right)$. Therefore, the equation of the diagonal $\overleftrightarrow{OC}$ is $(gh - af)x = (hf - bg)y$ that is, $c(-ph + aq)x = c(-hq + bp)y$ (using (1))

(or) $(aq - hp)x = (bp - hq)y$, since $c \neq 0$.

Exercise 4(b)

I. **1.** Find the angle between the lines represented by $2x^2 + xy - 6y^2 + 7y - 2 = 0$.

2. Prove that the equation $2x^2 + 3xy - 2y^2 + 3x + y + 1 = 0$ represents a pair of perpendicular lines.

II. **1.** Prove that the equation $3x^2 + 7xy + 2y^2 + 5x + 5y + 2 = 0$ represents a pair of straight lines and find the coordinates of the point of intersection.

2. Find the value of k, if the equation $2x^2 + kxy - 6y^2 + 3x + y + 1 = 0$ represents a pair of straight lines. Find the point of intersection of the lines and the angle between the straight lines for this value of k.

3. Show that the equation $x^2 - y^2 - x + 3y - 2 = 0$ represents a pair of perpendicular lines, and find their equations.

4. Show that the lines $x^2 + 2xy - 35y^2 - 4x + 44y - 12 = 0$ and $5x + 2y - 8 = 0$ are concurrent.

5. Find the distances between the following pairs of parallel straight lines :

(i) $9x^2 - 6xy + y^2 + 18x - 6y + 8 = 0$

(ii) $x^2 + 2\sqrt{3}\,xy + 3y^2 - 3x - 3\sqrt{3}\,y - 4 = 0$.

6. Show that the two pairs of lines $3x^2 + 8xy - 3y^2 = 0$ and

$3x^2 + 8xy - 3y^2 + 2x - 4y - 1 = 0$ form a square.

III. **1.** Find the product of the lengths of the perpendiculars drawn from (2, 1) upon the lines $12x^2 + 25xy + 12y^2 + 10x + 11y + 2 = 0$.

2. Show that the straight lines $y^2 - 4y + 3 = 0$ and $x^2 + 4xy + 4y^2 + 5x + 10y + 4 = 0$ form a parallelogram and find the lengths of its sides.

3. Show that the product of the perpendicular distances from the origin to the pair of straight lines represented by $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ is $\dfrac{|c|}{\sqrt{(a-b)^2 + 4h^2}}$.

4. If the equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of intersecting lines, then show that the square of the distance of their point of intersection from the origin is $\dfrac{c(a+b) - f^2 - g^2}{ab - h^2}$. Also show that the square of this distance is $\dfrac{f^2 + g^2}{h^2 + b^2}$ if the given lines are perpendicular.

4.6 Homogenising a second degree equation with a first degree equation in *x* and *y*

Generally, the locus of a second degree equation

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \qquad \text{... (1)}$$

is called a second degree curve. If the graph of the equation (1) contains more than one point, this second degree curve can be either a pair of straight lines or a circle or a conic. Any such curve has at most two points of intersection with a given straight line if the given line is not contained in the locus (1). In this section, we find the combined equation of the pair of lines joining the origin to the points of intersection of a second degree curve with a given straight line (see Fig. 4.10(a), 10(b), 10(c)).

Let the straight line $lx + my + n = 0 \;\; (n \neq 0)$ ... (2)

intersect the second degree curve (1) in two distinct points A and B. Then the equation of $\overleftrightarrow{AB}$ can be taken as $\dfrac{lx+my}{-n} = 1.$

We homogenise equation (1) with the help of (2) as follows:

We write (1) as

$$ax^2 + 2hxy + by^2 + 2(gx + fy).1 + c.1^2 = 0.$$

In this, we replace 1 by $\dfrac{lx+my}{-n}$. Then we get a homogeneous equation in x, y namely,

$$ax^2 + 2hxy + by^2 - 2(gx + fy)\left(\frac{lx+my}{n}\right) + c\left(\frac{lx+my}{n}\right)^2 = 0 \qquad \text{... (3)}$$

that is, $\left(a - \dfrac{2gl}{n} + c\dfrac{l^2}{n^2}\right)x^2 + 2\left[h - \dfrac{gm}{n} - \dfrac{fl}{n} + \dfrac{clm}{n}\right]xy + \left(b - \dfrac{2fm}{n} + \dfrac{cm^2}{n^2}\right)y^2 = 0$

that is, $a'x^2 + 2h'xy + b'y^2 = 0$ ... (4)

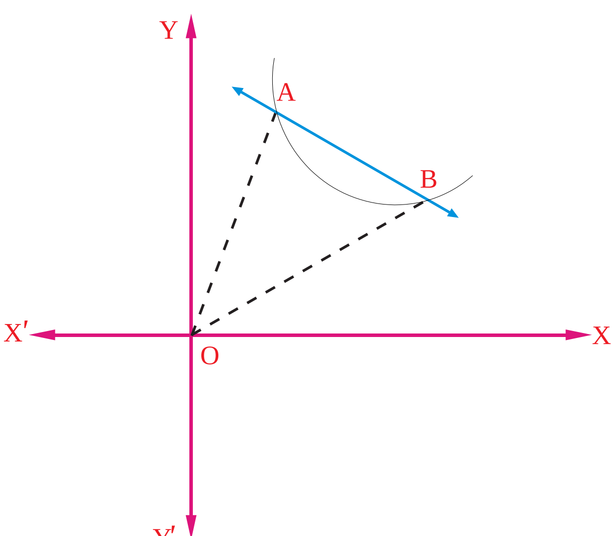


Fig. 4.10(a)

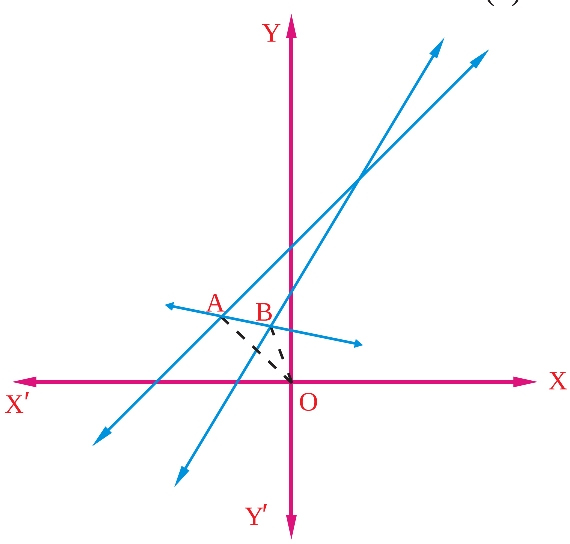


Fig. 4.10(b)

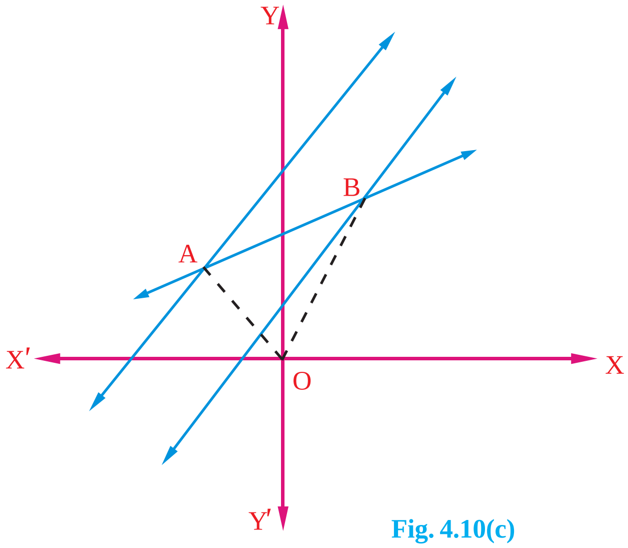


Fig. 4.10(c)

where $a' = a - \dfrac{2gl}{n} + \dfrac{cl^2}{n^2}$,

$h' = h - \dfrac{gm}{n} - \dfrac{fl}{n} + \dfrac{clm}{n}$,

$b' = b - \dfrac{2fm}{n} + \dfrac{cm^2}{n^2}$.

Since equation (3) is satisfied by both the (distinct) points A and B, it follows that $h'^2 \geq a'b'$.

Therefore, equation (3) represents the combined equation of $\overleftrightarrow{OA}$ and $\overleftrightarrow{OB}$.

4.6.1 Example : *Let us find the lines joining the origin to the points of intersection of the curve* $7x^2 - 4xy + 8y^2 + 2x - 4y - 8 = 0$ ... (1)

with the straight line $3x - y = 2$ *and also the angle between them.* ... (2)

Let the straight line $3x - y = 2$ meet the given curve in the points A and B. Then the equation of the lines $\overleftrightarrow{OA}$ and $\overleftrightarrow{OB}$ is obtained by homogenising the equation (1) with the help of equation (2).

From (2), $\dfrac{3x-y}{2} = 1$. Homogenising equation (1) with the above equation, we obtain

$$7x^2 - 4xy + 8y^2 + (x - 2y)(3x - y) - 2(3x - y)^2 = 0$$

that is, $-8x^2 + xy + 8y^2 = 0$ which is the equation of the pair of lines $\overleftrightarrow{OA}$ and $\overleftrightarrow{OB}$. Since the sum of the coefficients of x^2 and y^2 in this equation is zero, the lines $\overleftrightarrow{OA}$ and $\overleftrightarrow{OB}$ are mutually perpendicular.

Exercise 4(c)

I. 1. Find the equation of the lines joining the origin to the points of intersection of $x^2 + y^2 = 1$ and $x + y = 1$.

2. Find the angle between the lines joining the origin to the points of intersection of $y^2 = x$ and $x + y = 1$.

II. 1. Show that the lines joining the origin to the points of intersection of the curve $x^2 - xy + y^2 + 3x + 3y - 2 = 0$ and the straigth line $x - y - \sqrt{2} = 0$ are mutually perpendicular.

2. Find the values of k, if the lines joining the origin to the points of intersection of the curve $2x^2 - 2xy + 3y^2 + 2x - y - 1 = 0$ and the line $x + 2y = k$ are mutually perpendicular.

3. Find the angle between the lines joining the origin to the points of intersection of the curve $x^2 + 2xy + y^2 + 2x + 2y - 5 = 0$ and the line $3x - y + 1 = 0$.

III. 1. Find the condition for the chord $lx + my = 1$ of the circle $x^2 + y^2 = a^2$ (whose centre is the origin) to subtend a right angle at the origin.

2. Find the condition for the lines joining the origin to the points of intersection of the circle $x^2 + y^2 = a^2$ and the line $lx + my = 1$ to coincide.

3. Write down the equation of the pair of straight lines joining the origin to the points of intersection of the line $6x - y + 8 = 0$ with the pair of straight lines

$$3x^2 + 4xy - 4y^2 - 11x + 2y + 6 = 0.$$

Show that the lines so obtained make equal angles with the coordinate axes.

Key Concepts

- If a, b, h are not all zero, then the equation $H \equiv ax^2 + 2hxy + by^2 = 0$ represents a pair of straight lines if, and only if $h^2 \geq ab$.

- Let $ax^2 + 2hxy + by^2 = 0$ represent a pair of straight lines. Then the angle θ between them is given by

$$\cos\theta = \frac{a+b}{\sqrt{(a-b)^2 + 4h^2}}.$$

  The above lines are mutually perpendicular if and only if, $a + b = 0$.

- Let $ax^2 + 2hxy + by^2 = 0$ represent a pair of distinct straight lines. Then the equation of the pair of bisectors of the angles between these lines is $h(x^2 - y^2) = (a - b)xy$.

❖ Let $H \equiv ax^2 + 2hxy + by^2 = 0$ represent a pair of straight lines. Then

(i) the equation of the lines passing through the point (x_0, y_0) and parallel to the lines H = 0 is $a(x-x_0)^2 + 2h(x-x_0)(y-y_0) + b(y-y_0)^2 = 0$.

(ii) the equation of the lines passing through (x_0, y_0) and perpendicular to the lines H = 0 is $b(x-x_0)^2 - 2h(x-x_0)(y-y_0) + a(y-y_0)^2 = 0$.

❖ The general equation of second degree in x and y,

$S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of straight lines if and only if

(i) $\Delta \equiv abc + 2fgh - af^2 - bg^2 - ch^2 = 0$, and

(ii) $h^2 \geq ab, g^2 \geq ac, f^2 \geq bc.$

❖ Equation of the pair of straight lines passing through the origin and parallel to the pair of straight lines

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ is $ax^2 + 2hxy + by^2 = 0$.

❖ Let $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represent a pair of intersecting straight lines. Then their point of intersection is $\left(\dfrac{hf - bg}{ab - h^2}, \dfrac{gh - af}{ab - h^2}\right)$.

Historical Note

Coordinate Geometry gave us a means of classifying curves. All straight lines determine equations of first degree in x and y and all such equations determine straight lines. All equations of the second degree in x and y i.e., equations of the form.

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, coefficients being real numbers, determine curves including pairs of straight lines, which the ancient Greeks had studied and which result from cutting a solid circular cone or two equal cones with same axis, whose only point of contact is formed by the vertices. It may be noticed that long after these curves (circle, parabola, ellipse, hyperbola and pair of straight lines) were introduced as sections of a cone, Pappus (later part of 3rd century A.D.) disovered that they could be defined in a plane as loci of a point P moving in the plane subject to certain condition.

Answers

Exercise 4(a)

I. **1.** (i) $\text{Tan}^{-1}\left(\frac{1}{13}\right)$ (ii) $\frac{\pi}{4}$ (iii) 2α (iv) $\frac{\pi}{2}$

II. **2.** $\pm\frac{3\sqrt{3}}{2}$

5. $x^2+2xy\sec 2\alpha+y^2=0$ if $\alpha\neq\frac{\pi}{4}$ and $xy=0$ if $\alpha=\frac{\pi}{4}$

8. $(0, 0)$, $\left(-\frac{6}{5},-\frac{4}{5}\right)$, $\left(-\frac{11}{5},-\frac{9}{5}\right)$, $(-1, -1)$; $9x-11y=0$

9. (i) $\left(\frac{20}{9},-\frac{44}{9}\right)$, $\frac{56}{3}$ (ii) $(0, -4)$, 36

10. (i) $5x^2-13xy-6y^2-33x+14y+40=0$

(ii) $6x^2-13xy-5y^2-37x+16y+45=0$

11. $11x-3y+9=0$

12. $3x-y-9=0$

Exercise 4(b)

I. **1.** $\text{Cos}^{-1}\left(\frac{4}{\sqrt{65}}\right)$

II. **1.** $\left(-\frac{3}{5},-\frac{1}{5}\right)$

2. $k=4$, $\left(-\frac{5}{8},-\frac{1}{8}\right)$ and $\text{Cos}^{-1}\left(\frac{1}{\sqrt{5}}\right)$ (or) $k=-1$, $\left(-\frac{5}{7},\frac{1}{7}\right)$ and $\text{Cos}^{-1}\left(\frac{4}{\sqrt{65}}\right)$

3. $x+y-2=0$, $x-y+1=0$

5. (i) $\sqrt{\frac{2}{5}}$ (ii) $\frac{5}{2}$

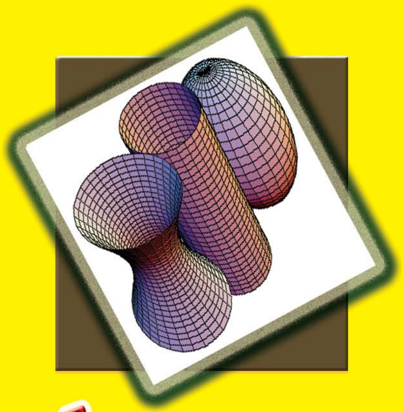
III. (i) $\frac{143}{25}$ (ii) $2\sqrt{5}$, 3

Exercise 4(c)

I. **1.** $xy=0$ **2.** 90^0

II. **2.** ± 1 **3.** $\text{Cos}^{-1}\left(\frac{13}{\sqrt{193}}\right)$

III. **1.** $a^2(l^2+m^2)=2$ **2.** $a^2(l^2+m^2)=1$ **3.** $4x^2-y^2=0$



Chapter 5

Three Dimensional Coordinates

"Geometry is the science of correct reasoning on incorrect figures "

- G. Polya

Introduction

Geometric shapes like spheres, cubes and cones do not exist in a single plane. These shapes require a third dimension to describe their location in space. To create this third dimension, a third axis is added to the co-ordinate system. Consequently, the location of each point in space is defined by three real numbers. Three dimensional geometry deals with geometry of solids like cone, sphere, and also planes, lines using algebraic equations. The study of analytical geometry is important because of its major applications.

In this chapter we learn how to determine the position of a point in space and the distance between two points. We derive a formula to find the coordinates of a point dividing a line segment in a certain ratio. As an application of this, we determine the coordinates of the centroid of a triangle and tetrahedron.

Pierre de Fermat

(1601 – 1665)

Fermat *was a French mathematician who is given credit for early developments that led to modern calculus. In particular, he is recognized for his discovery of an original method of finding the greatest and the smallest ordinates of curved lines, which is analogous to that of the then unknown differential calculus, as well as his research into the theory of numbers. He also made notable contributions to analytic geometry and probability.*

5.1 Coordinates

Let $\overleftrightarrow{X'OX}$, $\overleftrightarrow{Y'OY}$ be two mutually perpendicular straight lines passing through a fixed point 'O'. These two lines determine the XOY-plane or briefly XY-plane. Draw the line $\overleftrightarrow{Z'OZ}$ perpendicular to XY-plane and passing through O(this is unique). The fixed point O is called the *origin* and these three mutually perpendicular lines $\overleftrightarrow{X'OX}$, $\overleftrightarrow{Y'OY}$, $\overleftrightarrow{Z'OZ}$ are called *Rectangular Coordinate axes*. $\overrightarrow{OX}$, $\overrightarrow{OY}$, $\overrightarrow{OZ}$ are the positive directions of coordinate axes. In Figure 5.1, the positive directions of these axes are represented by arrow-heads.

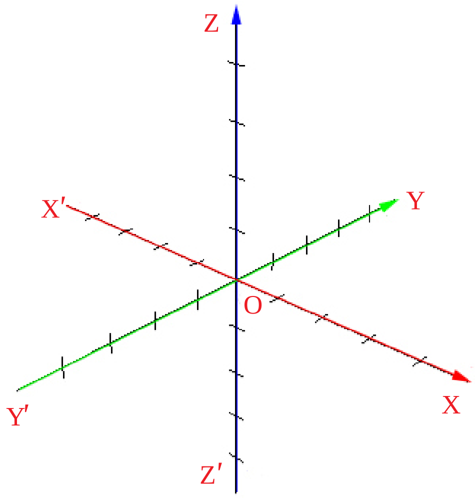


Fig. 5.1

The three coordinate axes taken two at a time determine three planes namely, XOY - plane, YOZ - plane and ZOX - plane or briefly XY, YZ, ZX-planes respectively. These planes are mutually perpendicular and they are called *Rectangular coordinate planes*. The triple of coordinate axes $\overleftrightarrow{X'OX}$, $\overleftrightarrow{Y'OY}$, $\overleftrightarrow{Z'OZ}$ are called the *rectangular frame of reference* and is written as OXYZ.

This frame of reference is said to be a *right handed system* if a right threaded screw advances in the direction of $\overrightarrow{OZ}$, when it is rotated from $\overrightarrow{OX}$ to $\overrightarrow{OY}$. Otherwise it is said to be a left handed system.

Given a point P in space other than O, through P, we can exactly draw three planes parallel to the coordinate planes so that they meet the axes $\overleftrightarrow{X'OX}$, $\overleftrightarrow{Y'OY}$ and $\overleftrightarrow{Z'OZ}$ in the points A, B, C respectively. Let x, y, z be real numbers such that $OA = |x|$, $OB = |y|$ and $OC = |z|$ and the signs of x, y, z are positive or negative according as A, B, C lie on the positive or negative directions of the axes. Then the real numbers x, y, z taken in this order are called the coordinates of P with respect to OXYZ. We write the coordinates of P as the ordered triad (x, y, z). The co-ordinates of the origin are (0, 0, 0).

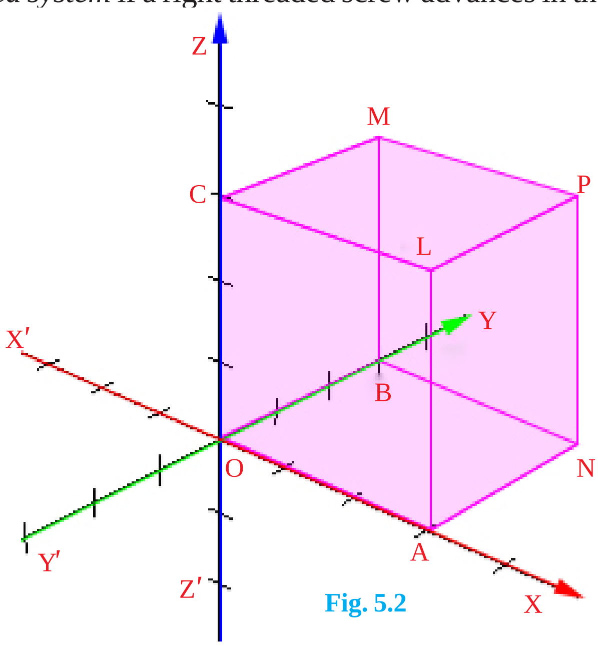


Fig. 5.2

Conversely, given an ordered triad of real numbers (x, y, z), we choose points A, B, C on the X, Y, Z - axes respectively so that $OA = |x|$, $OB = |y|$, $OC = |z|$. The positions of A, B, C on the positive or negative side of the axes are determined according as x, y, z are positive or negative respectively. Through

A, B, C draw planes parallel to YZ, ZX, XY planes respectively. These planes intersect at a unique point P in space. We observe that the coordinates of P are nothing but (x, y, z).

Thus, for every point P in space, we can associate an ordered triad (x, y, z) of real numbers formed by its coordinates and conversely, every ordered triad (x, y, z) of real numbers determines a unique point in space whose coordinates are (x, y, z). So we often refer to the triad (x, y, z) as the point P itself. The set of all points in space is referred to as *3-dimensional space* or $\mathbf{R}^3$ *-space*.

If P (x, y, z) is a point in space

x is called the X-coordinate of P

y is called the Y-coordinate of P

z is called the Z-coordinate of P.

5.1.1 Remark : Given a point P (x, y, z) othen than O in space, draw three planes PLCM, PLAN, PMBN parallel to XY, YZ, ZX planes respectively. (See Fig. 5.2). These three planes along with three coordinate planes constitute a rectangular parallelopiped. From Fig. 5.2, we have

$|x|$ = OA = CL = BN = MP = perpendicular distance of P from YZ - plane.

$|y|$ = OB = AN = CM = LP = perpendicular distance of P from ZX - plane.

$|z|$ = OC = BM = AL = NP = perpendicular distance of P from XY - plane.

$\therefore$ If the coordinates of P are (x, y, z), then its perpendicular distances from YZ, ZX, XY planes are $|x|, |y|, |z|$ respectively.

5.1.2 Remark : From Fig. 5.2, $\overline{OA}$ is perpendicular to the plane PLAN. So it is perpendicular to every line on the plane and in particular to $\overline{PA}$. that is $\overline{OA} \perp \overline{PA}$. Similarly, $\overline{OB} \perp \overline{PB}$ and $\overline{OC} \perp \overline{PC}$. Thus if the coordinates of P are (x, y, z), then $|x|, |y|, |z|$ are the perpendicular distances from the origin of the feet of the perpendiculars A, B, C from P to X, Y, Z-axes respectively.

5.1.3 Remark : From Fig. 5.2, NP = AL = OC = $|z|$,

AN = OB = $|y|$, OA = $|x|$.

Thus if $P(x, y, z)$ is a given point in space, from P draw PN perpendicular to the XY-plane meeting it at N. Draw NA parallel to Y-axis meeting OX at A. (see Fig. 5.3)

Then PN = $|z|$, NA = $|y|$, OA = $|x|$.

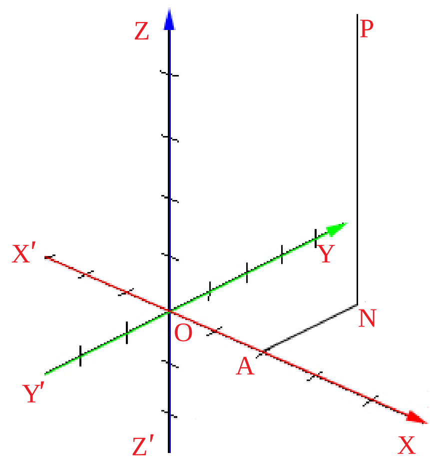


Fig. 5.3

5.1.4 Note

1. If a point $P(x, y, z)$ lies in the XY-plane then from Remark 5.1.1, $|z|$ = perpendicular distance of P from XY-plane = 0. i.e., $z = 0$.

$\therefore$ P is of the form $(x, y, 0)$.

Similarly, the coordinates of points in YZ and ZX planes may be taken as $(0, y, z)$ and $(x, 0, z)$ respectively.

2. If $P(x, y, z)$ lies on the X-axis, then its perpendicular distances from ZX and XY planes are zero.

So from Remark 5.1.1, $y = 0$, $z = 0$. Thus any point on X-axis is of the form $(x, 0, 0)$.

Similarly points on Y and Z-axes are of the form $(0, y, 0)$, $(0, 0, z)$ respectively.

5.1.5 Octants

The three coordinate planes divide the space into eight equal parts called Octants. The octant formed by the edges $\overrightarrow{OX}, \overrightarrow{OY}, \overrightarrow{OZ}$ is called the first octant. We write it as OXYZ. The octant whose bounding edges are OX, OY', OZ' is denoted by $OXY'Z'$. In a similar fashion the remaining six octants can be found. The following table shows the octants and the sign of coordinates in each octant.

Table 5.1

Octant	OXYZ	OX′YZ	OX′Y′Z	OXY′Z	OX′YZ′	OXYZ′	OXY′Z′	OX′Y′Z′
x - coordinates	+	–	–	+	–	+	+	–
y - coordinates	+	+	–	–	+	+	–	–
z - coordinates	+	+	+	+	–	–	–	–

5.1.6 Distance between two points in space

First we find the distance between the origin and any point in space. Using this we find the distance between any two points in space.

5.1.7 Theorem : *The distance between the origin 'O' and any point P (x, y, z) in space is* $OP = \sqrt{x^2 + y^2 + z^2}$.

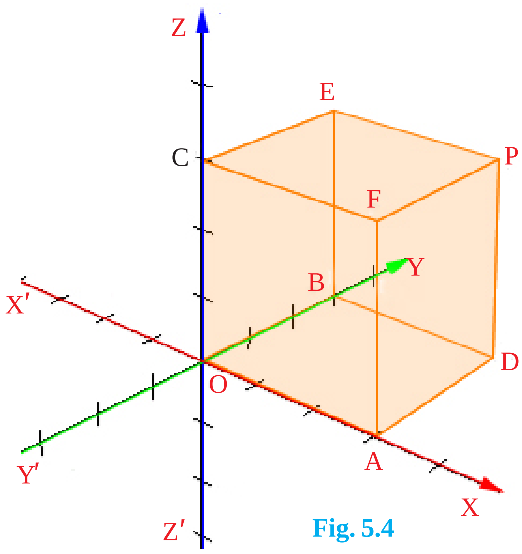


Fig. 5.4

Proof : We may assume $P \neq O$. Let the planes through P parallel to the coordinate planes intersect the X, Y, Z - axes respectively at A, B and C. (Fig. 5.4).

Since $\overline{PA} \perp \overline{OX}$,

in Δ OAP, $OP^2 = OA^2 + AP^2$.

Since $\overline{AP}$ is the diagonal of the rectangle PDAF,

$AP^2 = AF^2 + FP^2$.

From rectangle OAFC, AF = OC.

From rectangles PDAF and OBEC,

FP = AD = OB.

$OP^2 = OA^2 + AP^2 = OA^2 + AF^2 + FP^2 = OA^2 + OC^2 + OB^2$.

Since the coordinates of P are (x, y, z),

$$OA = |x|,\ OB = |y|,\ OC = |z|.$$

$$\therefore\quad OP^2 = x^2 + y^2 + z^2$$

$$\text{Hence}\quad OP = \sqrt{x^2 + y^2 + z^2}.$$

5.1.8 Note : Distance is a non negative number.

The distance of the point $(\sqrt{3}, 0, -1)$ from the origin is $\sqrt{3+0+1} = 2$.

5.1.9 Translation of axes

If we keep the direction of coordinate axes unchanged and shift the origin to some other point, the change is called translation of axes. The coordinates of a point in space change when the origin is shifted.

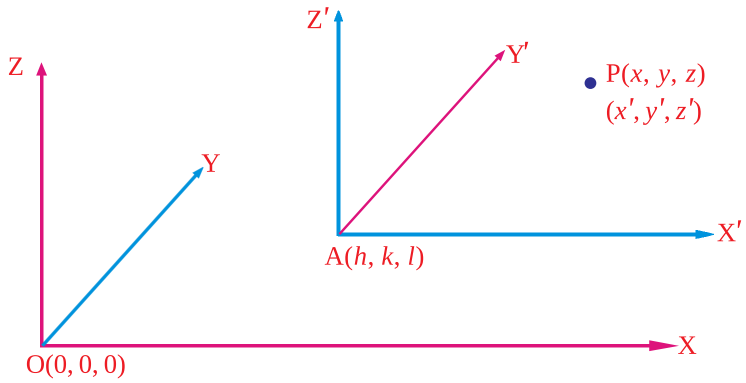


Fig. 5.5

Let P (x, y, z) and A (h, k, l) be two points in space with respect to the frame of reference OXYZ. Now treating A as the origin, let $\overrightarrow{AX'}, \overrightarrow{AY'}, \overrightarrow{AZ'}$ be the new axes parallel to $\overrightarrow{OX}, \overrightarrow{OY}, \overrightarrow{OZ}$ respectively (Fig. 5.5).

If (x', y', z') are coordinates of P with respect to AX′Y′Z′, then $x' = x - h,\ y' = y - k,\ z' = z - l$.

For example, suppose the coordinates of two points P and Q are (1, 2, –3) and (1, 0, –1) respectively, with reference to OXYZ frame. Shifting the origin to P, the new coordinates of Q are (1 – 1, 0 – 2, – 1 + 3) i.e., (0, –2, 2).

5.1.10 Theorem (Distance formula)

Distance between the points P (x_1, y_1, z_1) *and* Q (x_2, y_2, z_2) *is*

$$\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}\,.$$

Proof : Shifting the origin to P, the new coordinates of Q are $(x_2-x_1,\ y_2-y_1,\ z_2-z_1)$.

Using 5.1.7, Distance between P and Q = PQ

= Distance of Q from the new origin P

$$= \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$

Using the above formula, the distance between the points (1, –1, 1) and (3, –3, 2) is

$$PQ = \sqrt{(3-1)^2+(-3+1)^2+(2-1)^2} = \sqrt{4+4+1} = 3.$$

5.1.11 Note

1. Clearly QP = PQ.
2. The foot of the perpendicular from P(x, y, z) to X-axis is A$(x, 0, 0)$. Using 5.1.10 perpendicular distance of P from X - axis is

$$PA = \sqrt{(x-x)^2+(y-0)^2+(z-0)^2} = \sqrt{y^2+z^2}\,.$$

Similarly, perpendicular distances of P from Y-axis and Z-axis are $\sqrt{x^2+z^2}$ and $\sqrt{x^2+y^2}$ respectively.

5.1.12 Solved Problems

1. Problem : *Show that the points* A$(-4, 9, 6)$, B$(-1, 6, 6)$ *and* C$(0, 7, 10)$ *form a right angled isosceles triangle.*

Solution : Using distance formula (Theorem 5.1.10)

$$AB = \sqrt{(-1+4)^2+(6-9)^2+(6-6)^2} = \sqrt{9+9} = 3\sqrt{2}$$

$$BC = \sqrt{(0+1)^2+(7-6)^2+(10-6)^2} = \sqrt{1+1+16} = 3\sqrt{2}$$

$$AC = \sqrt{(0+4)^2+(7-9)^2+(10-6)^2} = \sqrt{16+4+16} = 6$$

$AB = BC = 3\sqrt{2}$

$\therefore$ The triangle is isosceles.

Also $AB^2+BC^2 = 18+18 = 36 = AC^2$ $\quad\therefore\ \underline{|B} = 90^0$

$\therefore$ Δ ABC is a right angled isosceles triangle.

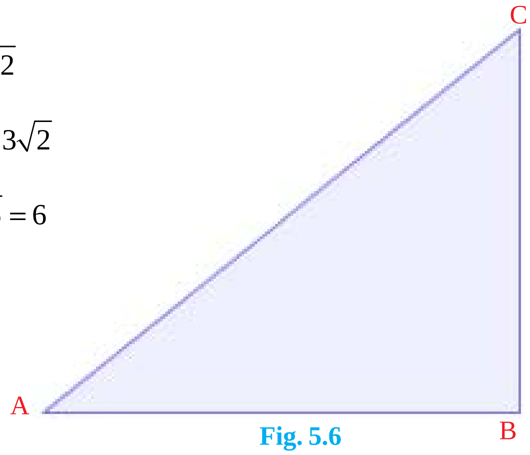


Fig. 5.6

2. Problem : *Show that locus of the point whose distance from Y-axis is thrice its distance from (1, 2, –1) is* $8x^2+9y^2+8z^2-18x-36y+18z+54=0$.

Solution

Let P (x, y, z) be any point on locus.

Distance of P from Y-axis = $\sqrt{x^2+z^2}$.

Distance of P from (1, 2 – 1) = $\sqrt{(x-1)^2+(y-2)^2+(z+1)^2}$.

Given that, $\sqrt{x^2+z^2}=3\sqrt{(x-1)^2+(y-2)^2+(z+1)^2}$

$$\Rightarrow x^2+z^2=9(x^2-2x+y^2-4y+z^2+2z+6)$$

$$\Rightarrow 8x^2+9y^2+8z^2-18x-36y+18z+54=0,$$

which is the required equation.

3. Problem : *A, B, C are three points on* $\overrightarrow{OX}$, $\overrightarrow{OY}$, $\overrightarrow{OZ}$ *respectively, at distances* a, b, c $(a \neq 0, b \neq 0, c \neq 0)$ *from the origin 'O'. Find the coordinates of the point which is equidistant from A, B, C and O.*

Solution : Let P (x, y, z) be the required point. The coordinates of A, B, C and O are $(a, 0, 0)$, $(0, b, 0)$, $(0, 0, c)$ and $(0, 0, 0)$ respectively.

Given that PA = PB = PC = PO.

$$\text{PA} = \text{PO} \Rightarrow \text{PA}^2=\text{PO}^2 \Rightarrow (x-a)^2+y^2+z^2=x^2+y^2+z^2$$

$$\Rightarrow a^2-2ax=0$$

$$\Rightarrow x=\frac{a}{2} \qquad (\because a \neq 0)$$

Similarly, we get $y=\frac{b}{2}$, $z=\frac{c}{2}$.

$\therefore$ $\text{P}=\left(\frac{a}{2},\frac{b}{2},\frac{c}{2}\right)$ is the point equidistant from A, B, C and O.

4. Problem : *Show that the points A(3, –2, 4), B(1, 1, 1) and C(–1, 4, –2) are collinear. (points are said to be collinear if they lie on the same line. See definition 5.2.1).*

Solution : By the distance formula,

$$\text{AB} = \sqrt{(1-3)^2+(1+2)^2+(1-4)^2}=\sqrt{4+9+9}=\sqrt{22}$$

$$\text{BC} = \sqrt{(-1-1)^2+(4-1)^2+(-2-1)^2}=\sqrt{4+9+9}=\sqrt{22}$$

and $$\text{AC} = \sqrt{(-1-3)^2+(4+2)^2+(-2-4)^2}=\sqrt{16+36+36}=\sqrt{88}.$$

Now, AB + BC = $2\sqrt{22}=\sqrt{88}$ = AC.

Therefore, A, B and C are collinear.

Exercise 5(a)

I. **1.** Find the distance of P(3, –2, 4) from the origin.

2. Find the distance between the points (3, 4, –2) and (1, 0, 7).

II. **1.** Find x if the distance between (5, –1, 7) and (x, 5, 1) is 9 units.

2. Show that the points (2, 3, 5), (–1, 5, –1,) and (4, –3, 2) form a right angled isosceles triangle.

3. Show that the points (1, 2, 3), (2, 3, 1) and (3, 1, 2) form an equilateral triangle.

4. P is a variable point which moves such that 3PA = 2PB. If A = (–2, 2, 3) and B = (13, –3, 13), prove that P satisfies the equation
$x^2+y^2+z^2+28x-12y+10z-247=0$.

5. Show that the points (1, 2, 3), (7, 0, 1) and (–2, 3, 4) are collinear.

6. Show that ABCD is a square where A, B, C, D are the points (0, 4, 1), (2, 3, –1), (4, 5, 0) and (2, 6, 2) respectively.

5.2 Section formula

Section formula gives the coordinates of a point that divides the line segment joining two given points in a given ratio. Using this we derive the coordinates of the centroid of a triangle and tetrahedron.

5.2.1 Definition

*If three or more points lie on the same line, they are said to be **collinear points**.*

5.2.2 Division of a line segment in space

Suppose A, B, P are three collinear points in space.

(i) If P lies on the segment $\overline{AB}$, we say P divides $\overline{AB}$ in the ratio AP : PB or P divides $\overline{AB}$ internally in the ratio AP : PB. (Fig. 5.7).

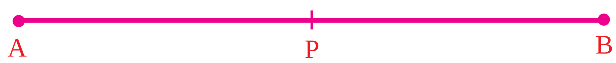


Fig. 5.7

(ii) If P lies on the line $\overleftrightarrow{AB}$ and outside the segment $\overline{AB}$, we say that P divides $\overline{AB}$ in the ratio - AP : PB (or AP : – PB) or P divides $\overline{AB}$ externally in the ratio AP : PB.

Fig. 5.8

5.2.3 Theorem (Section formula)

The point dividing the line segment joining the distinct points $A(x_1, y_1, z_1)$ *and* $B(x_2, y_2, z_2)$ *in the ratio* $m : n$ $(m+n \neq 0)$ *is given by*

$$\left(\frac{mx_2+nx_1}{m+n}, \frac{my_2+ny_1}{m+n}, \frac{mz_2+nz_1}{m+n}\right).$$

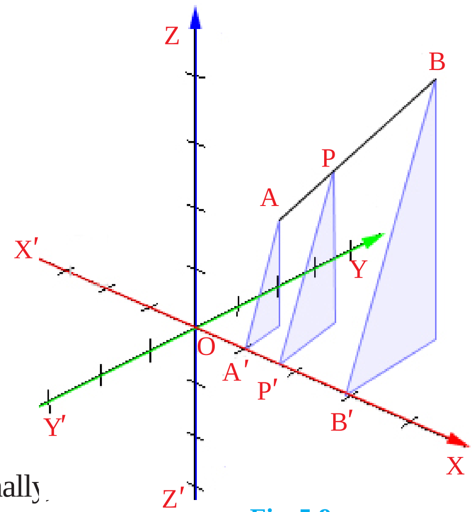


Fig. 5.9

Proof : Suppose P(x, y, z) divides $\overline{AB}$ in the ratio $m : n$. Draw planes through A, P, B parallel to the YZ - plane so as to meet $\overrightarrow{OX}$ in A′, P′, B′. Then, A′, P′, B′ are the feet of the perpendiculars of A, P, B on the X-axis (see Fig. 5.9).

$$\therefore \text{A}' = (x_1, 0, 0)$$

$$\text{P}' = (x, 0, 0) \text{ and } \text{B}' = (x_2, 0, 0).$$

Since parallel planes divide any two straight lines proportionally,

$$\frac{\overline{\text{A}'\text{P}'}}{\overline{\text{P}'\text{B}'}} = \frac{\overline{\text{AP}}}{\overline{\text{PB}}} = \frac{m}{n}$$

$$\Rightarrow \frac{m}{n} = \frac{x-x_1}{x_2-x} \quad \text{and so,} \quad x = \frac{mx_2+nx_1}{m+n}$$

Similarly, $$y = \frac{my_2+ny_1}{m+n}, \quad z = \frac{mz_2+nz_1}{m+n}.$$

$\therefore$ Coordinates of P are: $\left(\frac{mx_2+nx_1}{m+n}, \frac{my_2+ny_1}{m+n}, \frac{mz_2+nz_1}{m+n}\right)$.

5.2.4 Corollary

The midpoint of the segment $\overline{AB}$ where A = (x_1, y_1, z_1) and B = (x_2, y_2, z_2) is

$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right).$$

Proof: Since the mid point divides $\overline{AB}$ in the ratio $1 : 1$, taking $m = n = 1$ in Theorem 5.2.3, we get

The mid point is $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$.

5.2.5 Note

1. If $k \neq -1$ from 5.2.3, the point which divides $\overline{AB}$ in the ratio $k : 1$ is

$$\left(\frac{kx_2+x_1}{k+1}, \frac{ky_2+y_1}{k+1}, \frac{kz_2+z_1}{k+1}\right)$$

2. Further A = (x_1, y_1, z_1), B = (x_2, y_2, z_2) and C are collinear iff there exist $m, n, m \neq -n$ such that C = $\left(\frac{mx_2+nx_1}{m+n}, \frac{my_2+ny_1}{m+n}, \frac{mz_2+nz_1}{m+n}\right)$. In this case C divides $\overline{AB}$ in the ratio $m : n$.

5.2.6 Example

By section formula, the point which divides the line joining the points A(2, –3, 1) and B (3, 4, –5) in the ratio 1 : 3 is

$$\left(\frac{1\times3+3\times2}{1+3}, \frac{1\times4+3\times-3}{1+3}, \frac{1\times-5+3\times1}{1+3}\right)=\left(\frac{9}{4}, \frac{-5}{4}, \frac{-1}{2}\right).$$

5.2.7 Example

We can use section formula to find the ratio in which the line joining two points is divided by a given point on it.

Consider the points A(7, 0, –1), B(1, 2, 3) and C(–2, 3, 5). Suppose B divides $\overline{AC}$ in the ratio $k : 1$.

Then, by Note 5.2.5, B = $\left(\frac{k\times-2+1\times7}{k+1}, \frac{k\times3+1\times0}{k+1}, \frac{k\times5+1\times-1}{k+1}\right)$

$$= \left(\frac{7-2k}{k+1}, \frac{3k}{k+1}, \frac{5k-1}{k+1}\right)$$

But, B = (1, 2, 3).

$\therefore$ Equating the corresponding coordinates,

$$\frac{7-2k}{k+1} = 1, \quad \frac{3k}{k+1} = 2, \quad \frac{5k-1}{k+1} = 3.$$

Solving for k, we get $k = 2$.

$\therefore$ B Divides $\overline{AC}$ in the ratio 2 : 1

Note that B divides $\overline{AC}$ internally since the ratio is positive.

5.2.8 Example

Using section formula, we can verify whether the given points are collinear or not. Consider the points A(2, –4, 3), B(–4, 5, 6), C(4, –7, 2).

A, B, C are collinear iff C divides $\overline{AB}$ in some ratio say $m : n$. Then, the coordinates of C, according to section formula are $\left(\frac{-4m+2n}{m+n}, \frac{5m-4n}{m+n}, \frac{6m+3n}{m+n}\right)$.

But, C = (4, –7, 2).

Equating the corresponding coordinates we have

$$\frac{-4m+2n}{m+n} = 4, \quad \frac{5m-4n}{m+n} = -7, \quad \frac{6m+3n}{m+n} = 2.$$

From the above three relations we get a unique value $\frac{m}{n} = -\frac{1}{4}$.

So, we conclude that 'C' divides $\overline{AB}$ externally in the ratio 1 : 4.

$\therefore$ A, B, C are collinear.

5.2.9 Example

Let A, B, C be the points (5, 4, 6), (1, –1, 3) and (4, 3, 2) respectively. If these points are collinear, C must divide $\overline{AB}$ in some ratio say $m:n$. Then coordinates of C are $\left(\frac{m+5n}{m+n}, \frac{-m+4n}{m+n}, \frac{3m+6n}{m+n}\right)$. Since C is (4, 3, 2), equating the corresponding coordinates,

we get $\frac{m+5n}{m+n} = 4, \ \frac{-m+4n}{m+n} = 3, \ \frac{3m+6n}{m+n} = 2$.

These relations respectively give $\frac{m}{n} = \frac{1}{3}, \frac{1}{4}, \frac{-4}{1}$

we can see that there are no values of m and n that satisfy all the three equations simultaneously. So, we conclude that A, B, C are not collinear.

5.2.10 Definition

Three or more straight lines passing through a single point P *are called concurrent lines and the point* P *is called the point of concurrence.*

We know that in a triangle, the medians are concurrent and the point of concurrence is called its centroid. The centroid of a triangle trisects each median.

5.2.11 Theorem : *The centroid of a triangle whose vertices are* $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$, $C(x_3, y_3, z_3)$ is $\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}, \frac{z_1+z_2+z_3}{3}\right)$.

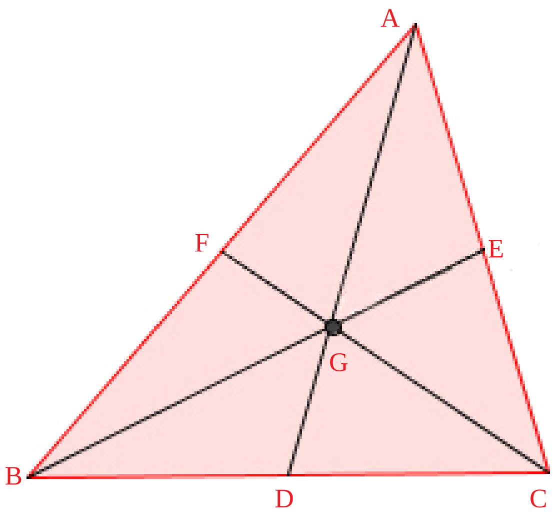


Fig. 5.10

Proof : Let D, E, F be the midpoints of the sides $\overline{BC}$, $\overline{AC}$, $\overline{AB}$ respectively. Then $\overline{AD}$, $\overline{BE}$, $\overline{CF}$ are medians of ΔABC. (see Fig. 5.10)

Since D is the mid point of $\overline{BC}$,

$$D = \left(\frac{x_2+x_3}{2}, \frac{y_2+y_3}{2}, \frac{z_2+z_3}{2}\right) \text{ by corollary 5.2.4.}$$

The centroid G divides $\overline{AD}$ in the ratio 2 : 1.

$$\therefore G = \left(\frac{x_1 + 2\left(\frac{x_2 + x_3}{2}\right)}{1+2}, \frac{y_1 + 2\left(\frac{y_2 + y_3}{2}\right)}{1+2}, \frac{z_1 + 2\left(\frac{z_2 + z_3}{2}\right)}{1+2} \right).$$

$$= \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_1 + z_2 + z_3}{3} \right).$$

5.2.12 Example : The centroid of the triangle whose vertices are (5, 4, 6), (1, –1, 3) and (4, 3, 2) is

$$\left(\frac{5+1+4}{3}, \frac{4-1+3}{3}, \frac{6+3+2}{3} \right) = \left(\frac{10}{3}, 2, \frac{11}{3} \right).$$

5.2.13 Tetrahedron

A tetrahedron is a closed figure formed by four planes not all passing through the same point. It has four vertices and six edges. Each vertex is obtained as the point of intersection of three planes. Each edge arises as the line of intersection of two of the four planes. If all edges of a tetrahedron are equal in length, it is called a *regular tetrahedron*.

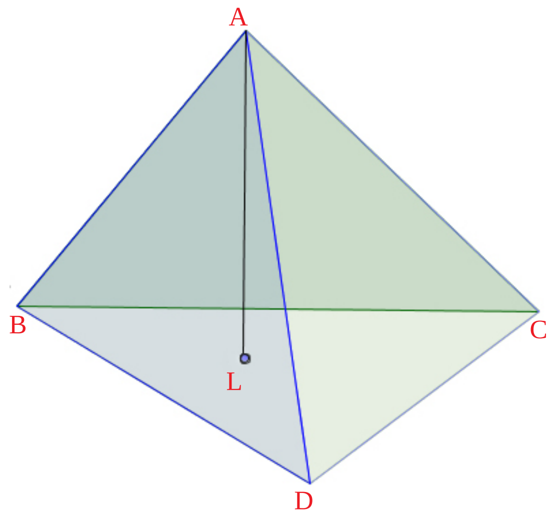


Fig. 5.11

In Fig. 5.11, A, B, C are three points and D is a point not lying in the plane of A, B, C. Now ABCD is a tetrahedron with vertices A, B, C, D. $\overline{AB}, \overline{AD}, \overline{AC}, \overline{BC}, \overline{BD}, \overline{CD}$ are its edges and $\Delta ABC, \Delta BCD, \Delta ACD$ and ΔABD are its faces. $\overline{AB}, \overline{CD}; \overline{BC}, \overline{AD}; \overline{CA}, \overline{DB}$ are called three pairs of opposite edges.

It is known that the line segments joining the vertices to the centroids of opposite faces are concurrent. The point of concurrence is called the centroid of the tetrahedron. It divides each line segment in the ratio 3 : 1.

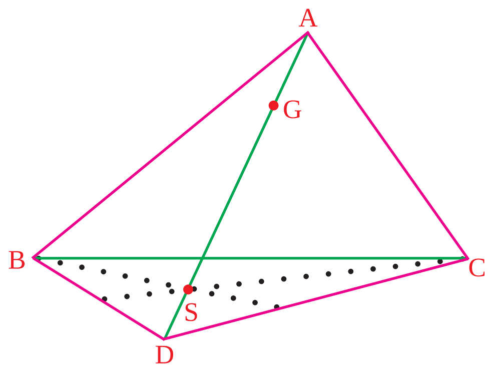


Fig. 5.12

5.2.14 Theorem : *The centroid of the tetrahedron whose vertices are* $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$, $C(x_3, y_3, z_3)$, $D(x_4, y_4, z_4)$ is

$$\left(\frac{x_1 + x_2 + x_3 + x_4}{4}, \frac{y_1 + y_2 + y_3 + y_4}{4}, \frac{z_1 + z_2 + z_3 + z_4}{4} \right).$$

Proof: Let S be the centroid of ΔBCD. Then by theorem 5.2.11

$$S = \left(\frac{x_2+x_3+x_4}{3}, \frac{y_2+y_3+y_4}{3}, \frac{z_2+z_3+z_4}{3}\right).$$

Let G be the centroid of the tetrahedron ABCD.

Then G divides $\overline{AS}$ in the ratio 3 : 1.

$$\therefore\ G = \left(\frac{\frac{3(x_2+x_3+x_4)}{3}+1.x_1}{3+1}, \frac{\frac{3(y_2+y_3+y_4)}{3}+1.y_1}{3+1}, \frac{\frac{3(z_2+z_3+z_4)}{3}+1.z_1}{3+1}\right).$$

$$= \left(\frac{x_1+x_2+x_3+x_4}{4}, \frac{y_1+y_2+y_3+y_4}{4}, \frac{z_1+z_2+z_3+z_4}{4}\right).$$

5.2.15 Example : The centroid of the tetrahedron whose vertices are (2, 3, –4), (–3, 3, –2), (–1, 4, 2), (3, 5, 1) is

$$\left(\frac{2-3-1+3}{4}, \frac{3+3+4+5}{4}, \frac{-4-2+2+1}{4}\right) = \left(\frac{1}{4}, \frac{15}{4}, \frac{-3}{4}\right).$$

5.2.16 Vector Method : The study of analytical geometry is so far confined to cartesian methods only. Though this gives a clear geometric and analytical picture of the situation, vector approach to 3D-geometry makes the study simpler and more elegant. Since the students are familiar with Vector Algebra, Vector methods are suggested for derivation of some formulae. According to convenience, either the classical method or the vector method may be used to solve the problems.

We know that if $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ are mutually orthogonal unit vectors along **OX, OY, OZ** in a right handed co-ordinate system, the position vector of a point P(x, y, z) in space with reference to the origin 'O' is given by $\mathbf{OP} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$.

Conversely, for every vector $x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$, there is a unique point P(x, y, z) in space whose position vector is $x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$. Thus there is a one-one correspondence between the set of points $\mathbf{R}^3$ and the set of position vectors. We identify the point (x, y, z) with its position vector $x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$.

Now it follows that

Distance of P (x, y, z) from origin O(0, 0, 0) is OP = $|\mathbf{OP}|$ = Magnitude of

$$\mathbf{OP} = |x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}| = \sqrt{x^2+y^2+z^2}.$$

Distance between the points A(x_1, y_1, z_1) and B(x_2, y_2, z_2) is AB = $|\mathbf{AB}|$ = Magnitude of

$$\mathbf{AB} = |\mathbf{OB} - \mathbf{OA}|$$

$$= |(x_2-x_1)\boldsymbol{i} + (y_2-y_1)\boldsymbol{j} + (z_2-z_1)\boldsymbol{k}|$$

$$= \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}.$$

Section formula

Suppose P(x, y, z) divides $\overline{AB}$ in the ratio $m:n$ where A(x_1, y_1, z_1) and B(x_2, y_2, z_2) are given points. Then A, P, B are collinear and $\dfrac{\text{AP}}{\text{PB}} = \dfrac{m}{n}$.

$$\therefore nAP = mPB$$

$$\Rightarrow n\,\mathbf{AP} = m\,\mathbf{PB}$$

$$\Rightarrow n\,[(x-x_1)\,\boldsymbol{i} + (y-y_1)\,\boldsymbol{j} + (z-z_1)\,\boldsymbol{k}] = m[(x_2-x)\,\boldsymbol{i} + (y_2-y)\,\boldsymbol{j} + (z_2-z)\,\boldsymbol{k}]$$

$$\Rightarrow n(x-x_1) = m(x_2-x),\ n(y-y_1) = m(y_2-y),\ n(z-z_1) = m(z_2-z)$$

$$\Rightarrow x = \frac{mx_2+nx_1}{m+n},\ y = \frac{my_2+ny_1}{m+n},\ z = \frac{mz_2+nz_1}{m+n}$$

$$\Rightarrow \text{P} = \left(\frac{mx_2+nx_1}{m+n}, \frac{my_2+ny_1}{m+n}, \frac{mz_2+nz_1}{m+n}\right).$$

5.3 Solved Problems

1. Problem : *Find the ratio in which YZ-plane divides the line joining A(2, 4, 5) and B(3, 5, –4). Also find the point of intersection.*

Solution : Suppose the line segment $\overline{AB}$ meets YZ-plane in P. Then, A, P, B are collinear. If P divides $\overline{AB}$ in the ratio $k:1$, then

$$\text{P} = \left(\frac{3k+2}{k+1}, \frac{5k+4}{k+1}, \frac{-4k+5}{k+1}\right)$$

Since P lies on the YZ plane, its X-coordinate is zero.

$$\therefore \frac{3k+2}{k+1} = 0 \Rightarrow k = \frac{-2}{3}.$$

Thus YZ plane divides $\overline{AB}$ in the ratio $\dfrac{-2}{3}:1$, i.e., in the ratio $-2:3$.

Substituting the value of k, the point of intersection P = (0, 2, 23).

2. Problem : *Show that the points A(3, –2, 4), B(1, 1, 1) and C(–1, 4, –2) are collinear.*

Solution : Suppose the point P divides $\overline{AB}$ in the ratio $k:1$.

$$\text{Then P} = \left(\frac{k+3}{k+1}, \frac{k-2}{k+1}, \frac{k+4}{k+1}\right) \quad \text{...(1)}$$

If C lies on $\overline{AB}$, then for some value of k, the coordinates of P must be same as those of C.

Equating the X-coordinates of P and C, $\dfrac{k+3}{k+1} = -1 \Rightarrow k = -2$.

Substituting $k = -2$ in (1)

$$\text{P} = (-1, 4, -2) = \text{C}.$$

$\therefore$ A, B, C are collinear.

3. Problem : *Find the fourth vertex of the parallelogram whose consecutive vertices are (2, 4, –1), (3, 6, –1) and (4, 5, 1).*

Solution : Let ABCD be the parallelogram where A = (2, 4, –1), B = (3, 6, – 1), C = (4, 5, 1) and D = (a, b, c).

Then, mid point of

AC = mid point of BD (see Fig. 5.13)

$$\Rightarrow \left(\frac{2+4}{2}, \frac{4+5}{2}, \frac{-1+1}{2}\right) = \left(\frac{3+a}{2}, \frac{6+b}{2}, \frac{-1+c}{2}\right)$$

$$\Rightarrow \frac{3+a}{2} = 3, \frac{6+b}{2} = \frac{9}{2}, \frac{-1+c}{2} = 0$$

$$\Rightarrow a = 3, b = 3, c = 1$$

$\therefore$ Fourth vertex D = (3, 3, 1).

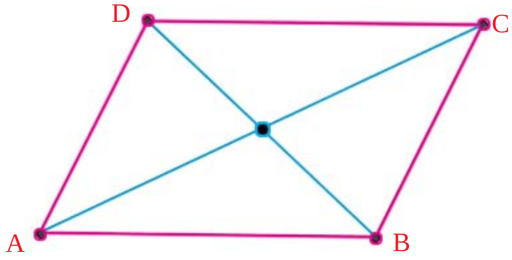


Fig. 5.13

4. Problem : A*(5, 4, 6),* B*(1, –1, 3),* C*(4, 3, 2) are three points. Find the coordinates of the point in which the bisector of* $\underline{|BAC}$ *meets the side* $\overline{BC}$.

Solution : We know that the bisector of $\underline{|BAC}$ divides $\overline{BC}$ in the ratio AB : AC. (see Fig. 5.14)

$$AB = \sqrt{(5-1)^2 + (4+1)^2 + (6-3)^2}$$
$$= 5\sqrt{2}$$
$$AC = \sqrt{(5-4)^2 + (4-3)^2 + (6-2)^2}$$
$$= 3\sqrt{2}$$

AB : AC = 5 : 3.

If D is the point where the bisector of $\underline{|BAC}$ meets $\overline{BC}$, then D divides $\overline{BC}$ in the ratio 5 : 3

$$\therefore D = \left(\frac{5\times4+3\times1}{5+3}, \frac{5\times3+3\times-1}{5+3}, \frac{5\times2+3\times3}{5+3}\right)$$
$$= \left(\frac{23}{8}, \frac{3}{2}, \frac{19}{8}\right).$$

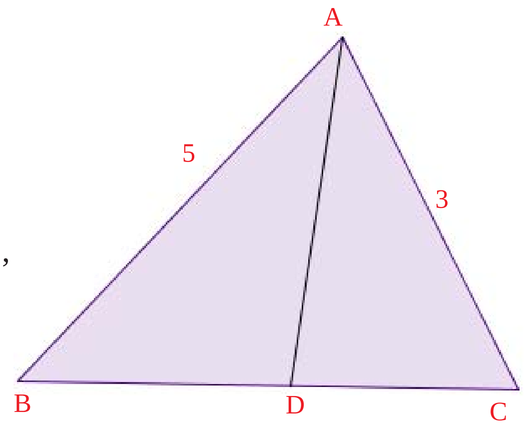


Fig. 5.14

5. Problem : *If* (x_1, y_1, z_1) and (x_2, y_2, z_2) *are two vertices and* (α, β, γ) *is the centroid of a triangle, find the third vertex of the triangle.*

Solution : Let A = (x_1, y_1, z_1) and B = (x_2, y_2, z_2) be the two vertices of the triangle ABC.

Let G = (α, β, γ) be the centroid.

If C = (x_3, y_3, z_3) is the third vertex, then we have

$$\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}, \frac{z_1+z_2+z_3}{3}\right) = (\alpha, \beta, \gamma).$$

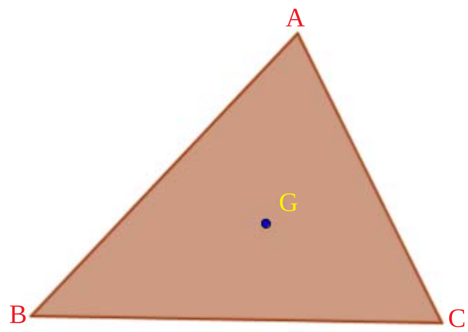


Fig. 5.15

$\Rightarrow\ x_1 + x_2 + x_3 = 3\alpha;\ y_1 + y_2 + y_3 = 3\beta;\ z_1 + z_2 + z_3 = 3\gamma.$

$\Rightarrow\ x_3 = 3\alpha - x_1 - x_2;\ y_3 = 3\beta - y_1 - y_2;\ z_3 = 3\gamma - z_1 - z_2.$

$\therefore$ The third vertex $C = (3\alpha - x_1 - x_2, 3\beta - y_1 - y_2, 3\gamma - z_1 - z_2)$.

6. Problem : *If* $D(x_1, y_1, z_1)$, $E(x_2, y_2, z_2)$ *and* $F(x_3, y_3, z_3)$ *are the midpoints of the sides BC, CA and AB respectively of a triangle, find its vertices A, B and C.*

Solution :

It is given that D is the mid point of the side BC, E is the mid point of the side CA and F is the mid point of the side AB. See Fig. 5.16.

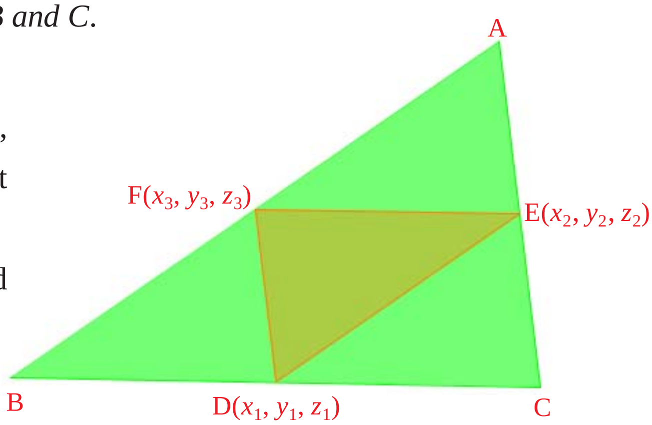


Fig. 5.16

$\therefore$ DEF is the triangle formed out of the mid points of the three sides.

Consider the parallelogram AEDF. Let $A = (h, k, s)$.

Mid point of AD = Mid point of EF

$$\Rightarrow \left(\frac{h+x_1}{2}, \frac{k+y_1}{2}, \frac{s+z_1}{2}\right) = \left(\frac{x_2+x_3}{2}, \frac{y_2+y_3}{2}, \frac{z_2+z_3}{2}\right)$$

$\Rightarrow\ h = x_2 + x_3 - x_1;\quad k = y_2 + y_3 - y_1;\ s = z_2 + z_3 - z_1.$

$\therefore$ Vertex A $= (x_2 + x_3 - x_1,\ y_2 + y_3 - y_1,\ z_2 + z_3 - z_1)$.

Similarly, the vertices B and C can be obtained as

$$B = (x_3 + x_1 - x_2,\ y_3 + y_1 - y_2,\ z_3 + z_1 - z_2)$$
$$C = (x_1 + x_2 - x_3,\ y_1 + y_2 - y_3,\ z_1 + z_2 - z_3).$$

7. Problem : If $M(\alpha, \beta, \gamma)$ *is the mid point of the line segment joining the points* $A(x_1, y_1, z_1)$ *and* B, *then find* B.

Solution :

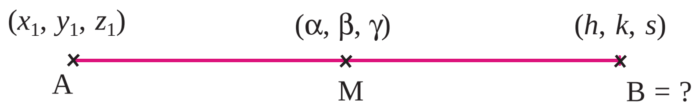


Fig. 5.17

Let $B(h, k, s)$ be the point required.

It is given that M is the mid point of AB.

$\therefore$ We have $(\alpha, \beta, \gamma) = \left(\dfrac{x_1+h}{2}, \dfrac{y_1+k}{2}, \dfrac{z_1+s}{2}\right)$.

$\Rightarrow\ 2\alpha = x_1 + h;\ 2\beta = y_1 + k;\ 2\gamma = z_1 + s$

$\Rightarrow\ h = 2\alpha - x_1;\quad k = 2\beta - y_1;\quad s = 2\gamma - z_1$

$\therefore$ Point B is $(2\alpha - x_1,\ 2\beta - y_1,\ 2\gamma - z_1)$.

8. Problem : *If H, G, S and I respectively denote orthocentre, centroid, circumcentre and in-centre of a triangle formed by the points* $(1, 2, 3)$, $(2, 3, 1)$ *and* $(3, 1, 2)$, *then find H, G, S, I.*

Solution :

AB = $\sqrt{(2-1)^2+(3-2)^2+(1-3)^2} = \sqrt{1+1+4} = \sqrt{6}$.

BC = $\sqrt{(3-2)^2+(1-3)^2+(2-1)^2} = \sqrt{1+4+1} = \sqrt{6}$.

CA = $\sqrt{(1-3)^2+(2-1)^2+(3-2)^2} = \sqrt{4+1+1} = \sqrt{6}$.

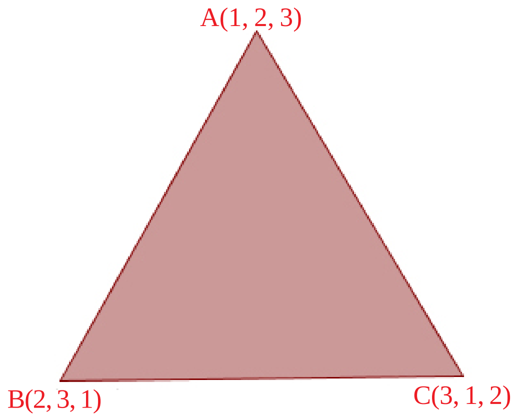


Fig. 5.18

Since AB = BC = CA, ABC is an equilateral triangle.

We know that orthocentre, centroid, circumcentre and incentre of an equilateral triangle are the same (i.e., all the four points coincide).

Now, centroid G $= \left(\dfrac{1+2+3}{3}, \dfrac{2+3+1}{3}, \dfrac{3+1+2}{3}\right)$

$= (2, 2, 2)$.

$\therefore$ H = (2, 2, 2), S = (2, 2, 2), I = (2, 2, 2).

9. Problem : *Find the incentre of the triangle formed by the points* (0, 0, 0), (3, 0, 0) *and* (0, 4, 0).

Solution : If a, b, c are the sides of the triangle ABC, where A = (x_1, y_1, z_1), B = (x_2, y_2, z_2), C = (x_3, y_3, z_3) are the vertices, then the in-centre of the triangle is given by

$$I = \left(\frac{ax_1+bx_2+cx_3}{a+b+c}, \frac{ay_1+by_2+cy_3}{a+b+c}, \frac{az_1+bz_2+cz_3}{a+b+c}\right).$$

Here, A = (0, 0, 0), B = (3, 0, 0), C = (0, 4, 0).

a = BC = $\sqrt{9+16+0}$ = 5;

b = CA = $\sqrt{0+16+0}$ = 4;

c = AB = $\sqrt{9+0+0}$ = 3.

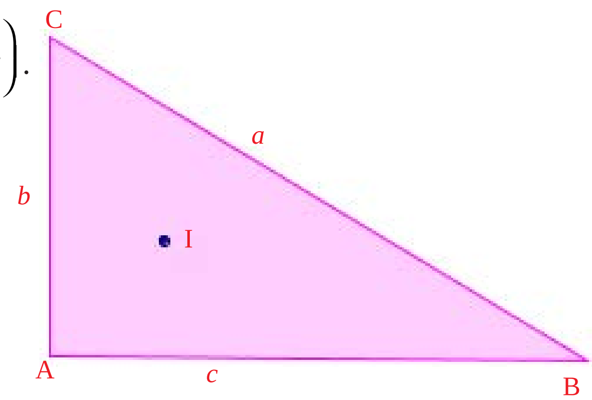


Fig. 5.19

$$\therefore I = \left(\frac{5(0)+4(3)+3(0)}{5+4+3}, \frac{5(0)+4(0)+3(4)}{5+4+3}, \frac{5(0)+4(0)+3(0)}{5+4+3}\right)$$

$= (1, 1, 0)$.

10. Problem : *If the point* (1, 2, 3) *is changed to the point* (2, 3, 1) *through translation of axes, find the new origin.*

Solution :

Let (x, y, z) be the co-ordinates of any point P w.r.t. the co-ordinate frame Oxyz and (X, Y, Z) be the co-ordinates of P w.r.t. the new frame of reference O'XYZ.

Let O'(h, k, s) be the new origin so that $x = X + h$, $y = Y + k$ and $z = Z + s$.

$\Rightarrow (h, k, s) = (x–X, y–Y, z– Z)$

$\Rightarrow (h, k, s) = (1– 2, 2–3, 3–1)$

$= (– 1, – 1, 2)$.

$\therefore$ O' = (–1, –1, 2) is the new origin.

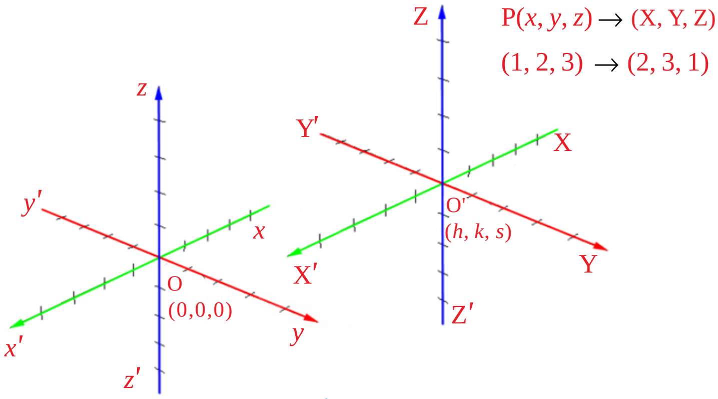


Fig. 5.20

11. Problem: *Find the ratio in which the point* P(5, 4, –6) *divides the line segment joining the points* A(3, 2,–4) *and* B(9, 8, –10). *Also, find the harmonic conjugate of P.*

Solution :

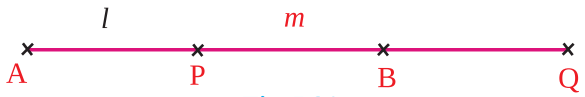


Fig. 5.21

Let P divide the line segment AB in the ratio $l : m$.

$\therefore$ We have $(5, 4, –6) = \left(\dfrac{9l+3m}{l+m}, \dfrac{8l+2m}{l+m}, \dfrac{-10l-4m}{l+m}\right)$.

$\Rightarrow l : m = 1 : 2$ or $2l = m$.

Now, Let Q divide AB in the ratio $l : –m$.

$$\text{Then} \quad Q = \left(\dfrac{9l-3m}{l-m}, \dfrac{8l-2m}{l-m}, \dfrac{-10l+4m}{l-m}\right)$$

$$= \left(\dfrac{9l-6l}{l-2l}, \dfrac{8l-4l}{l-2l}, \dfrac{-10l+8l}{l-2l}\right)$$

$$= (–3, –4, 2).$$

$\therefore$ Q(–3, –4, 2) is the harmonic conjugate of P(5, 4, –6).

Exercise 5(b)

I. **1.** Find the ratio in which the XZ-plane divides the line joining A(–2, 3, 4) and B(1, 2, 3).

2. Find the coordinates of the vertex 'C' of ΔABC if its centroid is the origin and the vertices A, B are (1, 1, 1) and (–2, 4, 1) respectively.

3. If (3, 2, –1), (4, 1, 1) and (6, 2, 5) are three vertices and (4, 2, 2) is the centroid of a tetrahedron, find the fourth vertex.

4. Find the distance between the mid point of the line segment $\overline{AB}$ and the point (3, –1, 2) where A = (6, 3, –4) and B = (–2, –1, 2).

II. **1.** Show that the points (5, 4, 2), (6, 2, –1) and (8, –2, –7) are collinear.

2. Show that the points A(3, 2, –4), B(5, 4, -6) and C(9, 8, –10) are collinear and find the ratio in which B divides $\overline{AC}$.

III. **1.** If A(4, 8, 12), B(2, 4, 6), C(3, 5, 4) and D(5, 8, 5) are four points, show that the lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ intersect.

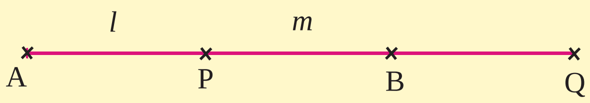
2. Find the point of intersection of the lines $\overleftrightarrow{AB}$ and $\overleftrightarrow{CD}$ where A = (7, –6, 1), B = (17, –18, –3), C = (1, 4, –5) and D = (3, –4, 11).

3. A(3, 2, 0), B(5, 3, 2), C(–9, 6, –3) are vertices of a triangle. $\overline{AD}$, the bisector of $\underline{|BAC}$ meets $\overline{BC}$ at D. Find the coordinates of D.

4. Show that the points O(0, 0, 0), A(2, –3, 3), B(–2, 3, –3) are collinear. Find the ratio in which each point divides the segment joining the other two.

Key Concepts

- Distance of P(*x*, *y*, *z*) from the origin = $\sqrt{x^2+y^2+z^2}$.
- The distance of the point P(*x*, *y*, *z*) from the *x*, *y* and *z* axes are respectively $\sqrt{y^2+z^2}$, $\sqrt{z^2+x^2}$ and $\sqrt{x^2+y^2}$.
- Distance between the points (x_1, y_1, z_1) and (x_2, y_2, z_2) is $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$.
- The point dividing the segment $\overline{AB}$ where A = (x_1, y_1, z_1), B = (x_2, y_2, z_2) in the ratio $m:n$ is $\left(\frac{mx_2+nx_1}{m+n}, \frac{my_2+ny_1}{m+n}, \frac{mz_2+nz_1}{m+n}\right)$.
- Point dividing the segment $\overline{AB}$ in $k:1$ ratio is $\left(\frac{kx_2+x_1}{k+1}, \frac{ky_2+y_1}{k+1}, \frac{kz_2+z_1}{k+1}\right)$.
- Mid point of $\overline{AB}$ is $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$.
- Centroid of a triangle with vertices (x_i, y_i, z_i), $i = 1, 2, 3$ is $\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}, \frac{z_1+z_2+z_3}{3}\right)$.

❖ 

AP : PB = $l : m$

AQ : QB = $l : -m$

P divides AB internally in the ratio $l : m$ whereas Q divides it externally in the same ratio. P and Q are harmonic conjugate points of A and B and vice-versa.

❖ Centroid of a tetrahedron with vertices (x_i, y_i, z_i), $i = 1, 2, 3, 4$ is :

$$\left(\frac{x_1+x_2+x_3+x_4}{4}, \frac{y_1+y_2+y_3+y_4}{4}, \frac{z_1+z_2+z_3+z_4}{4}\right).$$

Historical Note

Euclidean geometry is divided into two sub-sections: Plane geometry-dealing with figures in a plane and Solid geometry - dealing with solids such as sphere, cylinder, cone, polyhedra etc. in 3-dimensional space $\mathbf{R}^3$.

The basics of solid geometry are described by the Greek Mathematician *Euclid* (3rd century B.C.) in his book XI of "The Elements". This book covers definition of a solid, pyramid, sphere, cone, cube etc., inclination of figures in 3-dimensional space and similarity of figures. Many theorems related to these solids were proved.

Vachaspathi (850 A.D) anticipated in a rudimentary way the principle of coordinate solid geometry. According to him, the position of any simple atom in space with reference to another may be indicated with reference to three-axes. To conceive positions in space, *Vachaspati* takes three axes. The position of any point in space, relative to another point may now be given by measuring distances along these directions.

In 17th century *Rene Descartes* advanced solid geometry by inventing *cartesian* coordinates to express geometric relations, in algebraic form. *Pierre de Fermat* also made commendable contributions to the subject in its earlier days.

Answers

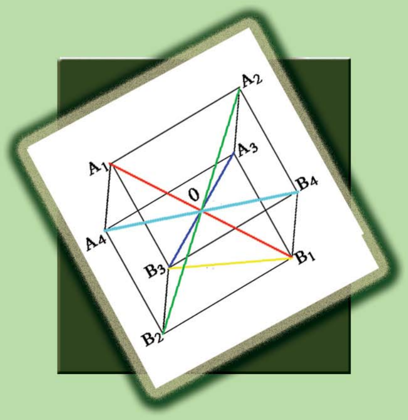
Exercise 5(a)

I. **1.** $\sqrt{29}$ **2.** $\sqrt{101}$ **II.** **1.** 8, 2.

Exercise 5(b)

I. **1.** $-3 : 2$ **2.** $(1, -5, -2)$ **3.** $(3, 3, 3)$ **4.** $\sqrt{14}$ **II.** **2.** $1 : 2$

III. **2.** $(2, 0, 3)$ **3.** $\left(\frac{38}{16}, \frac{57}{16}, \frac{17}{16}\right)$ **4.** $\frac{OA}{AB} = \frac{-1}{2}, \frac{AB}{BO} = \frac{-2}{1}, \frac{OB}{OA} = 1$



Chapter 6

Direction Cosines and Direction Ratios

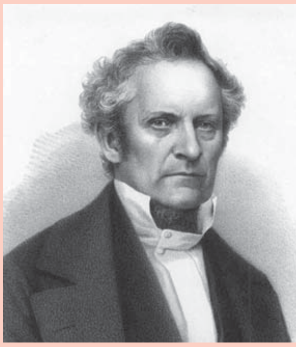
"Mathematics is the tool specially suited for dealing with abstract concept of any kind and there is no limit to its power in this field"

- P.A.M. Dirac

Introduction

Any two lines lying on the same plane are either parallel or intersecting. When two non-parallel lines on a plane meet at a point, an angle is formed and we know how to measure that angle. Some times we come across lines in space in space which are neither parallel nor intersecting. For example, the diagonal of the rectangle formed by the floor and the opposite diagonal of the rectangle formed by the roof of a room are two such lines, called skew lines. Measuring angle between such lines is very important.

In Analytical geometry of two dimensions the orientation of a line is given by slope. Whereas in 3-dimensional geometry it is measured in terms of direction cosines. In this chapter we learn about the direction cosines and direction ratios of a line and use them to derive a formula to find the angle between lines.



Julius Plucker

(1801 - 1868)

Julius Plucker *was a German Mathematician and Physicist. He made fundamental contributions to analytic and projective geometries. His first major work was"Analytisch-geometrische Eniwickelungen" published in two volumes, first in 1828 and the second in 1831. In each volume he discussed the plane, the line, circle and conic sections and many facts and theorems related to solid analytic geometry .*

6.1 Direction cosines

Consider a ray $\overrightarrow{OA}$ passing through O and making angles α, β, γ respectively with $\overrightarrow{OX}, \overrightarrow{OY}, \overrightarrow{OZ}$ (i.e., positive directions of X, Y, Z axes). The numbers $\cos\alpha, \cos\beta, \cos\gamma$ are called the **direction cosines (*d.c.'s*)** of the ray $\overrightarrow{OA}$. Usually they are denoted by (l, m, n) where $l = \cos\alpha$, $m = \cos\beta$ and $n = \cos\gamma$. (see Fig. 6.1)

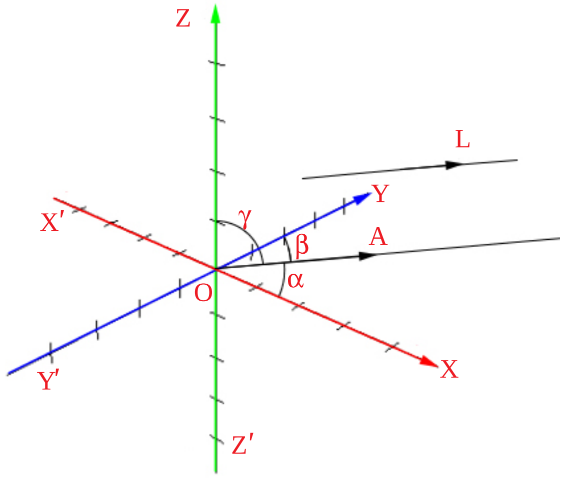


Fig. 6.1

By reversing the direction, we observe that the ray $\overrightarrow{AO}$ makes angles $\pi-\alpha, \pi-\beta, \pi-\gamma$ respectively with positive directions of X, Y, Z axes.

So $\cos(\pi-\alpha) = -\cos\alpha = -l$, $\cos(\pi-\beta) = -\cos\beta = -m$, $\cos(\pi-\gamma) = -\cos\gamma = -n$ are the *d.c.'s* of $\overrightarrow{AO}$. If L is a directed line in space, we draw a line parallel to L passing through origin so that the direction of this line is same as that of L. The direction cosines of L are defined as the direction cosines of this line through 'O'.

6.1.1 Note

1. Since a line in space has two directions, it has two sets of direction cosines, one for each direction. If (l, m, n) is one set of *d.c.'s* then $(-l, -m, -n)$ is the other set. So it is enough to mention any one set of *d.c.'s*, of a line.

2. It is clear from the definition and note (1) that if (l, m, n) are *d.c.'s* of a line then the *d.c.'s* of its parallel line L are $\pm(l, m, n)$.

6.1.2 Example : Since $\overrightarrow{OX}$ makes angles $0^0, 90^0, 90^0$ with $\overrightarrow{OX}, \overrightarrow{OY}, \overrightarrow{OZ}$ respectively, $\cos 0^0, \cos 90^0, \cos 90^0$ i.e., (1, 0, 0) are the *d.c.'s* of X-axis. Similarly *d.c.'s* of Y, Z-axes are (0, 1, 0), (0, 0, 1) respectively.

6.1.3 Theorem : *Suppose* P*(x, y, z) is any point in space such that* OP = *r.* *If (l, m, n) are d.c.'s of* $\overrightarrow{OP}$ *then* $x = lr$, $y = mr$, $z = nr$.

Proof: From P draw $\overline{PA}$ perpendicular to the X-axis. Let A be the foot of the perpendicular. Suppose $\overrightarrow{OP}$ makes angles α, β, γ with the positive directions of X, Y, Z axes respectively.

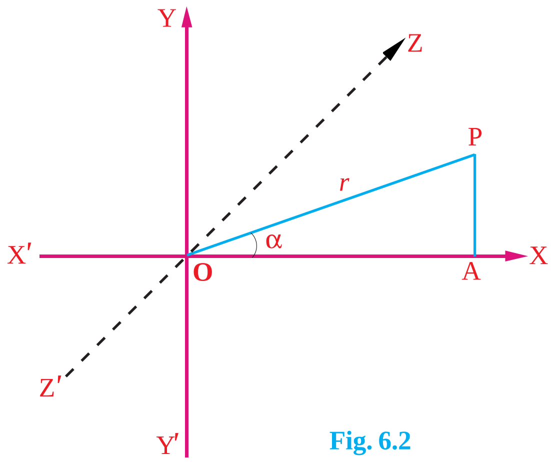


Fig. 6.2

In Fig. 6.2, ΔOAP is right angled.

Since A is the foot of the perpendicular from P to X-axis, A = (x, 0, 0).

If $x > 0$, A is on the positive side of X-axis.

$\therefore$ OA = x and $\cos\alpha = \dfrac{OA}{OP} = \dfrac{x}{r}$.

If $x < 0$, A is on the negative side of X-axis.

$\therefore$ OA = $-x$

$\therefore \cos(\pi - \alpha) = \dfrac{OA}{OP} = \dfrac{-x}{r} \Rightarrow \cos\alpha = \dfrac{x}{r}$.

Thus $l = \dfrac{x}{r}$ or $x = lr$.

Similarly, by dropping perpendiculars to Y and Z-axes respectively, we get $y = mr$, $z = nr$.

6.1.4 Note : If OP = r and *d.c.'s* of $\overrightarrow{OP}$ are (l, m, n) then the coordinates of P are (lr, mr, nr).

6.1.5 Example : Suppose P is a point in space such that OP = $\sqrt{3}$ and $\overrightarrow{OP}$ makes angles $\dfrac{\pi}{3}, \dfrac{\pi}{4}, \dfrac{\pi}{3}$ with $\overrightarrow{OX}$, $\overrightarrow{OY}$, $\overrightarrow{OZ}$ respectively.

Then *d.c.'s* of $\overrightarrow{OP}$ are: $\cos\dfrac{\pi}{3}, \cos\dfrac{\pi}{4}, \cos\dfrac{\pi}{3}$ i.e., $\left(\dfrac{1}{2}, \dfrac{1}{\sqrt{2}}, \dfrac{1}{2}\right)$.

$\therefore$ By 6.1.4 coordinates of P are $\left(\dfrac{\sqrt{3}}{2}, \dfrac{\sqrt{3}}{\sqrt{2}}, \dfrac{\sqrt{3}}{2}\right)$.

6.1.6 Corollary : If P(x, y, z) is a point in the space, then the *d.c.'s* of $\overrightarrow{OP}$ are

$$\left(\frac{x}{\sqrt{x^2+y^2+z^2}}, \frac{y}{\sqrt{x^2+y^2+z^2}}, \frac{z}{\sqrt{x^2+y^2+z^2}}\right).$$

Proof : If P = (x, y, z) then OP = r = $\sqrt{x^2+y^2+z^2}$.

By 6.1.3 *d.c.'s* of OP are $\left(\dfrac{x}{r}, \dfrac{y}{r}, \dfrac{z}{r}\right)$

i.e., $\left(\dfrac{x}{\sqrt{x^2+y^2+z^2}}, \dfrac{y}{\sqrt{x^2+y^2+z^2}}, \dfrac{z}{\sqrt{x^2+y^2+z^2}}\right)$.

6.1.7 Example : Consider the point P(2, 3, –1). By 6.1.6 direction cosines of $\overrightarrow{OP}$ are

$$\left(\frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}, \frac{-1}{\sqrt{14}}\right).$$

6.1.8 Corollary

If (l, m, n) are direction cosines of a line L, then $l^2 + m^2 + n^2 = 1$.

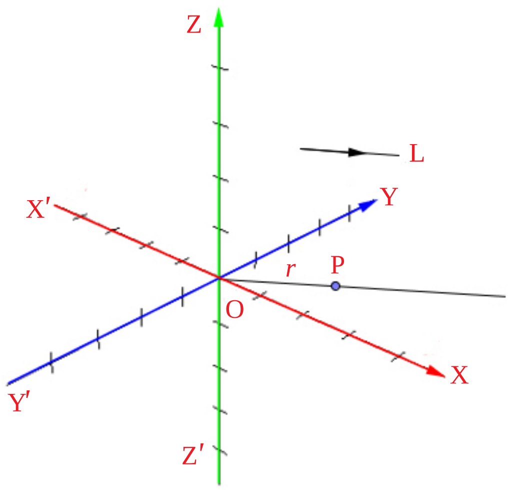


Fig. 6.3

Proof : Draw a line parallel to the given line and passing through 'O'. Let $P(x, y, z)$ be a point on this line such that $OP = r$. Then $r = \sqrt{x^2 + y^2 + z^2}$. By Theorem 6.1.3

$$x = \pm lr, \quad y = \pm mr, \quad z = \pm nr$$

where the sign should be taken positive or negative throughout, by Note 6.1.1.

Now $r^2 = x^2 + y^2 + z^2 = (l^2 + m^2 + n^2)r^2 \Rightarrow l^2 + m^2 + n^2 = 1$.

6.1.9 Example : We can not have a line whose direction cosines are $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ because $\left(\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{4}{3} \neq 1$.

6.1.10 Theorem : *The direction cosines of the directed line $\overrightarrow{PQ}$ joining the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are $\left(\frac{x_2 - x_1}{r}, \frac{y_2 - y_1}{r}, \frac{z_2 - z_1}{r}\right)$, where $r = \sqrt{\Sigma(x_2 - x_1)^2}$.*

Proof: Shifting the origin to $P(x_1, y_1, z_1)$ without changing the direction of axes, the coordinates of Q are $(x_2 - x_1, y_2 - y_1, z_2 - z_1)$. By corollary 6.1.6 d.c.'s of $\overrightarrow{PQ}$ are:

$$\left(\frac{x_2 - x_1}{\sqrt{\Sigma(x_2 - x_1)^2}}, \frac{y_2 - y_1}{\sqrt{\Sigma(x_2 - x_1)^2}}, \frac{z_2 - z_1}{\sqrt{\Sigma(x_2 - x_1)^2}}\right).$$

6.1.11 Solved Problems

1. Problem : *If P(2, 3, –6), Q(3, –4, 5) are two points, find the d.c.'s of $\overrightarrow{OP}$, $\overrightarrow{QO}$ and $\overrightarrow{PQ}$ where O is the origin.*

Solution : $OP = \sqrt{4 + 9 + 36} = 7$; $QO = \sqrt{9 + 16 + 25} = 5\sqrt{2}$.

$$PQ = \sqrt{(3-2)^2 + (-4-3)^2 + (5+6)^2} = \sqrt{1 + 49 + 121} = \sqrt{171}.$$

$\therefore$ *d.c.'s* of $\overrightarrow{OP}$ are : $\left(\frac{2}{7}, \frac{3}{7}, \frac{-6}{7}\right)$.

d.c.'s of $\overrightarrow{QO}$ are : $\left(\frac{0-3}{5\sqrt{2}}, \frac{0-(-4)}{5\sqrt{2}}, \frac{0-5}{5\sqrt{2}}\right)$ i.e., $\left(\frac{-3}{5\sqrt{2}}, \frac{4}{5\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$

d.c.'s of $\overrightarrow{PQ}$ are : $\left(\frac{3-2}{\sqrt{171}}, \frac{-4-3}{\sqrt{171}}, \frac{5+6}{\sqrt{171}}\right)$ i.e., $\left(\frac{1}{\sqrt{171}}, \frac{-7}{\sqrt{171}}, \frac{11}{\sqrt{171}}\right)$.

2. Problem : *Find the d.c.'s of a line that makes equal angles with the axes.*

Solution : Suppose the line makes an angle α with $\overrightarrow{OX}$. Since it makes equal angles with the axes, its *d.c.'s* are $(\cos\alpha, \cos\alpha, \cos\alpha)$.

But $\cos^2\alpha + \cos^2\alpha + \cos^2\alpha = 1$.

$$\Rightarrow 3\cos^2\alpha = 1 \Rightarrow \cos^2\alpha = \frac{1}{3} \Rightarrow \cos\alpha = \pm\frac{1}{\sqrt{3}}.$$

$\therefore$ *d.c.'s* of the line are: $\left(\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}}\right)$.

Hence there are eight such directions which reduce to four lines.

3. Problem : *If the d.c.'s of a line are* $\left(\frac{1}{c}, \frac{1}{c}, \frac{1}{c}\right)$ *find* c.

Solution : $\frac{1}{c^2} + \frac{1}{c^2} + \frac{1}{c^2} = 1 \Rightarrow \frac{3}{c^2} = 1 \Rightarrow c^2 = 3$

$$\Rightarrow c = \pm\sqrt{3}.$$

4. Problem : *Find the direction cosines of two lines which are connected by the relations* $l + m + n = 0$ *and* $mn - 2nl - 2lm = 0$.

Solution : Given that $l + m + n = 0$ ...(1)

and $mn - 2nl - 2lm = 0$ ...(2)

from (1) $l = -m - n$

Substituting in (2)

$$mn - 2n(-m-n) - 2m(-m-n) = 0$$

$$\Rightarrow mn + 2mn + 2n^2 + 2m^2 + 2mn = 0$$

$$\Rightarrow 2m^2 + 5mn + 2n^2 = 0$$

$$\Rightarrow (2m+n)(m+2n) = 0$$

$$\Rightarrow 2m + n = 0 \text{ or } m + 2n = 0$$

$$\Rightarrow \frac{m}{n} = \frac{-1}{2} \text{ or } \frac{m}{n} = \frac{-2}{1} \qquad ...(3)$$

from (1), $$\frac{l}{n} = \frac{-m}{n} - 1 \qquad ...(4)$$

When $\dfrac{m}{n} = \dfrac{-1}{2}$,

from (4), $\dfrac{l}{n} = \dfrac{1}{2} - 1 = \dfrac{-1}{2}$

$$\therefore \frac{m}{1} = \frac{l}{1} = \frac{n}{-2} = \frac{\sqrt{l^2+m^2+n^2}}{\sqrt{1^2+1^2+(-2)^2}} = \frac{1}{\sqrt{6}}.$$

$$\therefore\ l = \frac{1}{\sqrt{6}},\ m = \frac{1}{\sqrt{6}},\ n = \frac{-2}{\sqrt{6}}.$$

Again from (3) and (4) $\dfrac{m}{n} = -2$ gives $\dfrac{l}{n} = +2-1 = 1$.

$$\therefore \frac{l}{1} = \frac{m}{-2} = \frac{n}{1} = \frac{\sqrt{l^2+m^2+n^2}}{\sqrt{1+4+1}} = \frac{1}{\sqrt{6}}.$$

$$\therefore\ l = \frac{1}{\sqrt{6}},\ m = \frac{-2}{\sqrt{6}},\ n = \frac{1}{\sqrt{6}}$$

Thus the *d.c.'s* of the two lines are

$$\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}\right); \left(\frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right).$$

5. Problem : *A ray makes angles* $\dfrac{\pi}{3}, \dfrac{\pi}{3}$ *with* $\overrightarrow{OX}$ *and* $\overrightarrow{OY}$ *respectively. Find the angle made by it with* $\overrightarrow{OZ}$.

Solution : Let the angle made by the ray with $\overrightarrow{OZ}$ *be* γ.

d.c.'s of the ray are : $\left(\cos\dfrac{\pi}{3}, \cos\dfrac{\pi}{3}, \cos\gamma\right) = \left(\dfrac{1}{2}, \dfrac{1}{2}, \cos\gamma\right)$.

$$\frac{1}{4}+\frac{1}{4}+\cos^2\gamma = 1 \Rightarrow \cos^2\gamma = 1-\frac{1}{2} = \frac{1}{2} \Rightarrow \cos\gamma = \pm\frac{1}{\sqrt{2}}$$

$$\Rightarrow \gamma = \mathrm{Cos}^{-1}\left(\pm\frac{1}{\sqrt{2}}\right)$$

$$\Rightarrow \gamma = \frac{\pi}{4} \text{ or } \frac{3\pi}{4}.$$

Exercise 6(a)

I. 1. A line makes angles 90^0, 60^0 and 30^0 with the positive directions of X, Y, Z-axes respectively. Find its direction cosines.

2. If a line makes angles α, β, γ with the positive directions of X, Y, Z - axes, what is the value of $\sin^2\alpha + \sin^2\beta + \sin^2\gamma$?

3. If $P(\sqrt{3}, 1, 2\sqrt{3})$ is a point in space, find direction cosines of $\overrightarrow{OP}$.

4. Find the direction cosines of the line joining the points (–4, 1, 7) and (2, –3, 2).

II. 1. Find the direction cosines of the sides of the triangle whose vertices are (3, 5, –4), (–1, 1, 2) and (–5, –5, –2).

2. Show that the lines $\overleftrightarrow{PQ}$ and $\overleftrightarrow{RS}$ are parallel where P, Q, R, S are the points (2, 3, 4), (4, 7, 8), (–1, –2, 1) and (1, 2, 5) respectively.

III. 1. Find the direction cosines of two lines which are connected by the relations $l - 5m + 3n = 0$ and $7l^2 + 5m^2 - 3n^2 = 0$.

6.2 Direction ratios

Any three real numbers which are proportional to the direction cosines of a line are called *direction ratios* (*d.r.'s*) of that line.

If (a, b, c) are direction ratios of a line then for every $\lambda \neq 0$, $(\lambda a, \lambda b, \lambda c)$ are also its direction ratios. Thus a line may have infinite number of direction ratios.

6.2.1 Determining the direction cosines with given direction ratios

Let (a, b, c) be direction ratios of a line whose direction cosines are (l, m, n). Then (a, b, c) are proportional to (l, m, n).

$$\therefore \quad \frac{a}{l} = \frac{b}{m} = \frac{c}{n} = k \ (say)$$

$$\Rightarrow \quad a^2 + b^2 + c^2 = k^2(l^2 + m^2 + n^2) = k^2$$

$$\Rightarrow \quad k = \pm\sqrt{a^2 + b^2 + c^2}$$

$\therefore$ direction cosines of the line are :

$$(l, m, n) = \left(\frac{a}{k}, \frac{b}{k}, \frac{c}{k}\right) = \pm\left(\frac{a}{\sqrt{a^2 + b^2 + c^2}}, \frac{b}{\sqrt{a^2 + b^2 + c^2}}, \frac{c}{\sqrt{a^2 + b^2 + c^2}}\right).$$

6.2.2 Note

1. If (a, b, c) are direction ratios of a line, $a^2 + b^2 + c^2 \neq 1$ in general.
2. The directon cosines of a line are its direction ratios, but not vice versa.

6.2.3 Direction ratios of the line joining the points (x_1, y_1, z_1) and (x_2, y_2, z_2).

By Theorem 6.1.10, the direction cosines of the line joining (x_1, y_1, z_1) and (x_2, y_2, z_2) are

$$\left(\frac{x_2 - x_1}{\sqrt{\Sigma(x_2 - x_1)^2}}, \frac{y_2 - y_1}{\sqrt{\Sigma(x_2 - x_1)^2}}, \frac{z_2 - z_1}{\sqrt{\Sigma(x_2 - x_1)^2}}\right).$$

Since $(x_2 - x_1,\ y_2 - y_1,\ z_2 - z_1)$ are proportional to direction cosines of the line, they are direction ratios of the line.

6.2.4 Note : If $P(x, y, z)$ is a point in space, by corollary 6.1.6, direction cosines of $\overrightarrow{OP}$ are

$$\left(\frac{x}{\sqrt{x^2+y^2+z^2}}, \frac{y}{\sqrt{x^2+y^2+z^2}}, \frac{z}{\sqrt{x^2+y^2+z^2}}\right).$$

Since x, y, z are proportional to these values, direction ratios of $\overrightarrow{OP}$ are (x, y, z). Thus the coordinates of any point on a line through the origin may be taken are direction ratios of the line.

6.2.5 Example : If $P(-2, 4, -5)$ and $Q(1, 2, 3)$ are two points, direction ratios of line $\overleftrightarrow{PQ}$, are $(3, -2, 8)$.

Direction cosines of the line are $\left(\dfrac{3}{\sqrt{9+4+64}}, \dfrac{-2}{\sqrt{9+4+64}}, \dfrac{8}{\sqrt{9+4+64}}\right)$

i.e., $\left(\dfrac{3}{\sqrt{77}}, \dfrac{-2}{\sqrt{77}}, \dfrac{8}{\sqrt{77}}\right)$.

6.2.6 Angle between two lines

Let L_1, L_2 be two lines in space. Draw lines L_1', L_2' parallel to L_1, L_2 and passing through the origin. The angle between L_1' and L_2' which lies in $\left[0, \frac{\pi}{2}\right]$ is defined as the angle between L_1 and L_2.

6.2.7 Theorem : *If $(l_1, m_1, n_1), (l_2, m_2, n_2)$ are direction cosines of two lines, and θ is the angle between them, then $\cos\theta = |l_1 l_2 + m_1 m_2 + n_1 n_2|$.*

Proof : Let L_1, L_2 be the given lines with direction cosines (l_1, m_1, n_1) and (l_2, m_2, n_2) respectively.

Case (i) : If the lines L_1, L_2 are parallel then $\theta = 0^0$. So $\cos\theta = 1$.

From 6.1.1 Note (2),

$$l_2 = k\,l_1,\ m_2 = k\,m_1,\ n_2 = k\,n_1, \quad \text{where } k = \pm 1.$$

so that $\quad |l_1 l_2 + m_1 m_2 + n_1 n_2| = |l_1^2 + m_1^2 + n_1^2| = 1.$

$\therefore$ The result holds good in this case.

Case (ii) : Suppose L_1, L_2 are not parallel. Draw L_1', L_2' parallel to L_1 and L_2 and passing through the origin. Let A, B be points on L_1' and L_2' respectively at a distance of 1 unit from 'O'.

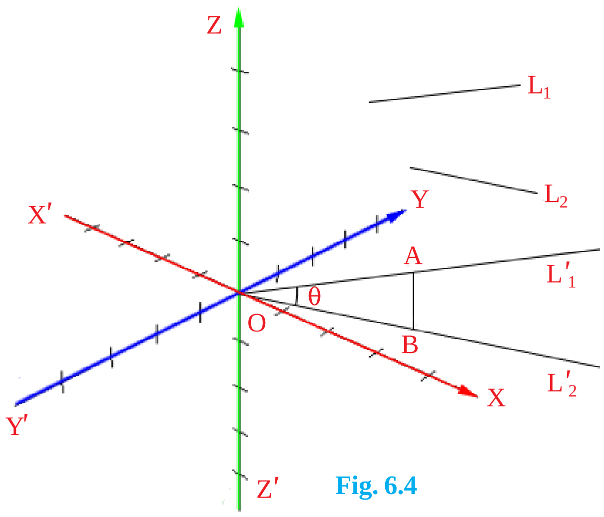


Fig. 6.4

Then $A = \pm\,(l_1, m_1, n_1)$ and

$$B = \pm\,(l_2, m_2, n_2)$$

$$\therefore \quad AB^2 = (l_1 - l_2)^2 + (m_1 - m_2)^2 + (n_1 - n_2)^2$$

$$\text{or} \quad (l_1 + l_2)^2 + (m_1 + m_2)^2 + (n_1 + n_2)^2$$

$$= (l_1^2 + m_1^2 + n_1^2) + (l_2^2 + m_2^2 + n_2^2) \pm 2(l_1l_2 + m_1m_2 + n_1n_2)$$

$$= 1 + 1 \pm 2(l_1l_2 + m_1m_2 + n_1n_2)$$

Using cosine rule, from ΔOAB,

$$\cos\theta = \frac{OA^2 + OB^2 - AB^2}{2OA.OB}$$

$$= \frac{1+1-[1+1 \pm 2(l_1l_2 + m_1m_2 + n_1n_2)]}{2} \qquad (\because\ OA = OB = 1)$$

$$= \pm\,(l_1l_2 + m_1m_2 + n_1n_2).$$

Since $\theta \in \left[0, \frac{\pi}{2}\right]$, $\cos\theta$ is non-negative.

$$\therefore \ \cos\theta = |\,l_1l_2 + m_1m_2 + n_1n_2\,|.$$

6.2.8 Note : If the lines are perpendicular, $\theta = \frac{\pi}{2}$, so $\cos\theta = 0$.

$$\therefore \text{ From } 6.2.7, \quad l_1l_2 + m_1m_2 + n_1n_2 = 0.$$

6.2.9 Lagrange's identity : For any two ordered triads of real numbers (a_1, b_1, c_1) and (a_2, b_2, c_2), then $(a_1^2 + b_1^2 + c_1^2)\,(a_2^2 + b_2^2 + c_2^2) - (a_1a_2 + b_1b_2 + c_1c_2)^2 = \Sigma(a_1b_2 - a_2b_1)^2$.

Notice that simplification and rearrangement of terms on the left yields the right side.

6.2.10 Note

1. If θ is the angle betwen two lines with direction cosines (l_1, m_1, n_1) and (l_2, m_2, n_2) then,

$$\sin^2\theta = 1-\cos^2\theta = (l_1^2+m_1^2+n_1^2)(l_2^2+m_2^2+n_1^2)-(l_1l_2+m_1m_2+n_1n_2)^2$$

$$= \Sigma(l_1m_2-m_1l_2)^2 \quad \text{(by Lagrange's identity)}$$

$$\therefore \sin\theta = \sqrt{\Sigma(l_1m_2-m_1l_2)^2}$$

and, $$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\sqrt{\Sigma(l_1m_2-m_1l_2)^2}}{|l_1l_2+m_1m_2+n_1n_2|} \quad \text{if } \theta \neq \frac{\pi}{2}.$$

2. If θ is the angle between the lines with direction ratios (a_1, b_1, c_1) and (a_2, b_2, c_2), then the direction cosines of the lines are

$$(l_1, m_1, n_1) = \pm\left(\frac{a_1}{\sqrt{a_1^2+b_1^2+c_1^2}}, \frac{b_1}{\sqrt{a_1^2+b_1^2+c_1^2}}, \frac{c_1}{\sqrt{a_1^2+b_1^2+c_1^2}}\right)$$

$$(l_2, m_2, n_2) = \pm\left(\frac{a_2}{\sqrt{a_2^2+b_2^2+c_2^2}}, \frac{b_2}{\sqrt{a_2^2+b_2^2+c_2^2}}, \frac{c_2}{\sqrt{a_2^2+b_2^2+c_2^2}}\right)$$

$\therefore$ from Theorem 6.2.7 and from above Note 6.2.10(1),

$$\cos\theta = \frac{|a_1a_2+b_1b_2+c_1c_2|}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$$

$$\tan\theta = \frac{\sqrt{\Sigma(a_1b_2-a_2b_1)^2}}{|a_1a_2+b_1b_2+c_1c_2|}.$$

3. If (a_1, b_1, c_1) and (a_2, b_2, c_2) are direction ratios of two lines and θ is the angle between them, then

(i) the lines are perpendicular $\Leftrightarrow \theta = \frac{\pi}{2} \Leftrightarrow \cos\theta = 0$

$\Leftrightarrow a_1a_2+b_1b_2+c_1c_2 = 0$ by 6.2.10 Note 2

(ii) the lines are parallel $\Leftrightarrow \theta = 0^0$

$\Leftrightarrow \Sigma(a_1b_2-a_2b_1)^2 = 0$ by 6.2.10 Note 2

$\Leftrightarrow a_1b_2-a_2b_1 = 0,\ b_1c_2-b_2c_1 = 0,$

$c_1a_2-c_2a_1 = 0$

$\Leftrightarrow \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}.$

6.2.11 Vector Method : If P (x, y, z) is a point in space such that $OP = r$ and (l, m, n) are the *d.c.'s* of **OP**, then from Theorem 6.1.3. we know that $x = lr$, $y = mr$, $z = nr$.

So, $\mathbf{OP} = x\,\boldsymbol{i} + y\,\boldsymbol{j} + z\,\boldsymbol{k} = lr\,\boldsymbol{i} + mr\,\boldsymbol{j} + nr\,\boldsymbol{k}$

A unit vector in the direction of **OP** is $\dfrac{\mathbf{OP}}{|\mathbf{OP}|} = \dfrac{\mathbf{OP}}{OP} = \dfrac{lr\,\boldsymbol{i} + mr\,\boldsymbol{j} + nr\,\boldsymbol{k}}{r}$

$= l\,\boldsymbol{i} + m\,\boldsymbol{j} + n\,\boldsymbol{k}$

Since this is a unit vector, $|l\,\boldsymbol{i} + m\,\boldsymbol{j} + n\,\boldsymbol{k}| = 1$

i.e., $l^2 + m^2 + n^2 = 1.$

Thus if (l, m, n) are direction cosines of line, a unit vector along the line is $l\,\boldsymbol{i} + m\,\boldsymbol{j} + n\,\boldsymbol{k}$ and $l^2 + m^2 + n^2 = 1.$

If (a, b, c) are the direction ratios of the line whose direction cosines are (l, m, n) then $a = \lambda l,\ b = \lambda m,\ c = \lambda n$ for some $\lambda \neq 0$.

$\therefore$ $\lambda(l\,\boldsymbol{i} + m\,\boldsymbol{j} + n\,\boldsymbol{k}) = a\,\boldsymbol{i} + b\boldsymbol{j} + c\,\boldsymbol{k}$ is a vector along the line.

Also notice that a unit vector along the line is $\dfrac{a\,\boldsymbol{i} + b\,\boldsymbol{j} + c\,\boldsymbol{k}}{|a\,\boldsymbol{i} + b\,\boldsymbol{j} + c\,\boldsymbol{k}|}$

$$= \frac{a\,\boldsymbol{i} + b\,\boldsymbol{j} + c\,\boldsymbol{k}}{\sqrt{a^2+b^2+c^2}} = \frac{a}{\sqrt{a^2+b^2+c^2}}\boldsymbol{i} + \frac{b}{\sqrt{a^2+b^2+c^2}}\boldsymbol{j} + \frac{c}{\sqrt{a^2+b^2+c^2}}\boldsymbol{k}$$

$\therefore$ The direction cosines of the line whose direction ratios are (a, b, c) are

$$\left(\frac{a}{\sqrt{a^2+b^2+c^2}}, \frac{b}{\sqrt{a^2+b^2+c^2}}, \frac{c}{\sqrt{a^2+b^2+c^2}}\right).$$

If P (x_1, y_1, z_1) and Q (x_2, y_2, z_2) are two points in space,

then $\mathbf{OP} = x_1\,\boldsymbol{i} + y_1\,\boldsymbol{j} + z_1\,\boldsymbol{k}$

$\mathbf{OQ} = x_2\,\boldsymbol{i} + y_2\,\boldsymbol{j} + z_2\,\boldsymbol{k}$

$\therefore$ $\mathbf{PQ} = \mathbf{OQ} - \mathbf{OP}$

$= (x_2 - x_1)\,\boldsymbol{i} + (y_2 - y_1)\,\boldsymbol{j} + (z_2 - z_1)\,\boldsymbol{k}$

$|\mathbf{PQ}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$.

$\therefore$ A unit vector along **PQ** is $\dfrac{x_2 - x_1}{\sqrt{\Sigma(x_2 - x_1)^2}}\boldsymbol{i} + \dfrac{y_2 - y_1}{\sqrt{\Sigma(x_2 - x_1)^2}}\boldsymbol{j} + \dfrac{z_2 - z_1}{\sqrt{\Sigma(x_2 - x_1)^2}}\boldsymbol{k}$.

Hence the direction cosines of the line joining P(x_1, y_1, z_1) and Q(x_2, y_2, z_2) are

$$\left(\frac{x_2 - x_1}{\sqrt{\Sigma(x_2 - x_1)^2}}, \frac{y_2 - y_1}{\sqrt{\Sigma(x_2 - x_1)^2}}, \frac{z_2 - z_1}{\sqrt{\Sigma(x_2 - x_1)^2}}\right).$$

Also the direction ratios of the line are $(x_2 - x_1,\ y_2 - y_1,\ z_2 - z_1)$.

Let $(l_1, m_1, n_1), (l_2, m_2, n_2)$ be the direction cosines of two lines and θ be the angle between the lines.

Unit vector along the line with $d.c.'s\ (l_1, m_1, n_1)$ is $\boldsymbol{a} = l_1\ \boldsymbol{i} + m_1\ \boldsymbol{j} + n_1\ \boldsymbol{k}$.

Unit vector along the line with $d.c.'s\ (l_2, m_2, n_2)$ is $\boldsymbol{b} = l_2\ \boldsymbol{i} + m_2\ \boldsymbol{j} + n_2\ \boldsymbol{k}$.

If θ is the angle between the lines, $\boldsymbol{a}.\boldsymbol{b} = |\boldsymbol{a}|\,|\boldsymbol{b}|\cos\theta$, by the definition of dot product.

$$\therefore (l_1\ \boldsymbol{i} + m_1\ \boldsymbol{j} + n_1\ \boldsymbol{k}) . (l_2\ \boldsymbol{i} + m_2\ \boldsymbol{j} + n_2\ \boldsymbol{k}) = 1.1.\cos\theta \qquad (\because |\boldsymbol{a}| = |\boldsymbol{b}| = 1)$$

$$\Rightarrow \cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2 .$$

Also $\boldsymbol{a}\times\boldsymbol{b} = |\boldsymbol{a}|\,|\boldsymbol{b}|\ \sin\theta\ \mathbf{N}$, where $\mathbf{N}$ is a unit vector perpendicular to both $\boldsymbol{a}$ and $\boldsymbol{b}$.

$$\therefore \boldsymbol{a}\times\boldsymbol{b} = \sin\theta\ \mathbf{N} \qquad (\because |\boldsymbol{a}| = |\boldsymbol{b}| = 1)$$

$$\Rightarrow (\boldsymbol{a}\times\boldsymbol{b})^2 = \sin^2\theta \qquad (\because\ \mathbf{N}^2 = \mathbf{N}.\mathbf{N} = 1)$$

But
$$\boldsymbol{a}\times\boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}$$

$$= (m_1 n_2 - m_2 n_1)\ \boldsymbol{i} + (n_1 l_2 - n_2 l_1)\ \boldsymbol{j} + (l_1 m_2 - m_1 l_2)\boldsymbol{k} .$$

$$\therefore \quad \sin^2\theta = (\boldsymbol{a}\times\boldsymbol{b})^2 = (\boldsymbol{a}\times\boldsymbol{b}).(\boldsymbol{a}\times\boldsymbol{b})$$
$$= \Sigma(m_1 n_2 - m_2 n_1)^2 .$$

Condition for perpendicularity

The lines are perpendicular

$$\Leftrightarrow \quad \boldsymbol{a}.\boldsymbol{b} = 0$$

$$\Leftrightarrow \quad (l_1\ \boldsymbol{i} + m_1\ \boldsymbol{j} + n_1\ \boldsymbol{k}).(l_2\ \boldsymbol{i} + m_2\ \boldsymbol{j} + n_2\ \boldsymbol{k}) = 0$$

$$\Leftrightarrow \quad l_1 l_2 + m_1 m_2 + n_1 n_2 = 0 .$$

Condition for parallelism

The lines are parallel $\Leftrightarrow$ the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are collinear.

$\Leftrightarrow$ $\boldsymbol{a} = \lambda\boldsymbol{b}$ for some real number $\lambda \neq 0$

$$\Leftrightarrow \quad (l_1\ \boldsymbol{i} + m_1\ \boldsymbol{j} + n_1\ \boldsymbol{k}) = \lambda(l_2\ \boldsymbol{i} + m_2\ \boldsymbol{j} + n_2\ \boldsymbol{k})$$

$$\Leftrightarrow \quad l_1 = \lambda l_2,\ m_1 = \lambda m_2,\ n_1 = \lambda n_2$$

$$\Leftrightarrow \quad \frac{l_1}{l_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2} .$$

Let $(a_1, b_1, c_1), (a_2, b_2, c_2)$ be the direction ratios of two lines and θ be the angle between them.

Then, a vector along the line with *d.r.'s* (a_1, b_1, c_1) is $\mathbf{A} = a_1\boldsymbol{i} + b_1\boldsymbol{j} + c_1\boldsymbol{k}$ and a vector along the line with *d.r.'s* (a_2, b_2, c_2) is $\mathbf{B} = a_2\boldsymbol{i} + b_2\boldsymbol{j} + c_2\boldsymbol{k}$.

$$|\mathbf{A}| = \sqrt{a_1^2 + b_1^2 + c_1^2}\ ; \quad |\mathbf{B}| = \sqrt{a_2^2 + b_2^2 + c_2^2}$$

Proceeding as above, we find that

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}.$$

The lines are perpendicular if $a_1a_2 + b_1b_2 + c_1c_2 = 0$.

The lines are parallel if $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$.

6.2.12 Solved Problems

1. Problem : *Find the d.r.'s and d.c.'s of the line joining the points (4, – 7, 3), (6, – 5, 2).*

Solution : A set of *d.r.'s* of the given line are: (6 – 4, – 5 + 7, 2 –3) i.e., (2, 2, – 1) (by 6.2.3)

$\therefore$ *d.c.'s* of the given line are $\pm\left(\dfrac{2}{\sqrt{4+4+1}}, \dfrac{2}{\sqrt{4+4+1}}, \dfrac{-1}{\sqrt{4+4+1}}\right)$

$$= \pm\left(\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}\right).$$

2. Problem : *If the d.c.'s of a line are proportional to (1, –2, 1) find its d.c.'s*

Solution : By 6.2.1 d.c.'s of the line are $\pm\left(\dfrac{1}{\sqrt{1+4+1}}, \dfrac{-2}{\sqrt{1+4+1}}, \dfrac{1}{\sqrt{1+4+1}}\right)$

i.e., $\pm\left(\dfrac{1}{\sqrt{6}}, \dfrac{-2}{\sqrt{6}}, \dfrac{1}{\sqrt{6}}\right)$.

3. Problem : *Show that the line joining the points P (0, 1, 2) and Q (3, 4, 8) is parallel to the line joining the points* $R\left(-2, \dfrac{3}{2}, -3\right)$ *and* $S\left(\dfrac{5}{2}, 6, 6\right)$.

Solution : *d.r.'s* of $\overline{PQ}$ = (3, 3, 6)

d.r.'s of $\overline{RS}$: $\left(\dfrac{5}{2}+2,\ 6-\dfrac{3}{2},\ 6+3\right)$

i.e., $\left(\dfrac{9}{2}, \dfrac{9}{2}, 9\right)$

Observe that $\dfrac{2}{3}\left(\dfrac{9}{2}, \dfrac{9}{2}, 9\right) = (3, 3, 6)$.

$\therefore$ *d.r.'s* of $\overline{PQ}$ are proportional to the *d.r.'s* of $\overline{RS}$.

$\therefore$ $\overline{PQ}$ is parallel to $\overline{RS}$ (by 6.2.10, Note 3).

4. Problem : *Show that the line joining the points A(2, 3, –1) and B(3, 5, –3) is perpendicular to the line joining C(1, 2, 3) and D(3, 5, 7).*

Solution : *d.r.'s* of $\overline{AB}$: (1, 2, –2)

d.r.'s of $\overline{CD}$: (2, 3, 4)

$2.1 + 2.3 - 2.4 = 0$

$\therefore$ The lines are perpendicular to each other (by 6.2.10 Note 3.).

5. Problem : *For what value of x the line joining A(4, 1, 2), B(5, x, 0) is perpendicular to the line joining C(1, 2, 3), D(3, 5, 7)?*

Solution : *d.r.'s* of $\overline{AB}$: $(1, x-1, -2)$

d.r.'s of $\overline{CD}$: (2, 3, 4)

If $\overline{AB} \perp \overline{CD}$, $2 + 3(x-1) - 8 = 0$

$$\Rightarrow x - 1 = \frac{6}{3} = 2 \Rightarrow x = 3.$$

6. Problem : *Show that the points A(1, 2, 3), B(4, 0, 4), C(–2, 4, 2) are collinear.*

Solution : *d.r.'s* of $\overline{AB}$ are : (3, –2, 1)

d.r.'s of $\overline{BC}$ are : $(-6, 4, -2) = -2(3, -2, 1)$.

Thus *d.r.'s* of $\overline{AB}$ and $\overline{BC}$ are proprotional. $\overleftrightarrow{AB}$, $\overleftrightarrow{BC}$ are parallel and B is a common point. So the points A, B, C lie on the same line i.e., they are collinear.

7. Problem : *A(1, 8, 4), B(0, –11, 4), C(2, –3, 1) are three points and D is the foot of the perpendicular from A to BC. Find the coordinates of D.*

Solution : Suppose D divides $\overline{BC}$ in the ratio $m : n$.

$$\text{Then} \quad D = \left(\frac{2m}{m+n}, \frac{-3m-11n}{m+n}, \frac{m+4n}{m+n}\right) \qquad ...(1)$$

Direction ratios of $\overline{AD}$: $\left(\frac{m-n}{m+n}, \frac{-11m-19n}{m+n}, \frac{-3m}{m+n}\right)$

Direction ratios of $\overline{BC}$: (2, 8, –3)

$$\overline{AD} \perp \overline{BC} \quad \Rightarrow 2\left(\frac{m-n}{m+n}\right) + 8\left(\frac{-11m-19n}{m+n}\right) - 3\left(\frac{-3m}{m+n}\right) = 0$$

$$\Rightarrow 2m - 2n - 88m - 152n + 9m = 0$$

$$\Rightarrow m = -2n.$$

Substituting in (1), D = (4, 5, –2).

8. Problem : *Lines* $\overleftrightarrow{OA}$, $\overleftrightarrow{OB}$ *are drawn from O with direction cosines proportional to (1, –2, –1); (3, –2, 3). Find the direction cosines of the normal to the plane AOB.*

Solution: Let (a, b, c) be the direction ratios of a normal to the plane AOB. Since $\overleftrightarrow{OA}$, $\overleftrightarrow{OB}$ lie on the plane, they are perpendicular to the normal to the plane. Using the condition of perpendicularity,

$$a.1 + b(-2) + c(-1) = 0 \text{ and} \quad \text{...(1)}$$

$$a.3 + b(-2) + c(3) = 0 \quad \text{...(2)}$$

Solving (1) and (2) $\dfrac{a}{-8} = \dfrac{b}{-6} = \dfrac{c}{4}$ or $\dfrac{a}{4} = \dfrac{b}{3} = \dfrac{c}{-2}$

$\therefore$ The *d.c.'s* of the normal are :

$$\left(\frac{4}{\sqrt{16+9+4}}, \frac{3}{\sqrt{16+9+4}}, \frac{-2}{\sqrt{16+9+4}}\right)$$

i.e., $\left(\dfrac{4}{\sqrt{29}}, \dfrac{3}{\sqrt{29}}, \dfrac{-2}{\sqrt{29}}\right)$.

9. Problem : *Find the angle between two diagonals of a cube.*

Solution : Let 'O', one of the vertices of the cube, be taken as the origin and the three coterminus edges $\overline{OA}$, $\overline{OB}$, $\overline{OC}$ as coordinate axes. Let OA = OB = OC = a. The four diagonals are $\overline{OF}$, $\overline{AG}$, $\overline{BE}$ and $\overline{DC}$. (Fig. 6.5)

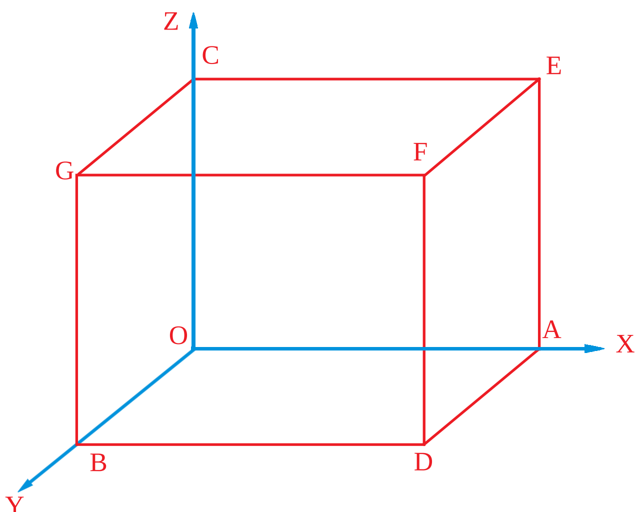


Fig. 6.5

The coordinates of the vertices of the cube are given by O(0, 0, 0), A(a, 0, 0), B(0, a, 0), C(0, 0, a), F(a, a, a), D(a, a, 0), E(a, 0, a), G(0, a, a).

The *d.r.'s* of diagonal $\overline{OF}$ are:

$(a – 0, a – 0, a – 0)$, i.e., (a, a, a).

The d.r.'s of diagonal $\overline{AG}$ are: $(– a, a, a)$.

If θ is the angle between $\overline{OF}$ and $\overline{AG}$, then

$$\cos\theta = \left|\frac{a(-a)+a(a)+a(a)}{\sqrt{a^2+a^2+a^2}\sqrt{a^2+a^2+a^2}}\right| = \frac{1}{3}$$

$$\therefore \theta = \text{Cos}^{-1}\left(\frac{1}{3}\right).$$

Similarly, the angle between any pair of diagonals can be found to be $\text{Cos}^{-1}\left(\dfrac{1}{3}\right)$.

10. Problem : *Show that the lines whose d.c.'s are proportional to* $(2, 1, 1), (4, \sqrt{3}-1, -\sqrt{3}-1)$ *are inclined to one another at angle* $\frac{\pi}{3}$.

Solution

The direction ratios of the given lines are $(2, 1, 1), \left(4, \sqrt{3}-1, -\sqrt{3}-1\right)$.

If θ is the angle between these lines,

$$\cos\theta = \left|\frac{2\times4+\sqrt{3}-1-\sqrt{3}-1}{\sqrt{4+1+1}\sqrt{16+(\sqrt{3}-1)^2+(-\sqrt{3}-1)^2}}\right| = \frac{6}{\sqrt{6}\sqrt{16+8}}$$

$$= \frac{6}{\sqrt{6}\sqrt{24}} = \frac{1}{2}$$

$$\therefore \theta = \text{Cos}^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}.$$

Exercise 6(b)

I. **1.** Find the direction ratios of the line joining the points (3, 4, 0) and (4, 4, 4).

2. The direction ratios of a line are (–6, 2, 3). Find its direction cosines.

3. Find the cosine of the angle between the lines whose direction cosines are $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ and $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$.

4. Find the angle between the lines whose directon ratios are (1, 1, 2), $\left(\sqrt{3}, -\sqrt{3}, 0\right)$.

5. Show that the lines with direction cosines $\left(\frac{12}{13}, \frac{-3}{13}, \frac{-4}{13}\right)$ and $\left(\frac{4}{13}, \frac{12}{13}, \frac{3}{13}\right)$ are perpendicular to each other.

6. O is the origin, P(2 ,3, 4) and Q(1, k, 1) are points such that $\overline{OP} \perp \overline{OQ}$. Find k.

II. **1.** If the direction ratios of a line are (3, 4, 0) find its direction cosines and also the angles made with the coordinate axes.

2. Show that the line through the points (1, –1, 2), (3, 4, –2) is perpendicular to the line through the points (0, 3, 2) and (3, 5, 6).

3. Find the angle between $\overline{DC}$ and $\overline{AB}$ where A = (3, 4, 5), B = (4, 6, 3), C = (–1, 2, 4) and D(1, 0, 5).

4. Find the direction cosines of a line which is perpendicular to the lines whose direction ratios are (1, –1, 2) and (2, 1, –1).

5. Show that the points (2, 3, –4), (1, –2, 3) and (3, 8, –11) are collinear.

6. Show that the points (4, 7, 8), (2, 3, 4), (–1, –2, 1), (1, 2, 5) are vertices of a parallelogram.

III. 1. Show that the lines whose *d.c.'s* are given by $l + m + n = 0$, $2mn + 3nl - 5lm = 0$ are perpendicular to each other.

2. Find the angle between the lines whose direction cosines satisfy the equations $l + m + n = 0$, $l^2 + m^2 - n^2 = 0$.

3. If a ray makes angles $\alpha, \beta, \gamma, \delta$ with the four diagonals of a cube find $\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta$.

4. If (l_1, m_1, n_1), (l_2, m_2, n_2) are *d.c.s* of two intersecting lines, show that *d.c.s* of two lines bisecting the angles between them are proportional to $l_1 \pm l_2$, $m_1 \pm m_2$, $n_1 \pm n_2$.

5. A(–1, 2, –3), B(5, 0, –6), C(0, 4, –1) are three points. Show that the direction cosines of the bisectors of $\underline{|BAC}$ are proportional to (25, 8, 5) and (–11, 20, 23).

6. If (6, 10, 10), (1, 0, –5), (6, –10, 0) are vertices of a triangle, find the direction ratios of its sides. Determine whether it is right angled or isosceles.

7. The vertices of a triangle are A(1, 4, 2), B(–2, 1, 2), C(2, 3, –4). Find $\underline{|A}$, $\underline{|B}$, $\underline{|C}$.

8. Find the angle between the lines whose direction cosines are given by the equations $3l + m + 5n = 0$ and $6mn - 2nl + 5lm = 0$.

9. If a variable line in two adjacent positions has direction cosines (l, m, n) and $(l + \delta l, m + \delta m, n + \delta n)$, show that the small angle $\delta\theta$ between the two positions is given by $(\delta\theta)^2 = (\delta l)^2 + (\delta m)^2 + (\delta n)^2$.

Key Concepts

- The direction cosines of a ray $\overrightarrow{OP}$ are $l = \cos\alpha,\ m = \cos\beta,\ n = \cos\gamma$ where α, β, γ are angles made by $\overrightarrow{OP}$ with positive directions of $\overrightarrow{OX}, \overrightarrow{OY}, \overrightarrow{OZ}$.
- If (l, m, n) are direction cosines of $\overrightarrow{OP}$ then direction cosines of $\overrightarrow{PO}$ are $(-l, -m, -n)$.
- The direction cosines of a line are the direction cosines of any line parallel to it and passing through the origin.
- Triad of numbers (a, b, c) proportional to direction cosines of a line are called its direction ratios.
- If (l, m, n) are direction cosines of a line, then $l^2 + m^2 + n^2 = 1$.
- Direction cosines of a line whose direction ratios are (a, b, c) are $\pm\left(\frac{a}{\sqrt{a^2+b^2+c^2}}, \frac{b}{\sqrt{a^2+b^2+c^2}}, \frac{c}{\sqrt{a^2+b^2+c^2}}\right)$.
- Direction ratios of the line joining (x_1, y_1, z_1) and (x_2, y_2, z_2) are $(x_2 - x_1, y_2 - y_1, z_2 - z_1)$ and its direction cosines are
$$\left(\frac{x_2 - x_1}{\sqrt{\Sigma(x_2 - x_1)^2}}, \frac{y_2 - y_1}{\sqrt{\Sigma(x_2 - x_1)^2}}, \frac{z_2 - z_1}{\sqrt{\Sigma(x_2 - x_1)^2}}\right).$$
- Angle between two lines in space is the angle between the lines parallel to the given lines and each passing through the origin.
- Angle between the lines whose direction cosines are (l_1, m_1, n_1) and (l_2, m_2, n_2) is $\mathrm{Cos}^{-1}|l_1l_2 + m_1m_2 + n_1n_2|$. The lines are perpendicular if $l_1l_2 + m_1m_2 + n_1n_2 = 0$

  The lines are parallel are equal if $l_1 = k\,l_2,\ m_1 = k\,m_2,\ n_1 = k\,n_2$, where $k = \pm 1$.
- Angle between the lines whose direction ratios are (a_1, b_1, c_1) and (a_2, b_2, c_2) is
$$\mathrm{Cos}^{-1}\left|\frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}\right|.$$

  The lines are perpendicular if $a_1a_2 + b_1b_2 + c_1c_2 = 0$.

  The lines are parallel if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$.

Historical Note

Hypatia (4th century A.D.) was distinguished in Mathematics, Medicine and Philosophy and is reported to have written commentaries on *Euclid's* elements, *Diophantus* Arithmetica and *Apollonius's* Conic sections. She is the first woman Mathematician to be mentioned in the history of Mathematics.

Books XII and XIII besides the Book XI of the Elements of *Euclid* concern themselves with solid geometry. Volumes are treated in Book XII and the five regular polyhedra (*Platonic* solids) are dealt with in Book XIII. *Julius Plucker* was a leader in the development of Modern geometry.

Answers

Exercise 6(a)

I. **1.** $\left(0, \frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ **2.** 2

3. $\left(\frac{\sqrt{3}}{4}, \frac{1}{4}, \frac{\sqrt{3}}{2}\right)$ **4.** $\left(\frac{6}{\sqrt{77}}, \frac{-4}{\sqrt{77}}, \frac{-5}{\sqrt{77}}\right)$

II. **1.** $\left(\frac{-2}{\sqrt{17}}, \frac{-2}{\sqrt{17}}, \frac{3}{\sqrt{17}}\right)$ $\left(\frac{-2}{\sqrt{17}}, \frac{-3}{\sqrt{17}}, \frac{-2}{\sqrt{17}}\right)$ $\left(\frac{4}{\sqrt{42}}, \frac{5}{\sqrt{42}}, \frac{-1}{\sqrt{42}}\right)$

III. **1.** $\left(\frac{-1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right)$ $\left(\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}\right)$

Exercise 6(b)

I. **1.** (1, 0, 4) **2.** $\left(\frac{-6}{7}, \frac{2}{7}, \frac{3}{7}\right)$

3. $\sqrt{\frac{2}{3}}$ **4.** $\frac{\pi}{2}$

6. $k = -2$

II. **1.** $\left(\frac{3}{5}, \frac{4}{5}, 0\right), \text{Cos}^{-1}\left(\frac{3}{5}\right), \text{Cos}^{-1}\left(\frac{4}{5}\right), \frac{\pi}{2}$

3. $\text{Cos}^{-1}\left(\frac{4}{9}\right)$

4. $\left(\frac{-1}{\sqrt{35}}, \frac{5}{\sqrt{35}}, \frac{3}{\sqrt{35}}\right)$

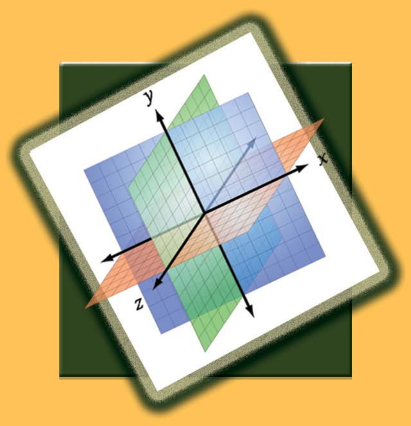
III. **2.** $\frac{\pi}{3}$.

3. $\frac{4}{3}$

6. $(-5, -10, -15)$, $(5, -10, 5)$, $(0, 20, 10)$ ΔABC is right angled

7. $\underline{|A} = \frac{\pi}{2}, \underline{|B} = \text{Cos}^{-1}\left(\frac{3}{2\sqrt{7}}\right), \underline{|C} = \text{Cos}^{-1}\left(\sqrt{\frac{19}{28}}\right)$

8. $\text{Cos}^{-1}\left(\frac{1}{6}\right)$



Chapter 7

The Plane

"Mathematicians do not study objects, but relations between objects"

- Jules Henri Poincare

Introduction

A plane is a proper subset of $\mathbf{R}^3$ which has at least three non-collinear points and is such that for any two points in it, the line joining them also lies in it.

Given a line L and a plane Π intersecting it at a point P, we say that L is normal or perpendicular to the plane at P if every line in the plane Π passing through P is perpendicular to L. Moreover every line that passes through P and perpendicular to L lies on the plane.

Poincare

(1854 - 1912)

Jules Henri Poincare *was born on 29th, April 1854 in France. He believed that the structure of the space can be known analytically. He wrote three books 'Science and Hypothesis', 'The value of Science' and 'Science and Method' that made his philosophies known. The famous 'Poincare conjucture' is one of the seven millennium prize problems. It is solved by Gregory Perelman in 2006.*

7.1 Cartesian equation of a plane - Simple illustrations

In this section we derive the equation of a plane in its general form. We also see various forms of equation of the plane. We determine the angle between two planes, using the concept of angle between lines.

7. 1.1 Equation of a plane under given conditions

Let us consider the following cases.

Case 1 : *Equation of the plane passing through a given point and perpendicular to a given line.*

Let $A(x_1, y_1, z_1)$ be a given point on the required plane, which is perpendicular to a given line L whose direction ratios are (a, b, c).

Let $P(x, y, z)$ be any point in the plane. Now d.r.'s of $\overline{AP}$ are $(x-x_1, y-y_1, z-z_1)$. Since A and P lie on the plane, $\overline{AP}$ lies on the plane. So, L and $\overline{AP}$ are perpendicular (Fig. 7.1(a)).

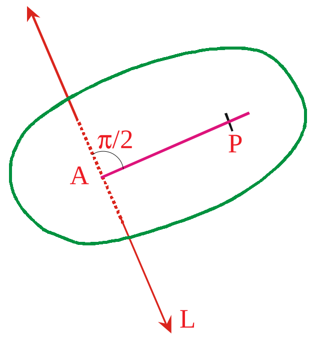


Fig. 7.1(a)

$\therefore \quad a(x-x_1)+b(y-y_1)+c(z-z_1)=0$ .... (1)

Since P is arbitrary, every point on the plane satisfies (1).

Conversely, every point satisfying (1) lies on the plane.

$\therefore$ (1) represents the equation of the required plane.

Case 2: *Equation of the plane when d.c.'s of the normal to the plane from origin and perpendicular distance from origin to the plane are given.*

Suppose the perpendicular drawn to the plane from origin 'O' meets the plane at P. Let $OP = p\,(>0)$.

Let (l, m, n) be the *d.c.'s* of the normal $\overline{OP}$. Hence atleast one of (l, m, n) is non-zero. Since $OP = p$, coordinates of P are (lp, mp, np).

Let R (x, y, z) be any point on the plane. Direction ratios of $\overline{PR}$ are:

$$(x-lp,\ y-mp,\ z-np).$$

Then, $OP \perp PR$ since $\overline{PR}$ lies on the plane. (See Fig. 7.1(b)).

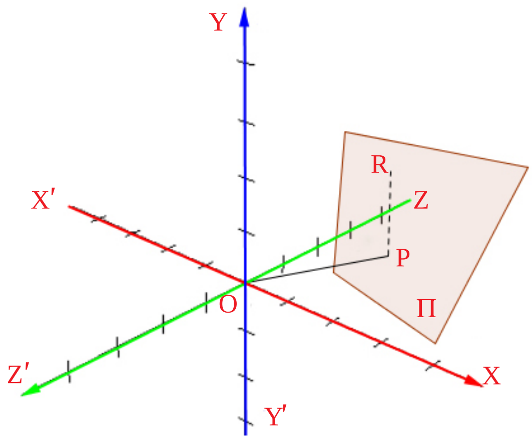


Fig. 7.1(b)

$$\therefore l(x-lp)+m(y-mp)+n(z-np)=0.$$

$$\Rightarrow lx+my+nz = p(l^2+m^2+n^2) = p. \quad \text{... (1)}$$

Since R is arbitrary, every point on the plane satisfies this equation of first degree.

Conversely, suppose A (x, y, z) is a point in space satisfying (1). Then $(x-pl,\ y-pm,\ z-pn)$ are direction ratios of $\overline{PA}$. $l(x-pl)+m(y-pm)+n(z-pn) = lx+my+nz-p(l^2+m^2+n^2)$

$= lx+my+nz-p = 0$ from (1)

$\therefore \quad \overline{PA} \perp \overline{OP}$.

$\therefore$ $\overline{PA}$ lies on the plane and so A is a point on the plane. Thus every point in space, satisfying (1) lies on the plane. Hence (1) represents the equation of the plane.

Case 3 : *Equation of the plane passing through three noncollinear points.*

Suppose the plane passes through three noncollinear points A (x_1, y_1, z_1), B(x_2, y_2, z_2) and C (x_3, y_3, z_3).

Since the plane passes through A, its equation may be taken as

$$a(x-x_1)+b(y-y_1)+c(z-z_1)=0 \quad \text{... (1)}$$

by case (1). Here (a, b, c) are *d.r.'s* of the normal to the plane.

Since B and C lie on this plane

$$a(x_2-x_1)+b(y_2-y_1)+c(z_2-z_1)=0 \quad \text{... (2)}$$

$$a(x_3-x_1)+b(y_3-y_1)+c(z_3-z_1)=0 \quad \text{... (3)}$$

Eliminating a, b, c from (1), (2) and (3) of case 3 above, the equation of the plane is obtained by expanding the determinant $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix}=0$.

7. 1.2 Note

1. The equation $lx+my+nz=p$ is called the *normal form*. Here (l, m, n) are the direction cosines of the normal to the plane from origin 'O' and $p\,(>0)$ is the perpendicular distance to the plane from origin.
2. The equation of the plane in case 3 can also be obtained by solving the simultaneous equations (1), (2) and (3) by taking two equations at a time.
3. Plane passing through the points A, B, C is denoted by $\overleftrightarrow{ABC}$.

7. 1.3 Theorem (General form of the equation of a plane): *Every equation of first degree in x, y, z represents a plane.*

Proof: Consider the general first degree equation in x, y, z as $ax+by+cz+d=0$, ... (1)

where a, b, c are real numbers, and at least one of a, b, c is non-zero.

Then there are at least three non collinear points satisfying (1).

Let A (x_1, y_1, z_1) and B (x_2, y_2, z_2) be two distinct points on the surface represented by (1).

Then $ax_1+by_1+cz_1+d=0$ ... (2)

$ax_2+by_2+cz_2+d=0$ ... (3)

Let 'P' be any point on the line segement $\overline{AB}$ and suppose 'P' divides $\overline{AB}$ in the ratio $m:n$.

Then $$P = \left(\frac{mx_2+nx_1}{m+n}, \frac{my_2+ny_1}{m+n}, \frac{mz_2+nz_1}{m+n}\right)$$

Multiplying (2) with n, (3) with m and adding

$$a(mx_2 + nx_1) + b(my_2 + ny_1) + c(mz_2 + nz_1) + d(m + n) = 0$$

$$\Rightarrow \; a\left(\frac{mx_2+nx_1}{m+n}\right)+b\left(\frac{my_2+ny_1}{m+n}\right)+c\left(\frac{mz_2+nz_1}{m+n}\right)+d=0.$$

This shows that 'P' lies on the surface given by (1). Since P is an arbitrary point on $\overline{AB}$, every point on $\overline{AB}$ lies on the surface represented by (1) and since there are at least three non-collinear points satisfying (1), it follows that the surface is a plane.

7.1.4 Note

1. $ax + by + cz + d = 0$ is called the *general form* of the equation of a plane.
2. In 7.1.3 we observe that direction ratios of $\overline{AB}$ are $(x_2 - x_1,\ y_2 - y_1,\ z_2 - z_1)$.
   From equations (2) and (3) above, we get
   $$a(x_2 - x_1) + b(y_2 - y_1) + c(z_2 - z_1) = 0.$$
   $\therefore$ The line with direction ratios (a, b, c) is perpendicular to $\overline{AB}$. But $\overline{AB}$ is any line on the plane. Therefore, the line with direction ratios (a, b, c) is perpendicular to any line on the plane. Hence it is normal to the plane.
3. If the point (x_1, y_1, z_1) lies on the plane, then $ax_1 + by_1 + cz_1 + d = 0$.
   $$\therefore\ d = -(ax_1 + by_1 + cz_1).$$
   $\therefore$ Equation of plane passing through (x_1, y_1, z_1) is given by
   $$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0.$$

7.1.5 Reduction of general form of the equation of a plane to its normal form

The general equation of the plane is

$$ax + by + cz + d = 0 \qquad \text{... (1)}$$

with at least one of a, b, c is non-zero.

Let (l, m, n) be the direction cosines of the normal to the plane and p be the perpendicular distance of the plane from 'O'.

Then, the equation of the plane in normal form is

$$lx + my + nz = p \qquad \text{... (2)}$$

Since (1) and (2) represent the same plane

$$\frac{l}{a}=\frac{m}{b}=\frac{n}{c}=\frac{p}{-d}=\pm\frac{\sqrt{l^2+m^2+n^2}}{\sqrt{a^2+b^2+c^2}}$$

$$=\pm\frac{1}{\sqrt{a^2+b^2+c^2}} \qquad (\because \; l,\; m,\; n \text{ are } d.c.'s \; l^2+m^2+n^2=1)$$

$$\therefore \quad l=\pm\frac{a}{\sqrt{a^2+b^2+c^2}}, \quad m=\pm\frac{b}{\sqrt{a^2+b^2+c^2}}, \quad n=\pm\frac{c}{\sqrt{a^2+b^2+c^2}}$$

$$p=\mp\frac{d}{\sqrt{a^2+b^2+c^2}}.$$

Substituting in (2), normal form of the equation of the plane is given by

$$\pm\frac{ax}{\sqrt{a^2+b^2+c^2}}\pm\frac{by}{\sqrt{a^2+b^2+c^2}}\pm\frac{cz}{\sqrt{a^2+b^2+c^2}}=\mp\frac{d}{\sqrt{a^2+b^2+c^2}} \qquad \ldots (3)$$

Sign in (3) should be chosen such that $p=\dfrac{\pm d}{\sqrt{a^2+b^2+c^2}}$ is always positive.

7.1.6 Note : From equation (3) of 7.1.5, the perpendicular distance of the plane represented by (1) from the origin is $\dfrac{|d|}{\sqrt{a^2+b^2+c^2}}$.

7.1.7 Example : Let us reduce the equation of the plane $x+2y-2z-9=0$ to the normal form and hence find the direction cosines of the normal to the plane and the length of the perpendicular drawn from the origin to the given plane.

The equation of the given plane is $x+2y-2z-9=0$.

Bringing the constant term to R.H.S.,

$$x+2y-2z=9 \qquad \ldots (1)$$

Square root of the sum of the squares of the coefficients of x, y, z in (1) is

$$\sqrt{1^2+2^2+2^2}=\pm 3$$

we observe that $\qquad p=\mp\left(\dfrac{-9}{\pm 3}\right)=3.$

dividing (1) by ± 3, $\pm\dfrac{1}{3}x\pm\dfrac{2}{3}y\mp\dfrac{2}{3}z=\pm 3$

Choosing the sign of the equation so that the constant on the right is positive, we get,

$$\frac{x}{3}+\frac{2}{3}y-\frac{2}{3}z=3 \qquad \ldots (2)$$

(2) represents the equation of the plane in the normal form. Hence *d.c.'s* of the normal to the plane are; $\left(\dfrac{1}{3}, \dfrac{2}{3}, -\dfrac{2}{3}\right)$ and the length of the perpendicular from the origin to the plane is 3.

7.1.8 Definition (Intercepts on the axes)

If a plane Π which does not pass through the origin, intersects the axes of coordinates at A*(a, 0, 0),* B*(0, b, 0),* C*(0, 0, c) then a, b, c are called* X, Y, Z*-intercepts of the plane respectively (Fig. 7.2).*

7.1.9 Equation of a plane in the intercept form

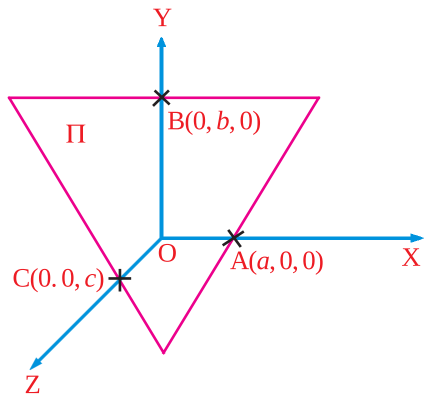


Fig. 7.2

Theorem: *The equation of the plane whose* X, Y, Z *- intercepts are a, b, c is* $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$.

Proof: Suppose the equation of the plane is $lx+my+nz=p$.

This passes through the points (a, 0, 0), (0, b, 0), (0, 0, c).

$\therefore\ la = p,\ mb = p,\ nc = p$

i.e., $l=\frac{p}{a},\quad m=\frac{p}{b},\quad n=\frac{p}{c}$.

$\therefore$ Equation of the plane is $\frac{xp}{a}+\frac{yp}{b}+\frac{zp}{c}=p$.

i.e., $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$.

Conversely, suppose the equation of the plane is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$. ... (1)

If this meets X-axis at A, then for A, $y=0,\ z=0$

Substituting in (1), $\frac{x}{a}=1 \Rightarrow x=a$.

$\therefore$ A = (a, 0, 0).

Similarly, we can show that the plane intersects Y and Z-axes respectively at B = (0, b, 0) and C = (0, 0, c).

$\therefore$ X, Y, Z - intercepts of the plane are a, b, c respectively.

7.1.10 Example: Suppose a plane makes intercepts 2,3,4 on the X,Y,Z-axes respectively. From 7.1.9, the equation of the plane is : $\frac{x}{2}+\frac{y}{3}+\frac{z}{4}=1$

i.e., $6x + 4y + 3z = 12$.

7.1.11 Example: Consider the plane whose equation is $x-3y+2z=9$.

Dividing by 9, $\frac{x}{9}+\frac{y}{-3}+\frac{z}{9/2}=1$.

Comparing this with $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

a = X – intercept = 9, b = Y – intercept = – 3, c = Z – intercept = $\frac{9}{2}$.

7.1.12 Angle between two planes

The angle between two planes is defined as the angle between their normals.

If $a_1x+b_1y+c_1z+d_1=0$ ... (1)

$a_2x+b_2y+c_2z+d_2=0$ ... (2)

are equations of two planes, then, direction ratios of the normal to the planes are (a_1, b_1, c_1) and (a_2, b_2, c_2) respectively.

Hence angle between the planes = angle between the normals

$$= \text{Cos}^{-1}\left(\frac{|a_1a_2+b_1b_2+c_1c_2|}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}\right).$$

The planes are perpendicular if their normals are perpendicular

i.e., if $a_1a_2+b_1b_2+c_1c_2=0$ (by 6.2.10, Note 3)

The planes are parallel if the normals are parallel

i.e., if $\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$ (by 6.2.10, Note 3)

Also note that equation of a plane parallel to $ax+by+cz+d=0$ is $ax+by+cz+k=0$ where k is a parameter.

7.1.13 Solved Problems

1. Problem : *Find the equation of the plane if the foot of the perpendicular from origin to the plane is (2, 3, –5).*

Solution : The plane passes through A and is perpendicular to $\overline{OA}$, so the line segment $\overline{OA}$ is normal to the plane.

The direction ratios of $\overline{OA}$ are $(2-0,\ 3-0,\ -5-0)$ i.e., $(2, 3, -5)$.

Let the equation of the plane be $ax+by+cz+d=0$. Since (2, 3, –5) are *d.r.'s* of the normal to the plane, the equation of the plane is $2x+3y-5z+d=0$.

Since (2, 3, –5) lies on the plane, $4+9+25+d=0$

$\therefore d=-38$.

Hence the equation of the plane is

$$2x + 3y - 5z - 38 = 0.$$

This can be obtained by using 7.1.1 case (1) also.

2. Problem : *Find the equation to the plane through the points* $(0, -1, -1)$, $(4, 5, 1)$ *and* $(3, 9, 4)$.

Solution : Any plane passing through $(0, -1, -1)$ is given by

$$a(x - 0) + b(y + 1) + c(z + 1) = 0 \quad \text{... (1)}$$

If this passes through (3, 9, 4) and (4, 5, 1), these two points satisfy the equation (1).

$$\therefore\ 3a + 10b + 5c = 0 \quad \text{... (2)}$$

$$4a + 6b + 2c = 0 \quad \text{... (3)}$$

Solving (2) and (3)

$$\frac{a}{30-20} = \frac{b}{6-20} = \frac{c}{-18+40}$$

$$\Rightarrow \frac{a}{5} = \frac{b}{-7} = \frac{c}{11}.$$

Substituting in (1), the required equation of the plane is $5x - 7y + 11z + 4 = 0$.

3. Problem : *Find the equation to the plane parallel to the* ZX *- plane and passing through* $(0, 4, 4)$.

Solution : Equation of the ZX - plane is $y = 0$.

Equation of the plane parallel to this is $y = k$.

Since (0, 4, 4) lies on the plane, $k = 4$.

$\therefore$ Equation of the required palne is $y = 4$.

4. Problem : *Find the equation of the plane through the point* (α, β, γ) *and parallel to the plane* $ax + by + cz = 0$.

Solution : The equation of any plane parallel to $ax + by + cz = 0$ is

$$ax + by + cz + k = 0 \quad \text{... (1)}$$

If (1) passes through (α, β, γ), then

$$a\alpha + b\beta + c\gamma + k = 0 \qquad \text{i.e.,} \quad k = -a\alpha - b\beta - c\gamma$$

Substituting the value of k in (1), the equation of the required plane is

$$ax + by + cz - a\alpha - b\beta - c\gamma = 0$$

$$\Rightarrow a(x-\alpha) + b(y-\beta) + c(z-\gamma) = 0.$$

5. Problem : *Find the angle between the planes* $2x - y + z = 6$ *and* $x + y + 2z = 7$.

Solution : If θ is the angle between the planes, $\theta = \text{Cos}^{-1}\left(\dfrac{|2.1 - 1.1 + 1.2|}{\sqrt{4+1+1}\,\sqrt{1+1+4}}\right)$

$$= \text{Cos}^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}.$$

6. Problem : *Find the equation of the plane passing through* (2, 0, 1) *and* (3, –3, 4) *and perpendicular to* $x - 2y + z = 6$.

Solution : Equation of any plane passing through (2, 0, 1) is

$$a(x-2) + b(y-0) + c(z-1) = 0 \qquad \text{... (1)}$$

If this passes through (3, –3, 4), then

$$a - 3b + 3c = 0 \qquad \text{... (2)}$$

If the plane (1) is perpendicular to the plane $x - 2y + z = 6$

$$a - 2b + c = 0 \qquad \text{... (3)}$$

Solving (2) and (3) for a, b, c we get

$$\frac{a}{3} = \frac{b}{2} = \frac{c}{1}$$

Substituting these values in (1), equation of the required plane is $3x + 2y + z = 7$.

Exercise 7(a)

I. **1.** Find the equation of the plane if the foot of the perpendicular from origin to the plane is (1, 3, –5).

2. Reduce the equation $x + 2y - 3z - 6 = 0$ of the plane to the normal form.

3. Find the equation of the plane whose intercepts on X, Y, Z- axes are 1, 2, 4 respectively.

4. Find the intercepts of the plane $4x + 3y - 2z + 2 = 0$ on the coordinate axes.

5. Find the *d.c.'s* of the normal to the plane $x + 2y + 2z - 4 = 0$.

6. Find the equation of the plane passing through (–2, 1, 3) and having (3, –5, 4) as *d.r.'s* of its normal.

7. Write the equation of the plane $4x - 4y + 2z + 5 = 0$ in the intercept form.

8. Find the angle between the planes $x + 2y + 2z - 5 = 0$ and $3x + 3y + 2z - 8 = 0$.

II. **1.** Find the equation of the plane passing through (1, 1, 1) and parallel to the plane $x + 2y + 3z - 7 = 0$.

2. Find the equation of the plane passing through (2, 3, 4) and perendicular to X-axis.

3. Show that $2x + 3y + 7 = 0$ represents a plane perpendicular to XY-plane.

4. Find the constant k so that the planes $x - 2y + kz = 0$ and $2x + 5y - z = 0$ are at right angles. Find the equation of the plane through $(1, -1, -1)$ and perpendicular to these planes.

5. Find the equation of the plane through $(-1, 6, 2)$ and perpendicular to the join of $(1, 2, 3)$ and $(-2, 3, 4)$.

6. Find the equation of the plane bisecting the line segment joining $(2, 0, 6)$ and $(-6, 2, 4)$ and perpendicular to it.

7. Find the equation of the plane passing through $(0, 0, -4)$ and perpendicular to the line joining the points $(1, -2, 2)$ and $(-3, 1, -2)$.

8. Find the equation of the plane through $(4, 4, 0)$, and perpendicular to the planes $2x + y + 2z + 3 = 0$ and $3x + 3y + 2z - 8 = 0$.

III. 1. Find the equation of the plane through the points $(2, 2, -1)$, $(3, 4, 2)$, $(7, 0, 6)$.

2. Show that the points $(0, -1, 0)$, $(2, 1, -1)$ $(1, 1, 1)$, $(3, 3, 0)$ are coplanar.

3. Find the equation of the planes through $(6, -4, 3)$, $(0, 4, -3)$ and cutting of intercepts whose sum is zero.

4. A plane meets the coordinate axes in A, B, C. If the centroid of ΔABC is (a, b, c) show that the equation to the plane is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 3$.

5. Show that the plane through $(1, 1, 1)$ $(1, -1, 1)$ and $(-7, -3, -5)$ is parallel to Y-axis.

6. Show that the equations $ax + by + r = 0$, $by + cz + p = 0$, $cz + ax + q = 0$ represent planes perpendicular to XY, YZ, ZX - planes respectively.

Key Concepts

- A plane is a surface with at least three noncollinear points such that the line joining any two points on the surface lies entirely on it.
- Equation of a plane is a first degree equation in x, y, z.
- Every first degree equation in x, y, z represents a plane.
- Equation of the plane in normal form is $lx + my + nz = p$ where (l, m, n) are $d.c.$'s of the normal to the plane and p is the perpendicular distance to the plane from the origin.
- General equation of the plane is $ax + by + cz + d = 0$, where (a, b, c) are direction ratios of the normal to the plane.

- Perpendicular distance from the origin to the plane $ax + by + cz + d = 0$ is $\dfrac{|d|}{\sqrt{a^2 + b^2 + c^2}}$.
- Equation of the plane in intercept form is $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 1$ where a, b, c are intercepts on X, Y, Z - axes respectively.
- Equation of the plane parallel to $ax + by + cz + d = 0$ is of the form $ax + by + cz + k = 0$, where k is a parameter.
- Equation of the plane passing through a point (x_1, y_1, z_1) is $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$.
- Angle between two planes is the angle between their normals. Thus if θ is the angle between the planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$

  then $\cos\theta = \dfrac{|a_1a_2 + b_1b_2 + c_1c_2|}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$.
- Above planes are perpendicular if $a_1a_2 + b_1b_2 + c_1c_2 = 0$.

  and are parallel if $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$.

Historical Note

Geometry, to start with, is concerned with the properties of space and objects such as - points, lines, angles, planes, surfaces, solids and curves in space. In the work of German mathematician *David Hilbert* (1862 - 1943) at the turn of the 20th century, the foundations of geometry were generalised. The classical concepts of space and objects, which depended very much on intuition, were replaced with abstract ideas. Before this, a decisive step towards generalization of classical geometry of *Euclid* was taken by the work of *Descartes* and others and thus the analytical geometry was born.

The basis of analytic geometry is the idea that a point in space can be specified by an ordered triad of numbers, giving its position. The notion that any point, for example, can be indicated by its latitude, longitude and height above the earth goes back to *Archimedes* and *Apollonius* (3rd century B.C. and 2nd century B.C.).

The 16th - 17th century German Astronomer *Johannes Kepler* conceived the idea of extending the Euclidean plane, and later the concept of projective plane was introduced. The projective geometry was enriched by *Girard Desargues, Poncelet* and several others there after.

In 1866 *Karl Theodor Reye* proposed using capital letters for points and small letters (Greek alphabet) for planes in a remarkable two volume work on geometry.

The designation of points, lines and planes by letters was in vogue among the ancient Greeks also and had been traced to the work of *Hippocrates of Chios* (ca. 440 B.C.)

Answers

Exercise 7(a)

I. **1.** $x + 3y - 5z = 35$

2. $\frac{x}{\sqrt{14}} + \frac{2y}{\sqrt{14}} - \frac{3z}{\sqrt{14}} = \frac{6}{\sqrt{14}}$

3. $4x + 2y + z = 4$

4. $-\frac{1}{2}, \frac{-2}{3}, +1$

5. $\left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$

6. $3x - 5y + 4z - 1 = 0$

7. $\frac{x}{(-5/4)} + \frac{y}{(5/4)} + \frac{z}{(-5/2)} = 1$

8. $\text{Cos}^{-1}\left(\frac{13}{3\sqrt{22}}\right)$

II. **1.** $x + 2y + 3z - 6 = 0$

2. $x = 2$

4. $k = -8,\ 42x - 15y + 9z - 48 = 0$

5. $3x - y - z + 11 = 0$

6. $-8x + 2y - 2z - 8 = 0$

7. $4x - 3y + 4z + 16 = 0$

8. $-4x + 2y + 3z + 8 = 0$

III. **1.** $5x + 2y - 3z = 17$

3. $\frac{x}{1} + \frac{y}{2} + \frac{z}{-3} = 1;\ \frac{x}{3} + \frac{y}{-2} + \frac{z}{-1} = 1$

Calculus

Chapter 8

Limits and Continuity

"With the Calculus as a key, Mathematics can be successfully applied to the explanation of the course of Nature"

- A.N. Whitehead

Introduction

Calculus can be considered as the subject that studies the problems of change. This mathematical discipline stems from the 17th century investigations of *Isaac Newton* (1642-1727) and *Gottfried Leibnitz* (1646-1716) and today it stands as the quantitative language of science and technology.

The very basic notion of a limit was conceived in 1680s by *Newton* and *Leibnitz* simultaneously, while they were struggling with the creation of calculus. They gave a loose verbal definition of limit which led to many problems. There were other mathematicians of the same era who proposed other definitions of the intuitive concept of limit. But none of these were adequate to provide a basis for rigorous proofs. Of course there are evidences that the idea of limit was first known to *Archemedes* (287 – 212 B.C.)

Cauchy

(1789 – 1857)

Augustin Louis Cauchy *was a French Mathematician. He started the project of formulating and proving the theorems of calculus in a rigorous manner and was thus an early pioneer of analysis. He wrote extensively and profoundly in both pure and applied mathematics and he can probably be ranked next to Euler in volume of output.*

It is *Augustin-Louis Cauchy* (1789 – 1857) who formulated the definition and presented the arguments with greater care than his predecessors in his monumental work 'Cours d' Analyse'. But the concept of a limit still remained elusive.

The precise definition of limit, that as we use today, was given by *Karl Weierstrass* (1815 - 1897).

8.1 Intervals and neighbourhoods

First we look into the concepts of intervals and neighbourhoods, which are very much useful in studying limits and continuity.

8.1.1 Intervals

Let $a, b \in \mathbf{R}$ such that $a \leq b$. Then the set

(i) $\{x \in \mathbf{R} : a \leq x \leq b\}$, denoted by $[a, b]$, is called a **closed interval**.

(ii) $\{x \in \mathbf{R} : a < x < b\}$, denoted by (a, b), is called an **open interval**.

In a similar way, some more intervals are given below.

(iii) $(a, b] = \{x \in \mathbf{R} :\ a < x \leq b\}$

(iv) $[a, b) = \{x \in \mathbf{R} :\ a \leq x < b\}$

(v) $[a, \infty) = \{x \in \mathbf{R} :\ x \geq a\}$

(vi) $(a, \infty) = \{x \in \mathbf{R} :\ x > a\}$

(vii) $(-\infty, a] = \{x \in \mathbf{R} :\ x \leq a\}$

(viii) $(-\infty, a) = \{x \in \mathbf{R} :\ x < a\}$

The intervals in (i), (ii), (iii) and (iv) are said to be intervals of finite length $b - a$. Others are intervals of infinite length.

8.1.2 Neighbourhoods

Let $a \in \mathbf{R}$. If $\delta > 0$, then the open interval $(a - \delta, a + \delta)$ is called **the δ - neighbourhood** of a. That is $\{x \in \mathbf{R} : a - \delta < x < a + \delta\}$. Figure 8.1 shows the location of $(a - \delta, a + \delta)$ on the number line.

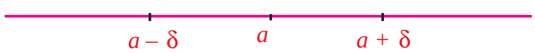


Fig. 8.1

The set obtained by deleting the point a from this neighbourhood is called **the deleted neighbourhood** of a. That is, the deleted δ - neighbourhood of a is

$$(a - \delta, a) \cup (a, a + \delta) \quad \text{or} \quad (a - \delta, a + \delta) \setminus \{a\}.$$

8.1.3 Note

1. Any interval (c, d) is a neighbourhood of some $a \in (c, d)$. In fact, take

$a = \dfrac{c+d}{2}$ and $\delta = \dfrac{d-c}{2} > 0$.

$$\text{Then } (a-\delta, a+\delta) = \left(\frac{c+d}{2} - \frac{d-c}{2}, \frac{c+d}{2} + \frac{d-c}{2}\right)$$
$$= (c, d).$$

Therefore (c, d) is the δ - neighbourhood of a.

2. The set $\{x \in \mathbf{R} : 0 < |x-a| < \delta\}$ is the deleted δ - neighbourhood of a, because

$$\begin{aligned} 0 < |x-a| < \delta &\Leftrightarrow |x-a| < \delta \text{ and } x \neq a \\ &\Leftrightarrow -\delta < x-a < \delta \text{ and } x \neq a \\ &\Leftrightarrow a-\delta < x < a+\delta \text{ and } x \neq a \\ &\Leftrightarrow x \in (a-\delta, a+\delta) \text{ and } x \neq a \\ &\Leftrightarrow x \in (a-\delta, a+\delta) \setminus \{a\}. \end{aligned}$$

3. If $\delta_1, \delta_2, ..., \delta_n$ are positive real numbers and $a \in \mathbf{R}$ then

$$\bigcap_{k=1}^{n}(a-\delta_k, a+\delta_k) \text{ is also a neighbourhood of } a.$$

For, take $\delta = \min\{\delta_1, \delta_2, ..., \delta_n\}$. Then clearly $\delta > 0$ and

$$(a-\delta, a+\delta) = \bigcap_{k=1}^{n}(a-\delta_k, a+\delta_k).$$

$x \in (a-\delta, a+\delta)$

$$\begin{aligned} &\Rightarrow a-\delta_k < x < a+\delta_k \quad (k = 1, 2, ... n) \\ &\Rightarrow x \in (a-\delta_k, a+\delta_k) \text{ for all } k \\ &\Rightarrow x \in \bigcap_{k=1}^{n}(a-\delta_k, a+\delta_k). \end{aligned}$$

Conversely, if $x \in \bigcap_{k=1}^{n}(a-\delta_k, a+\delta_k)$ then

$$a-\delta_k < x < a+\delta_k \text{ for all } k.$$

But $\delta = \min\{\delta_1, \delta_2, ..., \delta_n\}$ implies that $\delta = \delta_i$ for some i among 1, 2, ..., n.
Hence $x \in (a-\delta_i, a+\delta_i) = (a-\delta, a+\delta)$.

8.2 Limits

We illustrate some examples to get familiarity on the concept of limits.

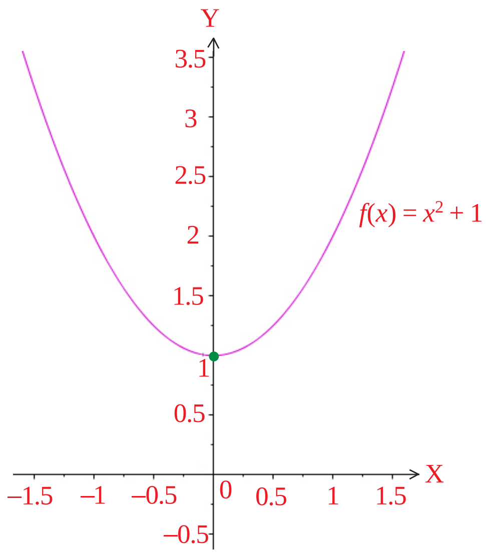


Fig. 8.2

1. Example

Let $f : \mathbf{R} \to \mathbf{R}$ be the function defined by $f(x) = x^2 + 1, x \in \mathbf{R}$. Here we observe that as x takes values very close to 0, the value of $f(x)$ approaches 1. (See Fig. 8.2). In this case, we say that $f(x)$ tends 1 as x tends to 0, and we write it as

$$\lim_{x \to 0} f(x) = 1.$$

That is, limit of $f(x)$ is 1 as x tends to 0.

2. Example

Let us define $f : (\mathbf{R} \setminus \{1\}) \to \mathbf{R}$ by

$$f(x) = \frac{x^2 - 1}{x - 1}, \ x \neq 1.$$

In the following table, we compute the values of $f(x)$ for certain values on either side of $x = 1$.

x	0.9	0.99	0.999	0.9999	1.0001	1.001	1.01	1.1
$f(x)$	1.9	1.99	1.999	1.9999	2.0001	2.001	2.01	2.1

Table 8.1

From Table 8.1, we observe that these values are very near to 2. This can be illustrated by considering the graph of the function f given in Fig. 8.3. Here we note that the limit of f at 1 exists even though f is not defined at 1.

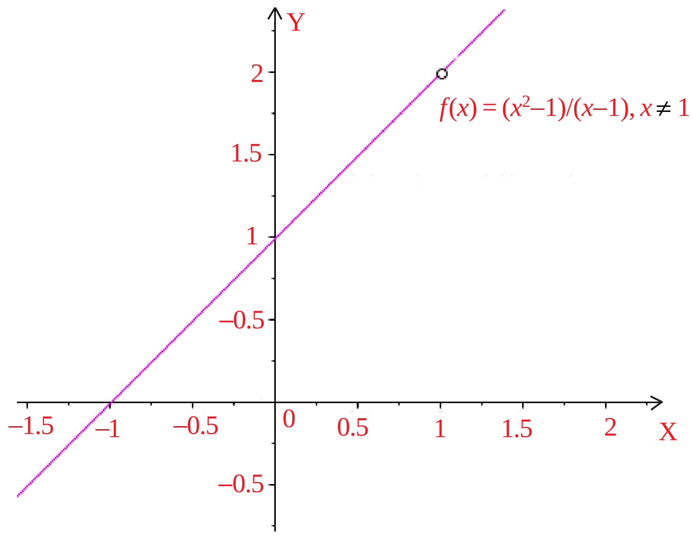


Fig. 8.3

3. Example

Let $f:(\mathbf{R}\setminus\{-2\})\to\mathbf{R}$ be defined by $f(x)=\dfrac{x-1}{x+2}$, for each $x\in(\mathbf{R}\setminus\{-2\})$.

Consider the following table of values of x close to $x=3$ on either side and the corresponding values of $f(x)$:

x	2.9	2.99	2.999	2.9999	2.99999	3.00001	3.0001	3.001	3.01	3.1
$f(x)$	0.38776	0.39880	0.39988	0.39999	0.399999	0.400001	0.40001	0.40012	0.40120	0.41176

Table 8.2

Table 8.2 shows that as x gets close to 3, the function $f(x)$ is approaching to 0.4.

4. Example

Let $f:(\mathbf{R}\setminus\{2\})\to\mathbf{R}$ be defined by $f(x)=\dfrac{x^2+3x-10}{x-2}$.

Here is a table of values of x near 2 and corresponding $f(x)$.

x	1.9	1.99	1.999	1.9999	2.0001	2.001	2.01	2.1
$f(x)$	6.9	6.99	6.999	6.9999	7.0001	7.001	7.01	7.1

Table 8.3

Though f is not defined at 2, but $f(x)$ is approaching to 7 as x is nearing to 2. The same can be seen from Table 8.3 and Fig. 8.4

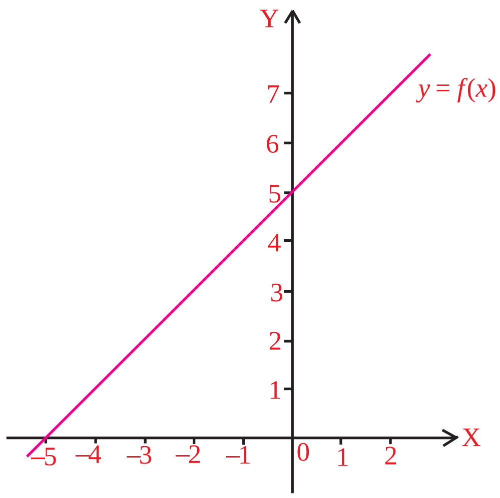


Fig. 8.4

5. Example

Let $f:(0,\infty)\to\mathbf{R}$ be defined by

$$f(x)=\sqrt{x}.$$

Let us look at the values of x near 0 and corresponding $f(x)$.

x	0.01	0.001	0.0001	0.00001
$f(x)$	0.1	0.0316	0.01	0.0031

Table 8.4

From Table 8.4, we observe that, $f(x)$ approaches zero, as x approaches zero.

In each of the above examples, it is clear that $f(x)$ approaches value l when x is nearing to a particular point c. This leads to an important concept called the limit of a function. In the third century (B.C) Archemedes of Greece (287 - 212 B.C.) was the first person who formulated this concept. But a precise definition of the limit of a function, that we use today, is due to the German mathematician Karl Weierstrass (1815 - 1897). We introduce this concept in the present section.

8.2.1 Definition of the limit

Let $E \subseteq \mathbf{R}$ and $f : E \to \mathbf{R}$. Let $a \in \mathbf{R}$ be such that $((a-r, a+r) \setminus \{a\}) \cap E$ is non empty for every $r > 0$. If there exists a real number l satisfying the condition below then l is said to be a limit of f at a:

Given $\varepsilon > 0$, there exists a $\delta > 0$ such that $|f(x) - l| < \varepsilon$ whenever $x \in E$ and $0 < |x-a| < \delta$. In this case, we say that limit of the function $f(x)$ as x tends to 'a' exists and it is 'l' and we write it as

$$\lim_{x \to a} f(x) = l \text{ or } f(x) \to l \text{ as } x \to a.$$

If such an l does not exist, we say that $\lim_{x \to a} f(x)$ does not exist.

8.2.2 Note

Let f, E, a, l be as given in Definition 8.2.1. Also let $m \in \mathbf{R}$ be such that $\lim_{x \to a} f(x) = m$. Then it can be shown that $l = m$. In other words, the limit of a function at a given point, if exists, is unique. This is proved as follows.

Given that $\lim_{x \to a} f(x) = l$, and $\lim_{x \to a} f(x) = m$.

In order to show that $l = m$ it is sufficient to show that $|l - m| < \varepsilon$ for every $\varepsilon > 0$.

Let $\varepsilon > 0$.

Since $\lim_{x \to a} f(x) = l$, for $\frac{\varepsilon}{2} > 0$, $\exists\ \delta_1 > 0$ such that

$$x \in E \text{ and } 0 < |x - a| < \delta_1 \Rightarrow |f(x) - l| < \frac{\varepsilon}{2}. \qquad \text{...(1)}$$

Since $\lim_{x \to a} f(x) = m$, for $\frac{\varepsilon}{2} > 0$, $\exists\ \delta_2 > 0$ such that

$$x \in \text{E and } 0 < |x-a| < \delta_2 \Rightarrow |f(x)-m| < \frac{\varepsilon}{2}. \qquad ...(2)$$

Let $\delta = \min\{\delta_1, \delta_2\}$. Then if $x \in$ E

and $0 < |x-a| < \delta$, we have, by (1) and (2),

$$|l-m| = |l - f(x) + f(x) - m|$$

$$\leq |l - f(x)| + |f(x) - m| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

i.e., $|l-m| < \varepsilon$.

Since $\varepsilon > 0$ is arbitrary, we get $l = m$.

8.2.3 Examples

In the following, we illustrate the definition of limit through examples with ε, δ notation.

1. Suppose $f:(0, \infty) \to \mathbf{R}$ is defined by $f(x) = \sqrt{x}$. Then $\lim_{x \to 0} f(x) = 0$.

Let $\varepsilon > 0$ be given. Choose $\delta = \varepsilon^2$. Then $\delta > 0$ and for all x with $x \in (0, \infty)$, $0 < |x| < \delta$ i.e., $0 < x < \delta$ we have

$$|f(x) - 0| = \sqrt{x} < \sqrt{\delta} = \varepsilon.$$

Hence $\lim_{x \to 0} \sqrt{x} = 0$.

2. Suppose $f: (\mathbf{R} \setminus \{0\}) \to \mathbf{R}$ is given by $f(x) = \frac{1}{x}$ for each $x \neq 0$. Then $\lim_{x \to 0} \frac{1}{x}$ does not exist.

If possible, suppose that $\lim_{x \to 0} \frac{1}{x}$ exists and is equal to, say, l.

Then for $\varepsilon = 1$, there is a $\delta > 0$ such that

$$0 < |x| < \delta \Rightarrow \left|\frac{1}{x} - l\right| < 1.$$

i.e., $$0 < |x| < \delta \Rightarrow \left|\frac{1}{x}\right| = \left|\frac{1}{x} - l + l\right|$$

$$\leq \left|\frac{1}{x} - l\right| + |l|$$

$$< 1 + |l|.$$

That is, $|x| > \frac{1}{1+|l|}$ whenever $0 < x < \delta$. ... (1)

But if we choose a y_0 such that $0 < y_0 < \min\left\{\delta, \dfrac{1}{1+|l|}\right\}$, then $0 < |y_0| = y_0 < \delta$ and $|y_0| = y_0 < \dfrac{1}{1+|l|}$, contradicting (1). Therefore $\lim\limits_{x\to 0} \dfrac{1}{x}$ does not exist.

8.2.4 Note

$\lim\limits_{x\to a} x = a$. (Try to give a proof!).

8.2.5 Theorem

We state the following theorem (without proof) which is helpful in finding the limits.

Let $f : E \to \mathbf{R}$, $g : E \to \mathbf{R}$ and let $a \in \mathbf{R}$ be such that $E \cap \big((a-r,\ a+r) \setminus \{a\}\big)$ is non-empty for every $r > 0$. Let $k \in \mathbf{R}$. Suppose that $\lim\limits_{x\to a} f(x) = l$ and $\lim\limits_{x\to a} g(x) = m$. Then the following are true.

(i) $\lim\limits_{x\to a} (f+g)(x) = l+m$, $\lim\limits_{x\to a} (f-g)(x) = l-m$, $\lim\limits_{x\to a} (fg)(x) = lm$,

$\lim\limits_{x\to a} (kf)(x) = kl$.

(ii) *If $h : E \to \mathbf{R}$ and $\lim\limits_{x\to a} h(x) = n \neq 0$ then h is never zero in $E \cap \big((a-r,\ a+r) \setminus \{a\}\big)$ for some $r > 0$,* $\lim\limits_{x\to a}\left(\dfrac{1}{h}\right)(x) = \dfrac{1}{n}$ *and*

$\lim\limits_{x\to a}\left(\dfrac{f}{h}\right)(x) = \dfrac{l}{n}$.

As an illustration we prove the following:

8.2.6 Theorem

If p is a polynomial function (i.e. a function $p(x)$ of the form $a_0 + a_1 x + \dots\dots + a_k x^k$, $k \geq 1$) then $\lim\limits_{x\to a} p(x) = p(a)$.

Proof: From Note 8.2.4, we have $\lim\limits_{x\to a} x = a$.

We use Theorem 8.2.5 to prove the conclusion.

$$\lim_{x\to a} x^2 = \lim_{x\to a}(x.x) = \lim_{x\to a} x \cdot \lim_{x\to a} x = a\cdot a = a^2.$$

Thus, by induction, it is easy to see that

$$\lim_{x\to a} x^n = a^n.$$

Let $\quad p(x) = a_0 + a_1x + a_2x^2 + ... + a_kx^k$.

Then by Theorem 8.2.5(i), we have

$$\lim_{x\to a} p(x) = \lim_{x\to a}[a_0 + a_1x + a_2x^2 + ... + a_kx^k]$$

$$= \lim_{x\to a} a_0 + \lim_{x\to a}(a_1x) + \lim_{x\to a}(a_2x^2) + ... + \lim_{x\to a}(a_kx^k)$$

$$= a_0 + a_1 \lim_{x\to a} x + a_2 \lim_{x\to a} x^2 + ... + a_k \lim_{x\to a} x^k.$$

$$= a_0 + a_1a + a_2a^2 + ... + a_ka^k$$

$$= p(a),$$

proving the theorem.

8.2.7 Remark

During the course of proof of Theorem 8.2.6, we proved that

$$\lim_{x\to a} x^n = a^n,\quad a \in \mathbf{R},\ n \in \mathbf{N}.$$

8.2.8 Theorem

Let $E \subseteq \mathbf{R}$, *Let* $f,\ g : E \to \mathbf{R}$ *be two functions. Let* $a \in \mathbf{R}$ *be such that* $((a-r,\ a+r)\setminus\{a\}) \cap E \neq \phi$ *for every* $r > 0$. *Assume that* $f(x) \le g(x)$ *for all* x *in* E *with* $x \neq a$. *If both* $\lim_{x\to a} f(x) = l$ *and* $\lim_{x\to a} g(x) = m$, *then* $l \le m$. *That is,* $\lim_{x\to a} f(x) \le \lim_{x\to a} g(x)$. *This is illustrated in Fig. 8.5.*

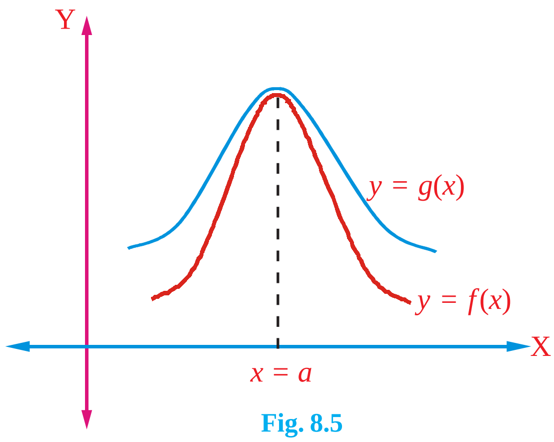


Fig. 8.5

8.2.9 Theorem (Sandwich theorem)

Let $E \subseteq \mathbf{R}$, $f, g, h : E \to \mathbf{R}$ *and let* $a \in \mathbf{R}$ *be such that* $\big((a-r, a+r)\setminus\{a\}\big)\cap E$ *is non-empty for every* $r > 0$. *If* $f(x) \le g(x) \le h(x)$ *for all* $x \in E, x \neq a$ *and if* $\lim_{x\to a} f(x) = l = \lim_{x\to a} h(x)$, *then* $\lim_{x\to a} g(x)$ *exists and is equal to* l.

Proof : Follows from Theorem 8.2.8.

8.2.10 Theorem

If F and G are polynomials such that $F(x) = (x-a)^k f(x)$, $G(x) = (x-a)^k g(x)$ for some $k \in N$ and for some polynomials $f(x)$ and $g(x)$ with $g(a) \neq 0$ then $\lim_{x \to a} \left(\frac{F}{G}\right)(x) = \frac{f(a)}{g(a)}$.

We shall now make use of these theorems to compute some limits.

Hereafter if the domain of a function f is not explicitly given, then, by convention, the domain of f is to be taken as the set of all those real x for which $f(x)$ is real.

8.2.11 Solved Problems

1. Problem : *Evaluate* $\lim_{x \to -3} \frac{1}{x+2}$.

Solution : Consider $f(x) = \frac{1}{x+2}$, $x \neq -2$.

Write $h(x) = x + 2$ for all $x \in \mathbf{R}$.

Then $f(x) = \frac{1}{h(x)}$ and $\lim_{x \to -3} h(x) = \lim_{x \to -3} (x+2) = -1$.

Therefore $\lim_{x \to -3} f(x) = \lim_{x \to -3} \frac{1}{x+2} = -1$.

2. Problem : *Compute* $\lim_{x \to 2} \frac{x-2}{x^3-8}$.

Solution : Write $f(x) = \frac{x-2}{x^3-8}$, $x \neq 2$ so that $f(x) = \frac{1}{x^2+2x+4}$.

Write $h(x) = x^2 + 2x + 4$ so that $\lim_{x \to 2} h(x) = 12$.

Hence $\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{1}{h(x)} = \frac{1}{12}$.

3. Problem : *Find* $\lim_{x \to 1} (x+2)(2x+1)$.

Solution : $\lim_{x \to 1} (x+2)(2x+1) = \lim_{x \to 1} (x+2) \lim_{x \to 1} (2x+1) = 3.3 = 9$.

4. Problem : *Compute* $\lim_{x \to 0} x^2 \sin\frac{1}{x}$.

Solution : For $x \neq 0$, we know that $-1 \leq \sin\frac{1}{x} \leq 1$.

Therefore $-x^2 \leq x^2 \sin\frac{1}{x} \leq x^2$. But $\lim_{x \to 0}(-x^2) = 0 = \lim_{x \to 0} x^2$.

Hence, by Sandwich theorem (Theorem 8.2.9), we have $\lim_{x \to 0} x^2 \sin\frac{1}{x} = 0$.

5. Problem : *Find* $\lim\limits_{x\to 2}\dfrac{x^2-5}{4x+10}$.

Solution : We define $f : \mathbf{R} \to \mathbf{R}$ by $f(x) = x^2 - 5$ and $g : \mathbf{R} \to \mathbf{R}$ by $g(x) = 4x + 10$.

$$\lim_{x\to 2} f(x) = \lim_{x\to 2}(x^2-5) = (2^2-5) = (4-5) = -1$$

and $$\lim_{x\to 2} g(x) = \lim_{x\to 2}(4x+10) = 8+10 = 18 \neq 0.$$

Hence, by Theorem 8.2.5 (ii), we have

$$\lim_{x\to 2}\frac{x^2-5}{4x+10} = -\frac{1}{18}.$$

6. Problem : *Find* $\lim\limits_{x\to 3}\dfrac{x^3-6x^2+9x}{x^2-9}$.

Solution: Write $F(x) = x^3 - 6x^2 + 9x = x(x-3)^2 = (x-3)\, f(x)$ where $f(x) = x(x-3)$.

Write $G(x) = x^2 - 9 = (x-3)(x+3) = (x-3)\, g(x)$ where $g(x) = x + 3$.

Therefore, $\dfrac{F(x)}{G(x)} = \dfrac{(x-3)\, f(x)}{(x-3)\, g(x)} = \dfrac{f(x)}{g(x)}$ and $g(3) = 6 \neq 0$.

Now, by applying Theorem 8.2.10, we get

$$\lim_{x\to 3}\frac{x^3-6x^2+9x}{x^2-9} = \lim_{x\to 3}\frac{F(x)}{G(x)} = \frac{f(3)}{g(3)} = \frac{3(3-3)}{3+3} = \frac{0}{6} = 0.$$

7. Problem : *Find* $\lim\limits_{x\to 3}\dfrac{x^3-3x^2}{x^2-5x+6}$.

Solution : We write $F(x) = x^3 - 3x^2 = x^2(x-3) = (x-3)\, f(x)$ where $f(x) = x^2$,

and $G(x) = x^2 - 5x + 6 = (x-3)(x-2) = (x-3)\, g(x)$ where $g(x) = x - 2$, with

$g(3) = 3 - 2 = 1 \neq 0$.

Therefore, by applying Theorem 8.2.10, we get

$$\lim_{x\to 3}\frac{x^3-3x^2}{x^2-5x+6} = \lim_{x\to 3}\frac{F(x)}{G(x)} = \frac{f(3)}{g(3)} = \frac{3^2}{3-2} = 9.$$

Exercise 8(a)

I. Compute the following limits.

1. $\lim\limits_{x\to a}\dfrac{x^2-a^2}{x-a}$

2. $\lim\limits_{x\to 1}(x^2+2x+3)$

3. $\lim\limits_{x\to 0}\dfrac{1}{x^2-3x+2}$

4. $\lim\limits_{x\to 3}\dfrac{1}{x+1}$

5. $\lim\limits_{x\to 1}\dfrac{2x+1}{3x^2-4x+5}$

6. $\lim\limits_{x\to 1}\dfrac{x^2+2}{x^2-2}$

7. $\lim_{x\to 2}\left(\frac{2}{x+1}-\frac{3}{x}\right)$

8. $\lim_{x\to 0}\left(\frac{x-1}{x^2+4}\right)$

9. $\lim_{x\to 0} x^{\frac{3}{2}}\ (x>0)$

10. $\lim_{x\to 0}\left(\sqrt{x}+x^{\frac{5}{2}}\right)\ (x>0)$

11. $\lim_{x\to 0} x^2\cos\frac{2}{x}$

12. $\lim_{x\to 3}\frac{x^2-9}{x^3-6x^2+9x+1}$

13. $\lim_{x\to 1}\left[\frac{x-1}{x^2-x}-\frac{1}{x^3-3x^2+2x}\right]$

14. $\lim_{x\to 3}\frac{x^4-81}{2x^2-5x-3}$

15. $\lim_{x\to 3}\frac{x^2-8x+15}{x^2-9}$

16. If $f(x)=-\sqrt{25-x^2}$ then find $\lim_{x\to 1}\frac{f(x)-f(1)}{x-1}$.

8.2.12 Right and left hand limits

We studied the limit of a function f at a given point $x=a$ as the approaching value of $f(x)$ when x tends to 'a'. Here we note that there are two ways x could approach 'a', either from the left of 'a' or from the right of 'a'. This naturally leads to two limits, namely 'the right hand limit' and 'the left hand limit'. Right hand limit of a function f at $x=a$ is the limit of the values of $f(x)$ as x tends to 'a' when x takes values greater than 'a'. We denote the right hand limit of f at 'a' by $\lim_{x\to a+} f(x)$. Similarly, we describe the left hand limit of f at 'a' and we denote it by $\lim_{x\to a-} f(x)$.

1. Example: Define $f:\mathbf{R}\to\mathbf{R}$ by $f(x)=\begin{cases}1 & \text{if } x\le 0\\ 1+x & \text{if } x>0\end{cases}$.

See Fig. 8.6. We observe that the limit of f at 0 which is defined by the values of $f(x)$ when $x<0$ is equal to 1 i.e., the left hand limit of $f(x)$ at '0' is $\lim_{x\to 0-} f(x)=1$.

Similarly, the limit of f at 0 which is defined by the values of $f(x)$ when $x>0$ is equal to 1 i.e., the right hand limit of $f(x)$ at '0' is $\lim_{x\to 0+} f(x)=1$.

Here we note that the right and left hand limits of f at 0 exist and are equal to 1 and in this case the limit of $f(x)$ as x tends to 0 exists and it is 1.

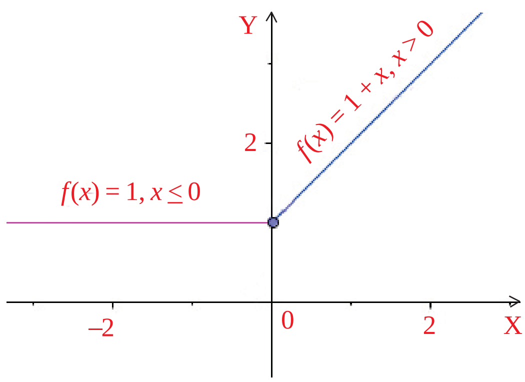


Fig. 8.6

2. Example: Define $f : \mathbf{R} \to \mathbf{R}$ by $f(x) = \begin{cases} 1 \text{ if } x \le 0 \\ -1 \text{ if } x > 0 \end{cases}$.

From Fig.8.7, we observe that $\lim\limits_{x \to 0-} f(x) = 1$ and $\lim\limits_{x \to 0+} f(x) = -1$.

Hence the right and left hand limits of f at 0 are different. We observe that the limit of $f(x)$ as x tends to 0 does not exist.

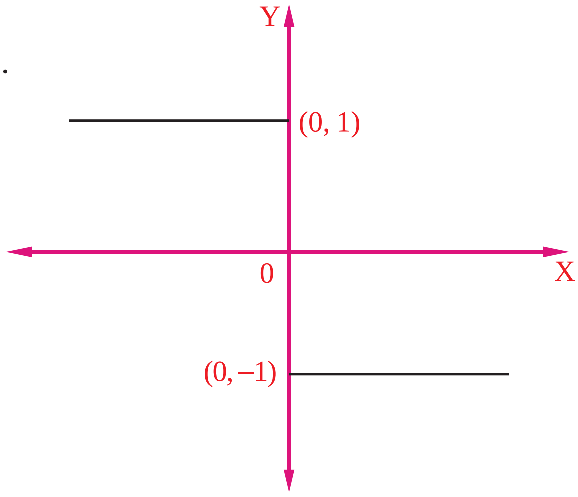


Fig. 8.7

To formulate these concepts analogous to the definition of a limit (as given in Definition 8.2.1) we have the following:

8.2.13 Definition (Right and Left hand limits)

Let $E \subseteq \mathbf{R}$ *and let* $f : E \to \mathbf{R}$.

(i) *Suppose* $a \in \mathbf{R}$ *is such that* $E \cap (a, a+r)$ *is non-empty for every* $r > 0$. *We say that* $l \in \mathbf{R}$ *is a right hand limit of* f *at* a, *and we write* $\lim\limits_{x \to a+} f(x) = l$,

if, given $\varepsilon > 0$, *there exists a* $\delta > 0$ *such that* $|f(x) - l| < \varepsilon$ *whenever* $0 < x - a < \delta$ *and* $x \in E$.

(ii) *Suppose* $a \in \mathbf{R}$ *is such that* $E \cap (a-r, a)$ *is non-empty for every* $r > 0$. *We say that* $m \in \mathbf{R}$ *is a left hand limit of* f *at* a, *and we write* $\lim\limits_{x \to a-} f(x) = m$

if, given $\varepsilon > 0$, *there exists a* $\delta > 0$ *such that* $|f(x) - m| < \varepsilon$ *whenever* $0 < a - x < \delta$ *and* $x \in E$.

The limits $\lim\limits_{x \to a+} f(x)$ and $\lim\limits_{x \to a-} f(x)$ are called **one-sided limits**. These limits (if exist) are unique.

8.2.14 Note

(i) If $E = (a, b)$ then it is clear from Definitions 8.2.13 and 8.2.1 that $f : E \to \mathbf{R}$ has limit at a if and only if it has right hand limit at a. In this case

$$\lim_{x \to a} f(x) = \lim_{x \to a+} f(x).$$

(ii) Also f has limit at b if and only if it has left hand limit at b. In this case

$$\lim_{x \to b} f(x) = \lim_{x \to b-} f(x).$$

(iii) If $a < c < b$, f has limit at c if and only if the left hand limit and the right hand limit both exist at c and are equal. In this case,

$$\lim_{x \to c} f(x) = \lim_{x \to c-} f(x) = \lim_{x \to c+} f(x).$$

The following theorem relates limit of a function to one sided limits.

8.2.15 Theorem

Let $E = (a - r,\ a + r) \setminus \{a\}$ *for all* $r > 0$ *and* $f : E \to \mathbf{R}$.

Then $\lim_{x \to a} f(x) = l \Leftrightarrow \lim_{x \to a+} f(x) = l = \lim_{x \to a-} f(x)$.

8.2.16 Note

If $\lim_{x \to a+} f(x)$ and $\lim_{x \to a-} f(x)$ exist then

$$\lim_{x \to a+} f(x) = \lim_{\substack{h \to 0 \\ h > 0}} f(a + h) \quad \text{and}$$

$$\lim_{x \to a-} f(x) = \lim_{\substack{h \to 0 \\ h > 0}} f(a - h).$$

8.2.17 Solved Problems

1. Problem: Show that $\lim_{x \to 0+} \frac{|x|}{x} = 1$ *and* $\lim_{x \to 0-} \frac{|x|}{x} = -1$ $(x \neq 0)$.

Solution: Here $\frac{|x|}{x} = \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$

Therefore $\lim_{x \to 0+} \frac{|x|}{x} = \lim_{x \to 0+} 1 = 1$ and $\lim_{x \to 0-} \frac{|x|}{x} = \lim_{x \to 0-} (-1) = -1$.

Remark : In Problem 1, we observe that $\lim_{x \to 0} \frac{|x|}{x}$ does not exist.

2. Problem: *Let* $f : \mathbf{R} \to \mathbf{R}$ *be defined by*

$$f(x) = \begin{cases} 2x - 1 & \text{if } x < 3 \\ 5 & \text{if } x \geq 3 \end{cases}$$

Show that $\lim_{x \to 3} f(x) = 5$.

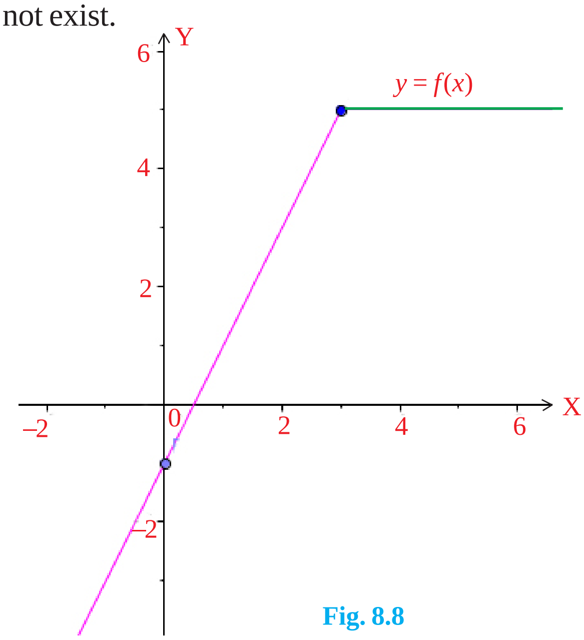


Fig. 8.8

Solution: $\lim_{x \to 3+} f(x) = 5$, since $f(x) = 5$

for $x > 3$ and $\lim_{x \to 3-} f(x) = \lim_{x \to 3-} (2x - 1) = 5$.

Hence $\lim_{x \to 3} f(x) = 5$.

This can be seen from Fig. 8.8.

3. Problem: Show that $\lim_{x\to -2} \sqrt{x^2-4} = 0 = \lim_{x\to 2} \sqrt{x^2-4}$.

Solution: Observe that $\sqrt{x^2-4}$ is not defined over (–2, 2).

But $\lim_{x\to 2+} \sqrt{x^2-4} = 0$ and $\lim_{x\to -2-} \sqrt{x^2-4} = 0$.

Therefore $\lim_{x\to 2} \sqrt{x^2-4} = 0 = \lim_{x\to -2} \sqrt{x^2-4}$.

4. Problem: *If* $f(x) = \begin{cases} x^2, & x \le 1 \\ 2x-1, & x > 1 \end{cases}$, *then find* $\lim_{x\to 1+} f(x)$ *and* $\lim_{x\to 1-} f(x)$. *Does* $\lim_{x\to 1} f(x)$ *exist ?*

Solution: $\lim_{x\to 1+} f(x) = \lim_{\substack{h\to 0 \\ h>0}} f(1+h) = \lim_{h\to 0} (2(1+h)-1) = 1$

and $\lim_{x\to 1-} f(x) = \lim_{\substack{h\to 0 \\ h>0}} f(1-h) = \lim_{h\to 0} (1-h)^2 = 1.$

Since $\lim_{x\to 1+} f(x) = \lim_{x\to 1-} f(x) = 1$, we get that

$\lim_{x\to 1} f(x)$ exists and $\lim_{x\to 1} f(x) = 1$.

Exercise 8(b)

Find the right and left hand limits of the functions in 1, 2, 3 of I and 1, 2 of II at the point a mentioned against them. Hence check whether the functions have limits at those a's.

I. **1.** $f(x) = \begin{cases} 1-x & \text{if } x \le 1 \\ 1+x & \text{if } x > 1 \end{cases}$; $a = 1$.

2. $f(x) = \begin{cases} x+2 & \text{if } -1 < x \le 3 \\ x^2 & \text{if } 3 < x < 5 \end{cases}$; $a = 3$.

3. $f(x) = \begin{cases} \dfrac{x}{2} & \text{if } x < 2 \\ \dfrac{x^2}{3} & \text{if } x \ge 2 \end{cases}$; $a = 2$.

II. **1.** $f(x) = \begin{cases} 1 & \text{if } x < 0 \\ 2x+1 & \text{if } 0 \le x < 1 \\ 3x & \text{if } x > 1 \end{cases}$; $a = 1$.

2. $f(x) = \begin{cases} x^2 & \text{if } x \le 1 \\ x & \text{if } 1 < x \le 2; \\ x-3 & \text{if } x > 2 \end{cases}$ $\quad a = 2.$

3. Show that $\lim\limits_{x \to 2-} \dfrac{|x-2|}{x-2} = -1.$

4. Show that $\lim\limits_{x \to 0+} \left(\dfrac{2|x|}{x} + x + 1 \right) = 3.$

5. Compute $\lim\limits_{x \to 2+} ([x]+x)$ and $\lim\limits_{x \to 2-} ([x]+x)$.

6. Show that $\lim\limits_{x \to 0-} x^3 \cos \dfrac{3}{x} = 0$.

III. 1. Find $\lim\limits_{x \to 0} f(x)$ where $f(x) = \begin{cases} x-1 \text{ if } x<0 \\ \;\;0 \;\;\text{ if } x=0. \\ x+1 \text{ if } x>0 \end{cases}$

2. Define $f: \left[-\dfrac{1}{2}, \infty\right) \to \mathbf{R}$ by $f(x) = \sqrt{1+2x}$, $x \in \left[-\dfrac{1}{2}, \infty\right)$.

Then compute $\lim\limits_{x \to \left(-\frac{1}{2}\right)+} f(x)$. Hence find $\lim\limits_{x \to -\frac{1}{2}} f(x)$.

8.3 Standard limits

We shall now obtain the limits of some standard functions in the following theorems. Using these we can find the limits of some functions easily.

8.3.1 Theorem

If $a > 0$, $n \in \mathbf{R}$ then $\lim\limits_{x \to a} x^n = a^n$.

Proof: We first prove the result for $n \le 1$.

Case (i) Suppose $n \le 1$.

$$x > a \Rightarrow x^{n-1} \le a^{n-1} \quad (\text{since } n-1 \le 0)$$

$$\Rightarrow x^n \le a^{n-1} x$$

$$\Rightarrow x^n - a^n \le a^{n-1}(x-a) \qquad \ldots (1)$$

$$0 < x < a \Rightarrow a^{n-1} \le x^{n-1} \quad (\text{since } n-1 \le 0)$$
$$\Rightarrow a^{n-1}x \le x^n$$
$$\Rightarrow a^{n-1}x - a^n \le x^n - a^n$$
$$\Rightarrow a^{n-1}(x-a) \le x^n - a^n$$
$$\Rightarrow a^n - x^n \le a^{n-1}(a-x) \qquad \ldots (2)$$

From (1) and (2), for $x \ne a$, $x > 0$ we have

$$\left|x^n - a^n\right| \le a^{n-1}\left|x-a\right|.$$

Since $\lim\limits_{x \to a} a^{n-1}\left|x-a\right| = 0$, we get by Sandwich theorem that

$$\lim_{x \to a}\left|x^n - a^n\right| = 0.$$

Therefore $\lim\limits_{x \to a} x^n = a^n$.

Case (ii) Suppose $n > 1$.

Write $n = m + \alpha$, $m \in \mathbf{N}$, $0 \le \alpha < 1$.

Then $x^n = x^m x^\alpha$. Now using **Case (i)** and Remark 8.2.7, we can see that the theorem is true in this case too.

8.3.2 Theorem

Let n be a rational number and a be a positive real number.

Then $\lim\limits_{x \to a} \dfrac{x^n - a^n}{x - a} = n a^{n-1}$.

Proof: We shall prove the theorem in each of the following cases.

Case (i) : Suppose $n = 0$. Then

$$\lim_{x \to a} \frac{x^n - a^n}{x - a} = 0 = n a^{n-1}.$$

Case (ii) : Suppose n is a positive integer. Then

$$x^n - a^n = (x-a)(x^{n-1} + x^{n-2} \cdot a + \ldots + xa^{n-2} + a^{n-1}) \text{ and hence}$$

$$\lim_{x \to a} \frac{x^n - a^n}{x - a} = \lim_{x \to a} (x^{n-1} + x^{n-2}a + \ldots + xa^{n-2} + a^{n-1})$$
$$= a^{n-1} + a^{n-1} + \ldots + a^{n-1} \quad (\text{by Remark } 8.2.7)$$
$$= n a^{n-1}.$$

Case (iii) : Suppose n is a negative integer. Then

$n = -m$, where m is a positive integer.

$$\text{Hence } \lim_{x\to a}\frac{x^n-a^n}{x-a} = \lim_{x\to a}\frac{x^{-m}-a^{-m}}{x-a}$$

$$= \lim_{x\to a}\left(\frac{1}{x^m}-\frac{1}{a^m}\right)\left(\frac{1}{x-a}\right)$$

$$= \lim_{x\to a}\frac{a^m-x^m}{x^m a^m(x-a)}$$

$$= \lim_{x\to a} -\left(\frac{x^m-a^m}{x-a}\right) . \lim_{x\to a}\frac{1}{x^m a^m}$$

$$= -ma^{m-1}.\frac{1}{a^m a^m}$$

$$= -ma^{-m-1} = na^{n-1}.$$

Case (iv) : Suppose n is a rational number, say $n = \frac{p}{q}$ where $p, q \in \mathbf{Z}$ and $q > 0$.

Let $y = x^{\frac{1}{q}}$ and $b = a^{\frac{1}{q}}$. Then $x = y^q$, $a = b^q$ and

$$\lim_{x\to a}\frac{x^n-a^n}{x-a} = \lim_{x\to a}\frac{x^{\frac{p}{q}}-a^{\frac{p}{q}}}{x-a}$$

$$= \lim_{y\to b}\frac{y^p-b^p}{y^q-b^q}$$

$$= \lim_{y\to b}\left(\frac{y^p-b^p}{y-b}.\frac{y-b}{y^q-b^q}\right)$$

$$= \frac{pb^{p-1}}{qb^{q-1}} = \frac{p}{q}b^{p-q}$$

$$= n\left(b^q\right)^{\frac{p}{q}-1} = na^{n-1}.$$

Note: The above theorem is valid for all real numbers n. The proof, in the case of irrational n, is more complicated and beyond the scope of this book.

8.3.3 Theorem

(i) $\lim_{x\to 0}\cos x = 1$

and (ii) $\lim_{x\to 0}\frac{\sin x}{x} = 1.$

Proof

First we prove the inequality

$$\sin x < x < \tan x \quad \text{for } x \in \left(0, \frac{\pi}{2}\right).$$

Consider the unit circle centered at O.

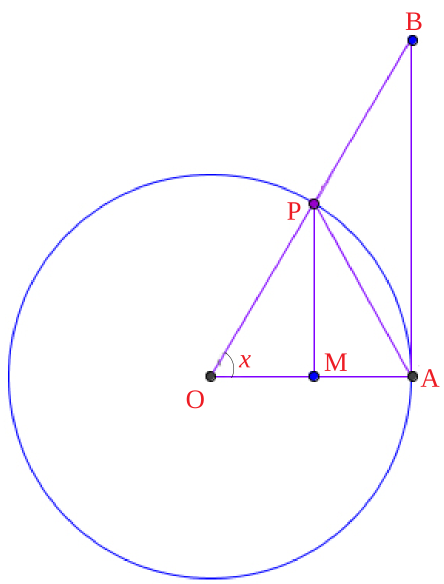


Fig. 8.9

Let $\angle AOP = x$, $0 < x < \frac{\pi}{2}$ (Fig. 8.9). Then

the area of Δ AOP < area of the sector OAP < area of the Δ OAB

$$\Rightarrow \frac{1}{2} \text{OA.PM} < \frac{1}{2}.\ x\ .1 < \frac{1}{2} \text{OA.AB}$$

$$\text{(since the area of the sector} = \frac{1}{2} lr = \frac{1}{2} r^2 \theta)$$

$$\Rightarrow \frac{1}{2} \sin x < \frac{x}{2} < \frac{1}{2} \tan x \Rightarrow \sin x < x < \tan x \quad \text{for } x \in \left(0, \frac{\pi}{2}\right).$$

Therefore $1 < \dfrac{x}{\sin x} < \dfrac{1}{\cos x}$ and hence $\cos x < \dfrac{\sin x}{x} < 1$.

The same inequalities hold good even if $-\frac{\pi}{2} < x < 0$,

$$\cos x \leq \frac{\sin x}{x} \leq 1.$$

Hence $0 \leq 1 - \dfrac{\sin x}{x} \leq 1 - \cos x = 2 \sin^2 \dfrac{x}{2} \leq 2\left(\dfrac{x}{2}\right)^2 = \dfrac{x^2}{2}$.

$$\text{Also } \lim_{x \to 0} \frac{x^2}{2} = 0.$$

Hence, by Sandwich theorem, we have

$$\lim_{x \to 0} \left(1 - \frac{\sin x}{x}\right) = 0 = \lim_{x \to 0} (1 - \cos x).$$

Now by Theorem 8.2.5, we get

$$\lim_{x \to 0} \cos x = 1 \text{ and } \lim_{x \to 0} \frac{\sin x}{x} = 1.$$

Now we state three more important theorems, whose proofs are beyond the scope of the book.

8.3.4 Theorem

$$\boxed{\lim_{x \to 0} (1 + x)^{\frac{1}{x}} = e.}$$

8.3.5 Theorem

$$\boxed{\lim_{x \to 0} \left(\frac{a^x - 1}{x}\right) = \log_e a.}$$

8.3.6 Corollary

$$\lim_{x \to 0} \left(\frac{e^x - 1}{x} \right) = 1.$$

Proof: Take $a = e$ in Theorem 8.3.5

8.3.7 Theorem

$$\lim_{x \to 0} \frac{\log_e (1+x)}{x} = 1$$

8.3.8 Solved Problems

1. Problem : *Show that* $\lim_{x \to 0} \frac{\tan x}{x} = 1.$

Solution : We have $\frac{\tan x}{x} = \frac{\sin x}{x} \cdot \frac{1}{\cos x} \quad (x \neq 0)$

Hence $\lim_{x \to 0} \frac{\tan x}{x} = \lim_{x \to 0} \left(\frac{\sin x}{x} \right) \lim_{x \to 0} \left(\frac{1}{\cos x} \right)$

$= 1 \, . \, 1 = 1.$

2. Problem : *Find* $\lim_{x \to 0} \left(\frac{\sqrt{1+x} - 1}{x} \right).$

Solution : For $0 < |x| < 1$, we have

$$\frac{\sqrt{x+1} - 1}{x} = \frac{\sqrt{1+x} - 1}{x} \cdot \frac{\sqrt{1+x} + 1}{\sqrt{1+x} + 1}$$

$$= \frac{1}{\sqrt{1+x} + 1}.$$

Therefore $\lim_{x \to 0} \frac{\sqrt{1+x} - 1}{x} = \lim_{x \to 0} \frac{1}{\sqrt{1+x} + 1} = \frac{1}{2}.$

3. Problem : Compute $\lim_{x \to 0} \left(\frac{e^x - 1}{\sqrt{1+x} - 1} \right).$

Solution : For $0 < |x| < 1$,

$$\frac{e^x - 1}{\sqrt{1+x} - 1} = \frac{e^x - 1}{\sqrt{1+x} - 1} \times \frac{\sqrt{1+x} + 1}{\sqrt{1+x} + 1}$$

$$= \left(\frac{e^x - 1}{x} \right) \left(\sqrt{1+x} + 1 \right)$$

Therefore, $\lim\limits_{x\to 0}\dfrac{e^x-1}{\sqrt{1+x}-1} = \lim\limits_{x\to 0}\dfrac{e^x-1}{x}\lim\limits_{x\to 0}\left(\sqrt{1+x}+1\right)$

$= 1\,.\,2 = 2.$

4. Problem : *Show that* $\lim\limits_{x\to 3}\dfrac{x-3}{\sqrt{|x^2-9|}} = 0.$

Solution : For $x^2 \neq 9$, $\left|\dfrac{x-3}{\sqrt{x^2-9}}\right| = \sqrt{\left|\dfrac{x-3}{x+3}\right|}$ ... (1)

But $\lim\limits_{x\to 3}\sqrt{|x-3|} = 0$ and $\lim\limits_{x\to 3}\sqrt{|x+3|} = \sqrt{6}$ so that $\lim\limits_{x\to 3}\sqrt{\left|\dfrac{x-3}{x+3}\right|} = 0.$

Therefore from (1), $\lim\limits_{x\to 3}\dfrac{x-3}{\sqrt{|x^2-9|}} = 0.$

5. Problem : *Compute* $\lim\limits_{x\to 0}\dfrac{a^x-1}{b^x-1}$, *$(a > 0, b > 0, b \neq 1)$.*

Solution : For $x \neq 0$, $\dfrac{a^x-1}{b^x-1} = \dfrac{\left(\dfrac{a^x-1}{x}\right)}{\left(\dfrac{b^x-1}{x}\right)}$ .... (1)

But, by Theorem 8.3.5, $\lim\limits_{x\to 0}\dfrac{a^x-1}{x} = \log_e a$

and $\lim\limits_{x\to 0}\dfrac{b^x-1}{x} = \log_e b.$

Therefore, from (1), $\lim\limits_{x\to 0}\dfrac{a^x-1}{b^x-1} = \dfrac{\log_e a}{\log_e b}.$

6. Problem : *Compute* $\lim\limits_{x\to 0}\dfrac{\sin ax}{\sin bx}$, *$b \neq 0, a \neq b$.*

Solution : $\lim\limits_{x\to 0}\dfrac{\sin ax}{\sin bx} = \lim\limits_{x\to 0}\left(\dfrac{\sin ax}{ax}\cdot\dfrac{bx}{\sin bx}\cdot\dfrac{a}{b}\right)$

$= \dfrac{a}{b}\lim\limits_{x\to 0}\dfrac{\sin ax}{ax}\cdot\lim\limits_{x\to 0}\dfrac{bx}{\sin bx}$

$= \dfrac{a}{b}\cdot 1\cdot 1$ (since, as $x \to 0$, $ax \to 0$ and $bx \to 0$)

$= \dfrac{a}{b}.$

7. Problem : *Compute* $\lim_{x\to 0}\frac{e^{3x}-1}{x}$.

Solution : We have $\lim_{x\to 0}\frac{e^{3x}-1}{x} = 3\lim_{3x\to 0}\frac{e^{3x}-1}{3x}$

$$= 3\lim_{y\to 0}\frac{e^{y}-1}{y}, \text{ where } y = 3x$$

$$= 3.1 = 3.$$

8. Problem : *Compute* $\lim_{x\to 0}\frac{e^{x}-\sin x-1}{x}$.

Solution : We have $\lim_{x\to 0}\frac{e^{x}-\sin x-1}{x} = \lim_{x\to 0}\left[\frac{e^{x}-1}{x}-\frac{\sin x}{x}\right]$

$$= \lim_{x\to 0}\left(\frac{e^{x}-1}{x}\right) - \lim_{x\to 0}\frac{\sin x}{x}$$

$$= 1 - 1 = 0.$$

9. Problem : *Evaluate* $\lim_{x\to 1}\frac{\log_e x}{x-1}$.

Solution : Put $y = x - 1$. Then $y \to 0$ as $x \to 1$.

Now $\lim_{x\to 1}\frac{\log_e x}{x-1} = \lim_{y\to 0}\frac{\log_e(1+y)}{y}$

$= 1$ (by Theorem 8.3.7)

Exercise 8(c)

Compute the following limits.

I.

1. $\lim_{x\to 1}\left(\frac{2x+1}{3x^2-4x+5}\right)$

2. $\lim_{x\to \frac{\pi}{2}}\frac{\cos x}{\left(x-\frac{\pi}{2}\right)}$

3. $\lim_{x\to 0}\frac{\sin ax}{x\cos x}$

4. $\lim_{x\to 1}\frac{\sin(x-1)}{\left(x^2-1\right)}$

5. $\lim_{x\to 0}\frac{\sin(a+bx)-\sin(a-bx)}{x}$

6. $\lim_{x\to a}\frac{\tan(x-a)}{x^2-a^2}$ $(a \neq 0)$

7. $\lim_{x\to 0}\frac{e^{7x}-1}{x}$

8. $\lim_{x\to 0}\frac{e^{3+x}-e^3}{x}$

9. $\lim_{x\to 3}\frac{e^{x}-e^3}{x-3}$

10. $\lim_{x\to 0}\frac{e^{\sin x}-1}{x}$

II. **1.** $\lim\limits_{x\to 1} \dfrac{(2x-1)(\sqrt{x}-1)}{(2x^2+x-3)}$ **2.** $\lim\limits_{x\to a} \left(\dfrac{x\sin a - a\sin x}{x-a}\right)$

3. $\lim\limits_{x\to 0} \left(\dfrac{\cos ax - \cos bx}{x^2}\right)$ **4.** $\lim\limits_{x\to 2} \dfrac{(2x^2-7x-4)}{(2x-1)\ (\sqrt{x}-2)}$

5. $\lim\limits_{x\to 0} \dfrac{\log_e(1+5x)}{x}$ **6.** $\lim\limits_{x\to 0} \dfrac{\sin(\pi\cos^2 x)}{x^2}$

III. **1.** $\lim\limits_{x\to 0} \left(\dfrac{(1+x)^{\frac{1}{8}}-(1-x)^{\frac{1}{8}}}{x}\right)$ **2.** $\lim\limits_{x\to 0} \left(\dfrac{3^x-1}{\sqrt{1+x}-1}\right)$

3. $\lim\limits_{x\to a} \left(\dfrac{\sqrt{a+2x}-\sqrt{3x}}{\sqrt{3a+x}-2\sqrt{x}}\right)$ **4.** $\lim\limits_{x\to 0} \dfrac{\sqrt[3]{1+x}-\sqrt[3]{1-x}}{x}$

5. $\lim\limits_{x\to a} \dfrac{\sin(x-a)\tan^2(x-a)}{(x^2-a^2)^2}$ **6.** $\lim\limits_{x\to 0} \dfrac{1-\cos 2mx}{\sin^2 nx}$ $(m, n \in \mathbf{Z})$

7. $\lim\limits_{x\to 0} \dfrac{1-\cos x}{x}$ **8.** $\lim\limits_{x\to 0} \dfrac{\sec x - 1}{x^2}$

9. $\lim\limits_{x\to 0} \dfrac{1-\cos mx}{1-\cos nx},\ n \neq 0$ **10.** $\lim\limits_{x\to 0} \dfrac{x(e^x-1)}{1-\cos x}$

11. $\lim\limits_{x\to 0} \dfrac{\log(1+x^3)}{\sin^3 x}$ **12.** $\lim\limits_{x\to 0} \dfrac{x\tan 2x - 2x\tan x}{(1-\cos 2x)^2}$

8.3.9 Infinite limits and limits at infinity

Consider $f(x) = x^{-4}$ for $x \neq 0$. At the points very close to zero the values of $f(x)$ would be increasing rapidly. Thus we can't have the concept of limit at zero for this function. We shall try to describe this nature of a function in the present section.

(i) Let $\mathrm{E} \subseteq \mathbf{R}$, $f : \mathrm{E} \to \mathbf{R}$ and $a \in \mathbf{R}$ be such that $\mathrm{E} \cap ((a-r, a+r) \setminus \{a\})$ is non-empty for every $r > 0$.

We say that $f(x)$ tends to ∞ as $x \to a$, and write $\lim\limits_{x\to a} f(x) = \infty$,

if given $\alpha \in \mathbf{R}$ there exists a $\delta > 0$ such that $f(x) > \alpha$ for all $x \in \mathrm{E}$ with $0 < |x-a| < \delta$.

(ii) $f(x)$ is said to tend to $-\infty$ as $x \to a$ and write $\lim\limits_{x\to a} f(x) = -\infty$, if given $\beta \in \mathbf{R}$ there exists a $\delta > 0$ such that $f(x) < \beta$ for all $x \in \mathrm{E}$ with $0 < |x-a| < \delta$.

(iii) Let $\mathrm{E} \subseteq \mathbf{R}$ and $f : \mathrm{E} \to \mathbf{R}$. Suppose $(a, \infty) \subseteq \mathrm{E}$ for some $a \in \mathbf{R}$. Then we say that $l \in \mathbf{R}$ is a limit of $f(x)$ as $x \to \infty$, and write $\lim\limits_{x\to\infty} f(x) = l$, if given $\varepsilon > 0$ there exists a $\mathrm{K} > a$ such that $|f(x) - l| < \varepsilon$ for all $x > \mathrm{K}$.

Such an l, if exists, is unique.

(iv) Let $E \subseteq \mathbf{R}$ and $f : E \to \mathbf{R}$. Suppose $(-\infty, a) \subseteq E$ for some $a \in \mathbf{R}$. Then we say that $l \in \mathbf{R}$ is a limit of $f(x)$ as $x \to -\infty$, and write $\lim\limits_{x \to -\infty} f(x) = l$, if given $\varepsilon > 0$ there exists a $K < a$ such that $|f(x) - l| < \varepsilon$ for all $x < K$. Such an l, if exists, is unique.

(v) Let $E \subseteq \mathbf{R}$ and $f : E \to \mathbf{R}$. Suppose $(a, \infty) \subseteq E$ for some $a \in \mathbf{R}$. Then we say that $f(x)$ tends to ∞ as $x \to \infty$, and write $\lim\limits_{x \to \infty} f(x) = \infty$, if given $\alpha \in \mathbf{R}$ there exists a $K > a$ such that $f(x) > \alpha$ for all $x > K$.

(vi) Let $E \subseteq \mathbf{R}$ and let $f : E \to \mathbf{R}$. Suppose $(a, \infty) \subseteq E$ for some $a \in \mathbf{R}$. Then we say that $f(x)$ tends to $-\infty$ as $x \to \infty$, and write $\lim\limits_{x \to \infty} f(x) = -\infty$, if given $\alpha \in \mathbf{R}$ there exists a $K > a$ such that $f(x) < \alpha$ for all $x > K$.

(vii) Let $E \subseteq \mathbf{R}$ and let $f : E \to \mathbf{R}$. Suppose $(-\infty, a) \subseteq E$ for some $a \in \mathbf{R}$. Then we say that $f(x)$ tends to ∞ as $x \to -\infty$, and write $\lim\limits_{x \to -\infty} f(x) = \infty$, if given $\alpha \in \mathbf{R}$ there exists a $K < a$ such that $f(x) > \alpha$ for all $x < K$.

(viii) Let $E \subseteq \mathbf{R}$ and let $f : E \to \mathbf{R}$. Suppose $(-\infty, a) \subseteq E$ for some $a \in \mathbf{R}$. Then we say that $f(x)$ tends to $-\infty$ as $x \to -\infty$, and write $\lim\limits_{x \to -\infty} f(x) = -\infty$, if given $\alpha \in \mathbf{R}$ there exists a $K < a$ such that $f(x) < \alpha$ for all $x < K$.

In order to compute the limits defined in (i) through (vii), the following theorem is of great use. We state the theorem without proof as the proof is beyond the scope of this book.

8.3.10 Theorem

Let $E \subseteq \mathbf{R}$, $f : E \to \mathbf{R}$ and $a \in \mathbf{R}$ be such that $((a - r, a + r) \setminus \{a\}) \cap E$ is non-empty for every $r > 0$.

(i) *Suppose $\lim\limits_{x \to a} f(x) = \infty$. Then $\lim\limits_{x \to a} \dfrac{1}{f(x)} = 0$.*

(ii) *Suppose $\lim\limits_{x \to a} f(x) = -\infty$. Then $\lim\limits_{x \to a} \dfrac{1}{f(x)} = 0$.*

(iii) *If $\lim\limits_{x \to a} f(x) = 0$ and f is positive in a deleted neighbourhood of a, then $\lim\limits_{x \to a} \dfrac{1}{f(x)} = \infty$.*

(iv) *If $\lim\limits_{x \to a} f(x) = 0$ and f is negative in a deleted neighbourhood of a, then*

$$\lim_{x \to a} \frac{1}{f(x)} = -\infty.$$

8.3.11 Theorem

Let $E \subseteq \mathbf{R}$. $f : E \to \mathbf{R}$, $g : E \to \mathbf{R}$, $h : E \to \mathbf{R}$ *and let* $(a, \infty) \subset E$ *for some* $a \in \mathbf{R}$. *If* $\lim_{x\to\infty} g(x) = l = \lim_{x\to\infty} h(x)$ *and* $g(x) \le f(x) \le h(x)$ *for all* $x \in E$ *then* $\lim_{x\to\infty} f(x) = l$.

8.3.12 Solved Problems

1. Problem : *Show that* $\lim_{x\to\infty} \frac{1}{x^2} = 0$.

Solution : Given $\varepsilon > 0$, choose $\alpha = \frac{1}{\sqrt{\varepsilon}} > 0$. Then

$$x > \alpha \Rightarrow x > \frac{1}{\sqrt{\varepsilon}} \Rightarrow x^2 > \frac{1}{\varepsilon} \Rightarrow \frac{1}{x^2} < \varepsilon \Rightarrow \left|\frac{1}{x^2} - 0\right| < \varepsilon.$$

Hence $\lim_{x\to\infty} \frac{1}{x^2} = 0$.

2. Problem : *Show that* $\lim_{x\to\infty} e^x = \infty$.

Solution : Given $K > 0$, let $\alpha = \log K$. Then $x > \alpha \Rightarrow e^x > e^\alpha = K$.

Hence $\lim_{x\to\infty} e^x = \infty$.

3. Problem : *Compute* $\lim_{x\to 2} \frac{x^2 + 2x - 1}{x^2 - 4x + 4}$.

Solution : Write $f(x) = \frac{x^2 - 4x + 4}{x^2 + 2x - 1} = \frac{(x-2)^2}{x^2 + 2x - 1}$.

Clearly $f(x) > 0$ in a deleted neighbourhood of 2.

Moreover $\lim_{x\to 2} f(x) = 0$ so that $\lim_{x\to 2} \frac{1}{f(x)} = \infty$.

Hence $\lim_{x\to 2} \frac{x^2 + 2x - 1}{x^2 - 4x + 4} = \infty$.

4. Problem : *Evaluate* $\lim_{x\to\infty} \frac{3x^5 - 1}{4x^2 + 1}$.

Solution : Both the numerator and the denominator approach ∞ as $x \to \infty$.

Now $\lim_{x\to\infty} \frac{3x^5 - 1}{4x^2 + 1} = \lim_{x\to\infty} \frac{x^3\left(3 - \frac{1}{x^5}\right)}{\left(4 + \frac{1}{x^2}\right)} = \infty$.

5. Problem : *If* $f(x) = \dfrac{a_n x^n + \ldots + a_1 x + a_0}{b_m x^m + \ldots + b_1 x + b_0}$ *with* $a_n > 0$, $b_m > 0$ *then show that* $\lim\limits_{x\to\infty} f(x) = \infty$ if $n > m$.

Solution : $$f(x) = \frac{x^n\left(a_n + \dfrac{a_{n-1}}{x} + \ldots + \dfrac{a_1}{x^{n-1}} + \dfrac{a_0}{x^n}\right)}{x^m\left(b_m + \dfrac{b_{m-1}}{x} + \ldots + \dfrac{b_1}{x^{m-1}} + \dfrac{b_0}{x^m}\right)}$$

$$= x^{n-m}\left(\frac{a_n + \left(\dfrac{a_{n-1}}{x}\right) + \ldots + \left(\dfrac{a_1}{x^{n-1}}\right) + \left(\dfrac{a_0}{x^n}\right)}{b_m + \left(\dfrac{b_{m-1}}{x}\right) + \ldots + \left(\dfrac{b_1}{x^{m-1}}\right) + \left(\dfrac{b_0}{x^m}\right)}\right)$$

As $x \to \infty$, all the quotients $\dfrac{a_{n-j}}{x^j}, \dfrac{b_{m-i}}{x^i}$ approach zero. Therefore the quantity in the big bracket above approaches $\dfrac{a_n}{b_m}\ (>0)$. But $\lim\limits_{x\to\infty} x^{n-m} = \infty$ (since $n > m$). Hence $\lim\limits_{x\to\infty} f(x) = \infty$.

6. Problem : *Compute* $\lim\limits_{x\to\infty} \dfrac{x^2 - \sin x}{x^2 - 2}$.

Solution : $-1 \le \sin x \le 1$ implies that $-1 \le -\sin x \le 1$.

Therefore $x^2 - 1 \le x^2 - \sin x \le x^2 + 1$.

Since $x \to \infty$, we can suppose without loss, that $x^2 - 2 > 0$.

Therefore $\dfrac{x^2 - 1}{x^2 - 2} \le \dfrac{x^2 - \sin x}{x^2 - 2} \le \dfrac{x^2 + 1}{x^2 - 2}$.

Here $$\lim_{x\to\infty} \frac{x^2 - 1}{x^2 - 2} = \lim_{x\to\infty}\left(\frac{1 - \dfrac{1}{x^2}}{1 - \dfrac{2}{x^2}}\right) = 1$$

and $$\lim_{x\to\infty} \frac{x^2 + 1}{x^2 - 2} = \lim_{x\to\infty}\left(\frac{1 + \dfrac{1}{x^2}}{1 - \dfrac{2}{x^2}}\right) = 1.$$

Hence by the Theorem 8.3.11, we get $\lim\limits_{x\to\infty} \dfrac{x^2 - \sin x}{x^2 - 2} = 1$.

Exercise 8(d)

Compute the following limits.

I. 1. $\lim\limits_{x\to 3} \dfrac{x^2+3x+2}{x^2-6x+9}$ 2. $\lim\limits_{x\to 1-} \dfrac{1+5x^3}{1-x^2}$

3. $\lim\limits_{x\to\infty} \dfrac{3x^2+4x+5}{2x^3+3x-7}$ 4. $\lim\limits_{x\to\infty} \dfrac{6x^2-x+7}{x+3}$

5. $\lim\limits_{x\to\infty} e^{-x^2}$ 6. $\lim\limits_{x\to\infty} \dfrac{\sqrt{x^2+6}}{2x^2-1}$

II. 1. $\lim\limits_{x\to\infty} \dfrac{8|x|+3x}{3|x|-2x}$ 2. $\lim\limits_{x\to\infty} \dfrac{x^2+5x+2}{2x^2-5x+1}$

3. $\lim\limits_{x\to-\infty} \dfrac{2x^2-x+3}{x^2-2x+5}$ 4. $\lim\limits_{x\to\infty} \dfrac{11x^3-3x+4}{13x^3-5x^2-7}$

5. $\lim\limits_{x\to 2} \left(\dfrac{1}{x-2}-\dfrac{4}{x^2-4}\right)$ 6. $\lim\limits_{x\to-\infty} \dfrac{5x^3+4}{\sqrt{2x^4+1}}$

7. $\lim\limits_{x\to\infty} \left(\sqrt{x+1}-\sqrt{x}\right)$ 8. $\lim\limits_{x\to\infty} \left(\sqrt{x^2+x}-x\right)$

III. 1. $\lim\limits_{x\to-\infty} \left(\dfrac{2x+3}{\sqrt{x^2-1}}\right)$ 2. $\lim\limits_{x\to\infty} \dfrac{2+\sin x}{x^2+3}$

3. $\lim\limits_{x\to\infty} \dfrac{2+\cos^2 x}{x+2007}$ 4. $\lim\limits_{x\to-\infty} \dfrac{6x^2-\cos 3x}{x^2+5}$

5. $\lim\limits_{x\to\infty} \dfrac{\cos x+\sin^2 x}{x+1}$

8.4 Continuity

In this section, we shall define one of the most important concepts of mathematical analysis, namely, the continuity of a function at a point and on a set. We shall also discuss the relation between limits and continuity.

We start this section with some examples.

1. Example : Define $f : \mathbf{R} \to \mathbf{R}$ by $f(x) = 2x + 1$, $x \in \mathbf{R}$. Note that this function is defined at every point of $\mathbf{R}$. The graph of this function is given in Fig. 8.10.

We observe that the left hand limit of f at $x = 0$ is 1 and also the right hand limit of f at $x = 0$ is 1.

Thus $\lim_{x \to 0+} f(x) = \lim_{x \to 0-} f(x) = 1$ and this value is equal to $f(0) = 1$.

Here, it is worth mentioning that it is possible to draw the graph of the function around the point $x = 0$ without lifting the pen from the plane of the paper. Since the same is true for every point in **R**, graphically the function $f(x) = 2x + 1$ defines a line without any breaks.

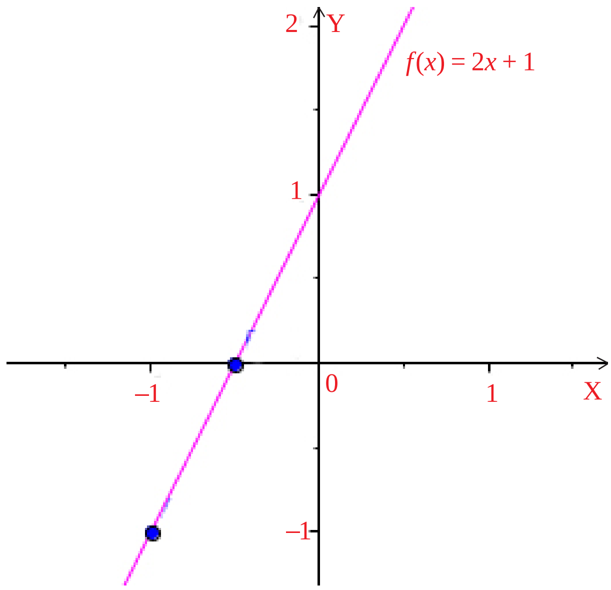


Fig. 8.10

2. Example : Define $f : \mathbf{R} \to \mathbf{R}$ by $f(x) = \begin{cases} 1 \text{ if } x \le 0 \\ 2 \text{ if } x > 0 \end{cases}$.

This function is defined at every point of **R**. The graph of this function is given in Fig. 8.11. It is easy to see that

$\lim_{x \to 0+} f(x) = 2 \ne f(0)$ and

$\lim_{x \to 0-} f(x) = 1$.

Here we note that it is not possible to draw the graph of the function on the plane of the paper without lifting the pen at $x = 0$. The graph of the function has two broken lines, and has a break at $x = 0$.

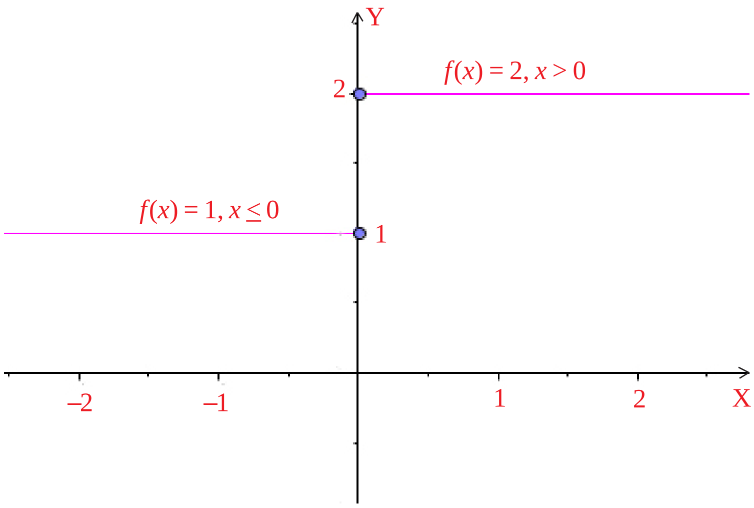


Fig. 8.11

3. Example : Define $f : \mathbf{R} \to \mathbf{R}$ by $f(x) = x^2, x \in \mathbf{R}$.

We observe that $\lim_{x \to 0+} f(x) = 0$ and $\lim_{x \to 0-} f(x) = 0$ with $f(0) = 0$, so that $\lim_{x \to 0} f(x) = \lim_{x \to 0+} f(x) = \lim_{x \to 0-} f(x) = 0 = f(0)$.

The graph $f(x) = x^2$ is a curved path which touches the X-axis at the origin, without any breaks (see Fig. 8.12).

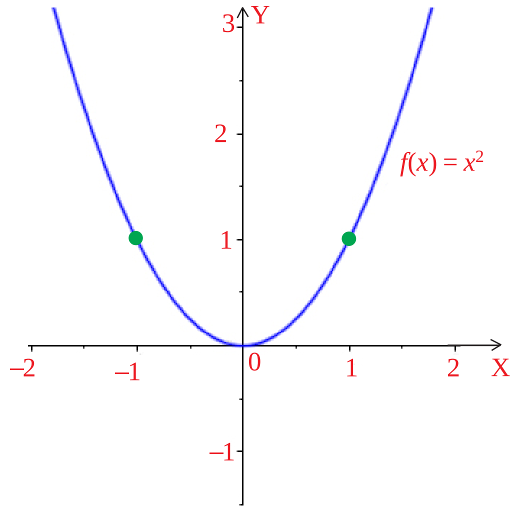


Fig. 8.12

8.4.1 Definition

Let $E \subseteq \mathbf{R}$, $f : E \to \mathbf{R}$ and $a \in E$. Then we say that f is continuous at a if given $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - f(a)| < \varepsilon$ whenever $x \in E$ and $|x - a| < \delta$.

If f is continuous at every point of E then we say that f is continuous on E.

If f is not continuous at a, we say that f is discontinuous at a.

Observe that, we talk of continuity or discontinuity of f at a point 'a' only when 'a' is in the domain of f.

8.4.2 Remark

Let $E \subseteq \mathbf{R}$, $f : E \to \mathbf{R}$ and $a \in E$. Suppose that there exists a positive real number r such that $(a - r, a + r) \cap E = \{a\}$. Then f is continuous at a. a being arbitrary, f is continuous on $\mathbf{N}$.

For, let $\varepsilon > 0$ be given. Then

$$x \in E \text{ and } |x - a| < r \;\Rightarrow\; x = a \text{ (since } (a - r, a + r) \cap E = \{a\})$$
$$\Rightarrow\; f(x) = f(a)$$
$$\Rightarrow\; |f(x) - f(a)| = 0 < \varepsilon.$$

As a particular case, any function $f : \mathbf{N} \to \mathbf{R}$ is continuous on $\mathbf{N}$. In fact, for any given $a \in \mathbf{N}$, $\left(a - \frac{1}{2}, a + \frac{1}{2}\right) \cap \mathbf{N} = \{a\}$. Hence f is continuous at a. As a being arbitrary, f is continuous on $\mathbf{N}$.

8.4.3 Geometric interpretation of continuity at a point

If $f : \mathbf{R} \to \mathbf{R}$ is a function, recall that the set $\{(x, f(x)) \in \mathbf{R} \times \mathbf{R} : x \in \mathbf{R}\}$ is called the graph of f.

If $[a, b]$ and $[c, d]$ are intervals in $\mathbf{R}$ then the cartesian product $[a, b] \times [c, d]$ is called a rectangle R in the plane. Infact, $R = [a, b] \times [c, d] = \{(x, y) : a \le x \le b,\ c \le y \le d\}$. (see Fig. 8.13). Note that $b - a$ is the width and $d - c$ is the height of the rectangle.

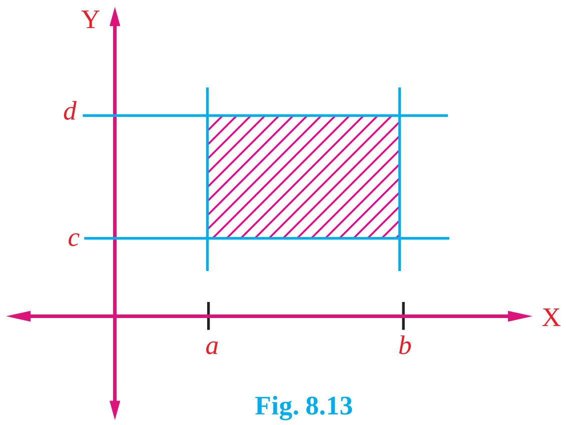


Fig. 8.13

Now recall that f is continuous at a point x_0 in $\mathbf{R}$ if and only if to each $\varepsilon > 0$, there is a $\delta > 0$ such that $x \in (x_0 - \delta, x_0 + \delta)$ implies $f(x) \in (f(x_0) - \varepsilon, f(x_0) + \varepsilon)$. That is f is continuous at x_0 if and only if to each $\varepsilon > 0$ there is a $\delta > 0$ such that $(x, f(x)) \in (x_0 - \delta, x_0 + \delta) \times (f(x_0) - \varepsilon, f(x_0) + \varepsilon) = R_{\varepsilon\delta}$ (say).

Thus f is continuous at $x_0 \in \mathbf{R}$ if and only if to each $\varepsilon > 0$ there is a $\delta > 0$ such that the part of the graph of f given by

$\{(x, f(x)) : x \in (x_0 - \delta, x_0 + \delta)\} \subseteq R_{\varepsilon\delta}$ (see Fig. 8.14).

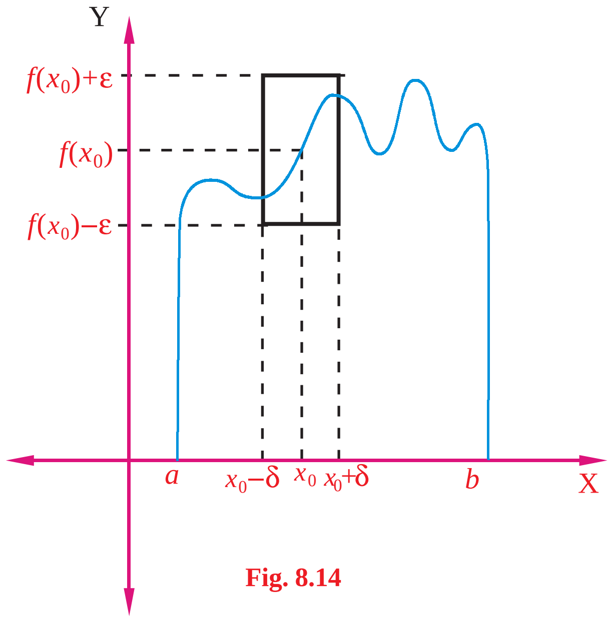


Fig. 8.14

That is, as the height 2ε of the rectangle $R_{\varepsilon\delta}$ is sufficiently small, a part of the graph of f is contained in $R_{\varepsilon\delta}$.

8.4.4 Remark

Let $E \subseteq \mathbf{R}$, $f : E \to \mathbf{R}$ and $a \in E$ be such that $((a-r, a+r) \setminus \{a\}) \cap E$ is non - empty for every $r > 0$. Then f is continuous at a if and only if $\lim\limits_{x \to a} f(x) = f(a)$.

If $((a-r, a+r) \setminus \{a\}) \cap E$ is non-empty for every $r > 0$ then for f to be continuous at a the following conditions should be valid.

(i) f should be defined at a,

(ii) $\lim\limits_{x \to a} f(x)$ must exist,

(iii) $f(a) = \lim\limits_{x \to a} f(x)$.

Theorem analogous to Theorem 8.2.5 can also be had for continuous functions.

8.4.5 Theorem

Let $E \subseteq \mathbf{R}$, f, g, be functions from E into $\mathbf{R}$ and $c \in \mathbf{R}$. Suppose $a \in E$ and f, g are continuous at a. Then

(i) $f + g$, $f - g$, fg, cf *are all continuous at* a.

(ii) *if, in addition,* $g(a) \neq 0$ *then the quotient* $\frac{f}{g}$ *is continuous at* a.

8.4.6 Observation

When f and g are continuous at 'a', we have $\lim\limits_{x \to a} f(x) = f(a)$ and $\lim\limits_{x \to a} g(x) = g(a)$. Now, by Theorem 8.2.5, we get

(i) $\lim\limits_{x \to a}(f+g)(x) = f(a) + g(a)$

(ii) $\lim\limits_{x \to a}(f-g)(x) = f(a) - g(a)$

(iii) $\lim\limits_{x \to a}(fg)(x) = f(a)g(a)$

(iv) $\lim\limits_{x \to a}(cf)(x) = cf(a)$ for any $c \in \mathbf{R}$

(v) Also, if $g(a) \neq 0$ then $\lim\limits_{x \to a}\left(\frac{f}{g}\right)(x) = \frac{f(a)}{g(a)}$.

8.4.7 Theorem

Let $E \subseteq \mathbf{R}$, f, g be real valued continuous functions on E and $c \in \mathbf{R}$. Then

(i) $f + g$, $f - g$, fg *and* cf *are all continuous on* E.

(ii) *if, in addition,* $g(x) \neq 0$ *for all* $x \in E$ *then* $\frac{f}{g}$ *is continuous on* E.

Proofs of these two theorems are not given, as they are beyond the scope of this book.

8.4.8 Definition (Right and Left continuities)

Let $E \subseteq \mathbf{R}$, $f : E \to \mathbf{R}$ *be a function. Let* $a \in E$ *and* $((a-r, a+r) \setminus \{a\}) \cap E \neq \phi$ *for every* $r > 0$.

We say that a function f is right continuous at 'a' if $\lim_{x \to a+} f(x)$ *exists and is equal to f(a).*

Similarly, we say that f is left continuous at 'a' if $\lim_{x \to a-} f(x)$ *exists and is equal to f(a).*

8.4.9 Theorem

Let $E \subseteq \mathbf{R}$ *and* $f : E \to \mathbf{R}$ *be a function. Let* $a \in E$. *Then f is continuous at a if and only if* $\lim_{x \to a+} f(x)$ *and* $\lim_{x \to a-} f(x)$ *both exist and* $\lim_{x \to a+} f(x) = \lim_{x \to a-} f(x) = f(a)$.

The proof of this theorem is beyond the scope of the book.

8.4.10 Note

1. f is continuous on the closed interval $[a, b]$ if

(i) f is continuous in (a, b). (ii) $\lim_{x \to a+} f(x) = f(a)$.

(iii) $\lim_{x \to b-} f(x) = f(b)$.

2. Let $E \subseteq \mathbf{R}$. Let $a \in E$, f is discontinuous at a point $x = a$ in any one of the following cases.

(i) $\lim_{x \to a+} f(x)$ and $\lim_{x \to a-} f(x)$ exist, but are not equal.

(ii) $\lim_{x \to a+} f(x)$ and $\lim_{x \to a-} f(x)$ exist and are equal, but not equal to $f(a)$.

(iii) One or both of the two limits $\lim_{x \to a+} f(x)$ and $\lim_{x \to a-} f(x)$ fail to exist.

8.4.11 Examples

1. Polynomial functions are continuous on **R**. For more details, we see Theorem 8.2.6.

2. Rational functions are continuous at all points on **R** where the denominator is not zero. If p and q are polynomial functions on **R** and if $q(c) \neq 0$ then

$$\lim_{x \to c} \frac{p(x)}{q(x)} = \frac{p(c)}{q(c)}.$$

3. The function $f: \mathbf{R} \to \mathbf{R}$ defined by $f(x) = |x|, x \in \mathbf{R}$ is continuous on **R**.

Let $\varepsilon > 0$ be given. Let $c \in \mathbf{R}$. Then we choose $\delta > 0$ such that $\delta < \varepsilon$. For any $x \in \mathbf{R}$, with $|x - c| < \delta$ we have

$$|f(x) - f(c)| = ||x| - |c|| \le |x - c| < \delta < \varepsilon.$$

Hence, it follows that f is continuous at c. Since $c \in \mathbf{R}$ is arbitrary, we have f is continuous on **R**.

4. Sine function is continuous on **R**. We use the following properties of sine function.

For all $x, y, w \in \mathbf{R}$, we have

$$|\sin w| \le |w|, |\cos w| \le 1 \text{ and } \sin x - \sin y = 2\sin\left(\frac{x-y}{2}\right)\cos\left(\frac{x+y}{2}\right).$$

Hence, if $c \in \mathbf{R}$ then we have

$$|\sin x - \sin c| = 2\cdot\left|\sin\left(\frac{x-c}{2}\right)\right|\left|\cos\left(\frac{x+c}{2}\right)\right| \le 2\left|\frac{x-c}{2}\right| = |x-c|.$$

Therefore sine is continuous at c. Since $c \in \mathbf{R}$ is arbitrary, it follows that sine is continuous on **R**.

5. The cosine function is continuous on **R**. We make use of the following properties of sine and cosine functions. For all w, x, y in **R**,

$$|\sin w| \le |w|, |\sin w| \le 1$$

and $\cos x - \cos y = -2\sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$.

Hence, if $c \in \mathbf{R}$, then we have

$$|\cos x - \cos c| \le 2\cdot 1\cdot\frac{1}{2}|x-c| = |x-c|.$$

Therefore cosine function is continuous at c. Since $c \in \mathbf{R}$ is arbitrary, it follows that cos is continuous on **R**.

6. The functions tan, cot, sec, cosec are continuous where they are defined.

For example, the cotangent function is defined by $\cot x = \dfrac{\cos x}{\sin x}$ provided $\sin x \neq 0$ (that is, provided $x \neq n\pi, n \in \mathbf{Z}$). Since sine and cosine are continuous on **R**, it follows that the function cot is continuous on its domain, wherever it is defined. We treat the other trigonometric functions similarly.

We state the following theorem without proof.

8.4.12 Theorem

Let $A \subseteq \mathbf{R}$, $f : A \to \mathbf{R}$ *and let* $f(x) \geq 0$ *for all x in A. Let* $\sqrt{f}$ *be defined for* $x \in A$ *by* $(\sqrt{f})(x) = \sqrt{f(x)}$. *Then the following conclusions hold.*

(i) *If* f *is continuous at a point c in A, then* $\sqrt{f}$ *is continuous at c.*

(ii) *If f is continuous on A, then* $\sqrt{f}$ *is continuous on A.*

The continuity behaviour of the composition of two continuous functions is given in the following two theorems.

8.4.13 Theorem

Let $A, B \subseteq \mathbf{R}$, *and let* $f : A \to \mathbf{R}$ *and* $g : B \to \mathbf{R}$ *be two functions such that* $f(A) \subseteq B$. *If f is continuous at a point c in A and g is continuous at b = f(c) in B then the composition gof :* $A \to \mathbf{R}$ *is continuous at c.*

8.4.14 Theorem

Let $A, B \subseteq \mathbf{R}$. *Let* $f : A \to \mathbf{R}$ *be continuous on A and let* $g : B \to \mathbf{R}$ *be continuous on B. If* $f(A) \subseteq B$ *then the composite function gof :* $A \to \mathbf{R}$ *is continuous on A.*

8.4.15 Solved Problems

1. Problem: *Show that* $f(x) = [x]$ $(x \in \mathbf{R})$ *is continuous at only those real numbers that are not integers.*

Solution

Case (i) If $a \in \mathbf{Z}$ then $f(a) = [a] = a$. Now,

$$\lim_{x \to a-} f(x) = \lim_{h \to 0} [a - h] = a - 1, \quad \lim_{x \to a+} f(x) = \lim_{h \to 0} [a + h] = a.$$

Hence, $\lim_{x \to a-} f(x) \neq \lim_{x \to a+} f(x)$ so that $\lim_{x \to a} f(x)$ does not exist.

Hence f is not continuous at $x = a$.

Case (ii) If $a \notin \mathbf{Z}$. then there exists $n \in \mathbf{Z}$ such that $n < a < n + 1$ and $f(a) = [a] = n$.

Now, $\lim_{x \to a-} f(x) = \lim_{h \to 0} [a - h] = n$, $\lim_{x \to a+} f(x) = \lim_{h \to 0} [a + h] = n$.

So $\lim_{x \to a} f(x) = n = f(a)$.

Hence f is continuous at $x = a$.

2. Problem: *If* $f : \mathbf{R} \to \mathbf{R}$ *is such that* $f(x + y) = f(x) + f(y)$ *for all* $x, y \in \mathbf{R}$, *then f is continuous on* $\mathbf{R}$ *if it is continuous at a single point in* $\mathbf{R}$.

Solution: Let f be continuous at $x_0 \in \mathbf{R}$.

Then $\lim_{t \to x_0} f(t) = f(x_0)$ or $\lim_{h \to 0} f(x_0 + h) = f(x_0)$.

Let $x \in \mathbf{R}$. Now, since $f(x+h) - f(x) = f(h) = f(x_0 + h) - f(x_0)$, we have

$$\lim_{h \to 0} (f(x+h) - f(x)) = \lim_{h \to 0} (f(x_0 + h) - f(x_0)) = 0.$$

Therefore f is continuous at x. Since $x \in \mathbf{R}$ is arbitrary, f is continuous on $\mathbf{R}$.

3. Problem: *Check the continuity of the function f given below at 1 and 2.*

$$f(x) = \begin{cases} x+1 & \text{if } x \le 1 \\ 2x & \text{if } 1 < x < 2 \\ 1+x^2 & \text{if } x \ge 2 \end{cases}$$

Solution

We have $\lim\limits_{x \to 1+} f(x) = \lim\limits_{x \to 1+} 2x = 2 = f(1)$ and $\lim\limits_{x \to 1-} f(x) = \lim\limits_{x \to 1-} (x+1) = 2$.

Hence $\lim\limits_{x \to 1} f(x) = f(1)$.

Therefore f is continuous at 1.

Similarly, $\lim\limits_{x \to 2+} f(x) = \lim\limits_{x \to 2+} (1 + x^2) = 5 = f(2)$, and $\lim\limits_{x \to 2-} f(x) = \lim\limits_{x \to 2-} 2x = 4$.

Hence $\lim\limits_{x \to 2+} f(x)$ and $\lim\limits_{x \to 2-} f(x)$ exist, but are not equal so that f is not continuous at 2.

Note : *Let $E \subseteq \mathbf{R}$ and $f : E \to \mathbf{R}$ be a function. Suppose $a \in E$ is such that $f(a) \ne 0$ and f is continuous at a. Then we prove that there exists $r > 0$ such that $f(x) \ne 0$ for any x in* $(a - r, a + r) \cap E$.

For, consider $\dfrac{|f(a)|}{2} > 0$. Corresponding to this positive number, there exists $r > 0$ such that

$$x \in (a - r, a + r) \cap E \Rightarrow |f(x) - f(a)| < \frac{|f(a)|}{2} \Rightarrow f(x) \ne 0.$$

4. Problem : *Show that the function f defined on **R** by $f(x) = \cos x^2$, $x \in$ **R** is a continuous function.*

Solution : We define $h : \mathbf{R} \to \mathbf{R}$ by $h(x) = x^2$ and $g : \mathbf{R} \to \mathbf{R}$ by $g(x) = \cos x$. Now, for $x \in \mathbf{R}$, we have $(goh)(x) = g(h(x)) = g(x^2) = \cos x^2 = f(x)$.

Since g and h are continuous on their respective domains, by Theorem 8.4.14, it follows that f is a continuous function on **R**.

5. Problem : *Show that the function f defined on **R** by $f(x) = |1 + 2x + |x||$, $x \in$ **R** is a continuous function.*

Solution : We define $g : \mathbf{R} \to \mathbf{R}$ by $g(x) = 1 + 2x + |x|$, $x \in \mathbf{R}$,

and $h : \mathbf{R} \to \mathbf{R}$ by $h(x) = |x|$, $x \in \mathbf{R}$. Then

$$(hog)(x) = h(g(x)) = h(1 + 2x + |x|) = |1 + 2x + |x|| = f(x).$$

By Example 8.4.11(3), we have h is a continuous function. Since g is the sum of the polynomial function $1 + 2x$ and the modulus function $|x|$ and since both are continuous functions, by Theorem 8.4.7(i), g is continuous.

Since f is the composition of two continuous functions h and g, by Theorem 8.4.14, it follows that f is continuous.

Exercise 8(e)

I. **1.** Is the function f, defined by $f(x) = \begin{cases} x^2 & \text{if } x \le 1 \\ x & \text{if } x > 1 \end{cases}$, continuous on $\mathbf{R}$?

2. Is f defined by $f(x) = \begin{cases} \dfrac{\sin 2x}{x} & \text{if } x \ne 0 \\ 1 & \text{if } x = 0 \end{cases}$ continuous at 0?

3. Show that the function $f(x) = [\cos(x^{10} + 1)]^{1/3}, x \in \mathbf{R}$ is a continuous function.

II. **1.** Check the continuity of the following function at 2.

$$f(x) = \begin{cases} \frac{1}{2}\left(x^2 - 4\right) & \text{if } 0 < x < 2 \\ 0 & \text{if } x = 2 \\ 2 - 8x^{-3} & \text{if } x > 2 \end{cases}$$

2. Check the continuity of f given by

$$f(x) = \begin{cases} \left(x^2 - 9\right)\big/\left(x^2 - 2x - 3\right) & \text{if } 0 < x < 5 \text{ and } x \ne 3 \\ 1.5 & \text{if } x = 3 \end{cases}$$ at the point 3.

3. Show that f, given by $f(x) = \dfrac{x - |x|}{x}$ $(x \ne 0)$, is continuous on $\mathbf{R} \setminus \{0\}$.

4. If f is a function defined by $f(x) = \begin{cases} \dfrac{x-1}{\sqrt{x} - 1} & \textit{if} \quad x > 1 \\ 5 - 3x & \textit{if} \quad -2 \le x \le 1, \\ \dfrac{6}{x - 10} & \textit{if} \quad x < -2 \end{cases}$

then discuss the continuity of f.

5. If f, given by $f(x) = \begin{cases} k^2 x - k & \textit{if} \quad x \ge 1 \\ 2 & \textit{if} \quad x < 1 \end{cases}$, is a continuous function on $\mathbf{R}$, then find the values of k.

6. Prove that the functions $\sin x$ and $\cos x$ are continuous on $\mathbf{R}$.

III. **1.** Check the continuity of f given by $f(x) = \begin{cases} 4-x^2 & \text{if } x \le 0 \\ x-5 & \text{if } 0 < x \le 1 \\ 4x^2-9 & \text{if } 1 < x < 2 \\ 3x+4 & \text{if } x \ge 2 \end{cases}$

at the points 0, 1 and 2.

2. Find real constants a, b so that the function f given by

$$f(x) = \begin{cases} \sin x & \text{if } x \le 0 \\ x^2+a & \text{if } 0 < x < 1 \\ bx+3 & \text{if } 1 \le x \le 3 \\ -3 & \text{if } x > 3 \end{cases}$$

is continuous on **R**.

3. Show that $f(x) = \begin{cases} \dfrac{\cos ax - \cos bx}{x^2} & \text{if } x \ne 0 \\ \dfrac{1}{2}\left(b^2 - a^2\right) & \text{if } x = 0 \end{cases}$

where a and b are real constants, is continuous at 0.

Key Concepts

- Intervals and neighbourhoods.
- Definition of the limit of a function.
- If $\lim\limits_{x\to a} f(x) = l$, $\lim\limits_{x\to a} g(x) = m$ and $k \in \mathbf{R}$ then

  $\lim\limits_{x\to a} (f+g)(x) = l+m$, $\lim\limits_{x\to a} (f-g)(x) = l-m$,

  $\lim\limits_{x\to a} (fg)(x) = lm$ and $\lim\limits_{x\to a} (kf)(x) = kl$.
- If $\lim\limits_{x\to a} f(x) = l$, $\lim\limits_{x\to a} h(x) = n \ne 0$ then $\lim\limits_{x\to a} \left(\dfrac{1}{h}\right)(x) = \dfrac{1}{n}$, and

  $\lim\limits_{x\to a} \left(\dfrac{f}{h}\right)(x) = \dfrac{l}{n}$.
- If $p(x) = a_0 + a_1 x + \ldots\ldots\ldots + a_k x^k$, $k \ge 1$ is a polynomial function then

  $\lim\limits_{x\to a} p(x) = p(a)$.

- Let $f, g, h : E \to \mathbf{R}$, $a \in \mathbf{R}, E \subset \mathbf{R}$. If
  $f(x) \le g(x) \le h(x)\ \forall\, x \in E, x \ne a$ and
  if $\lim\limits_{x\to a} f(x) = l = \lim\limits_{x\to a} h(x)$ then $\lim\limits_{x\to a} g(x) = l$.

- If F and G are polynomials such that $F(x) = (x-a)^k f(x)$, $G(x) = (x-a)^k g(x)$ for some $k \in \mathbf{N}$ and some polynomials $f(x)$ and $g(x)$ where $g(a) \ne 0$ then $\lim\limits_{x\to a}\left(\dfrac{F}{G}\right)(x) = \dfrac{f(a)}{g(a)}$.

- Definition of the right and left hand limits of a function.

- Let $E = (a-r, a+r) \setminus \{a\}$ for all $r > 0$ and $f : E \to \mathbf{R}$. Then
  $\lim\limits_{x\to a} f(x) = l \Leftrightarrow \lim\limits_{x\to a+} f(x) = l = \lim\limits_{x\to a-} f(x)$.

- If $\lim\limits_{x\to a+} f(x)$ and $\lim\limits_{x\to a-} f(x)$ exist then $\lim\limits_{x\to a+} f(x) = \lim\limits_{\substack{h\to 0\\ h>0}} f(a+h)$ and
  $\lim\limits_{x\to a-} f(x) = \lim\limits_{\substack{h\to 0\\ h>0}} f(a-h)$.

- If $n \in \mathbf{N}, a \in \mathbf{R}$ then $\lim\limits_{x\to a} x^n = a^n$.

- If $a > 0, n \in \mathbf{R}$ then $\lim\limits_{x\to a} x^n = a^n$.

- If $n \in \mathbf{R}$, $a > 0$ then $\lim\limits_{x\to a} \dfrac{x^n - a^n}{x-a} = na^{n-1}$.

- $\lim\limits_{x\to 0} \dfrac{\sin x}{x} = 1$.

- $\lim\limits_{x\to 0} (1+x)^{1/x} = e$.

- $\lim\limits_{x\to 0} \left(\dfrac{a^x - 1}{x}\right) = \log_e a$.

- $\lim\limits_{x\to 0} \left(\dfrac{e^x - 1}{x}\right) = 1$.

- $\lim\limits_{x\to 0} \dfrac{\log_e(1+x)}{x} = 1$.

- The limits at infinity and the infinite limits.

- (i) $\lim_{x\to a} f(x)=\infty \Rightarrow \lim_{x\to a}\left(\frac{1}{f}\right)(x)=0.$

  (ii) $\lim_{x\to a} f(x)=-\infty \Rightarrow \lim_{x\to a}\left(\frac{1}{f}\right)(x)=0.$

  (iii) If $f(x)>0$ in a deleted neighbourhood of a and $\lim_{x\to a} f(x)=0$, then $\lim_{x\to a}\frac{1}{f}(x)=\infty.$

  (iv) If $f(x)<0$ in a deleted neighbourhood of a and $\lim_{x\to a} f(x)=0$, then

  $$\lim_{x\to a}\left(\frac{1}{f}\right)(x)=-\infty.$$

- Let $f, g, h : \mathrm{E}\to\mathbf{R}, \mathrm{E}\subseteq\mathbf{R}$ and $(a,\infty)\subset \mathrm{E}$ for some $a\in\mathbf{R}$. If $\lim_{x\to\infty} g(x)=l=\lim_{x\to\infty} h(x)$ and $g(x)\le f(x)\le h(x)$ for all $x\in \mathrm{E}$ then $\lim_{x\to\infty} f(x)=l$.

- Definition of continuity of a function $f:\mathrm{E}\to\mathbf{R}$ at a point $a\in \mathrm{E}$ and the definition of a continuous function.

- Let $f:\mathrm{E}\to\mathbf{R}$, $a\in \mathrm{E}$. If $(a-r, a+r)\cap \mathrm{E}=\{a\}$ for some $r>0$ then f is continuous at a.

- If $f:\mathbf{N}\to\mathbf{R}$ is a function then f is continuous on $\mathbf{N}$.

- Let $f:\mathrm{E}\to\mathbf{R}, ((a-r, a+r)\setminus\{a\})\cap \mathrm{E}$ be non-empty for every $r>0$. Then f is continuous at $a \Leftrightarrow \lim_{x\to a} f(x)=f(a)$.

- Let $\mathrm{E}\subseteq\mathbf{R}$ and $f:\mathrm{E}\to\mathbf{R}$ be a function. Let $a\in \mathrm{E}$. Then f is continuous at a if and only if $\lim_{x\to a+} f(x)$ and $\lim_{x\to a-} f(x)$ both exist and

  $$\lim_{x\to a+} f(x)=\lim_{x\to a-} f(x)=f(a).$$

❖ f is continuous on the closed interval $[a, b]$ if

(i) f is continuous in (a, b)

(ii) $\lim\limits_{x \to a+} f(x) = f(a)$

and (iii) $\lim\limits_{x \to b-} f(x) = f(b)$.

❖ Let $E \subseteq \mathbf{R}$, $a \in E$. f is discontinuous at a point $x = a$ in any one of the following cases:

(i) $\lim\limits_{x \to a+} f(x)$ and $\lim\limits_{x \to a-} f(x)$ exist, but are not equal.

(ii) $\lim\limits_{x \to a+} f(x)$ and $\lim\limits_{x \to a-} f(x)$ exist and are equal, but not equal to $f(a)$.

(iii) One of the two limits $\lim\limits_{x \to a+} f(x)$ and $\lim\limits_{x \to a-} f(x)$ fails to exist.

❖ Let $A, B \subseteq \mathbf{R}$. Let $f : A \to \mathbf{R}$ be continuous on A and $g : B \to \mathbf{R}$ be continuous on B. If $f(A) \subseteq B$ then the composite function $gof : A \to \mathbf{R}$ is continuous on A.

Historical Note

Here are some interesting historical milestones regarding the notations in Calculus that are in use today.

Concept	Notation	First introduced	Country	Year
Limit	$\lim\limits_{x \to a} f(x)$	*G.H. Hardy* (1877 – 1947)	English	1908
Definition of limit	$\varepsilon - \delta$	*Weierstrass* (1815 – 1897)	German	1875
Derivative	$\dfrac{dy}{dx}$	*Gottfried Wilhelm Leibnitz* (1646 – 1716)	German	1675
Derivative	y'	*Joseph Louis Lagrange* (1736 – 1813)	French	1772
Partial Derivative	$\dfrac{\partial u}{\partial x}$	*Adrin Mary Legendre* (1752 – 1833)	French	1786
Definite Integral	$\int_a^b f(x)\,dx$	*Joseph Fourier* (1768 – 1830) *Cauchy* (1789 – 1857)	French	1822

Answers

Exercise 8(a)

I

1. $2a$	**2.** 6	**3.** $\frac{1}{2}$	**4.** $\frac{1}{4}$
5. $\frac{3}{4}$	**6.** -3	**7.** $-\frac{5}{6}$	**8.** $-\frac{1}{4}$
9. 0	**10.** 0	**11.** 0	**12.** 0
13. 2	**14.** $\frac{108}{7}$	**15.** $-\frac{1}{3}$	**16.** $\frac{1}{\sqrt{24}}$

Exercise 8(b)

I. **1.** $\lim_{x\to1+} f(x)=2$, $\lim_{x\to1-} f(x)=0$ and $\lim_{x\to1} f(x)$ does not exist.

2. $\lim_{x\to3+} f(x)=9$, $\lim_{x\to3-} f(x)=5$ and $\lim_{x\to3} f(x)$ does not exist.

3. $\lim_{x\to2+} f(x)=\frac{4}{3}$, $\lim_{x\to2-} f(x)=1$ and $\lim_{x\to2} f(x)$ does not exist.

II. **1.** $\lim_{x\to1-} f(x)=3$, $\lim_{x\to1+} f(x)=3$ and $\lim_{x\to1} f(x)=3$.

2. $\lim_{x\to2-} f(x)=2$, $\lim_{x\to2+} f(x)=-1$ and $\lim_{x\to2} f(x)$ does not exist.

5. 4 and 3

III. **1.** $\lim_{x\to0} f(x)$ does not exist **2.** 0 and 0

Exercise 8(c)

I.

1. $\frac{3}{4}$	**2.** -1	**3.** a
4. $\frac{1}{2}$	**5.** $2b \cos a$	**6.** $\frac{1}{2a}$
7. 7	**8.** e^3	**9.** e^3
10. 1		

II.

1. $\frac{1}{10}$	**2.** $\sin a - a\cos a$	**3.** $\frac{1}{2}(b^2 - a^2)$

4. $\frac{5}{3}(2+\sqrt{2})$ 5. 5 6. π

III. 1. $\frac{1}{4}$ 2. 2 log 3 3. $\frac{2}{3\sqrt{3}}$

4. $\frac{2}{3}$ 5. 0 6. $\frac{2m^2}{n^2}$

7. 0 8. $\frac{1}{2}$ 9. $\left(\frac{m}{n}\right)^2$

10. 2 11. 1 12. $\frac{1}{2}$

Exercise 8(d)

I. 1. ∞ 2. ∞ 3. 0

4. ∞ 5. 0 6. 0

II. 1. 11 2. $\frac{1}{2}$ 3. 2 4. $\frac{11}{13}$

5. $\frac{1}{4}$ 6. $-\infty$ 7. 0 8. $\frac{1}{2}$

III. 1. -2 2. 0 3. 0 4. 6

5. 0

Exercise 8(e)

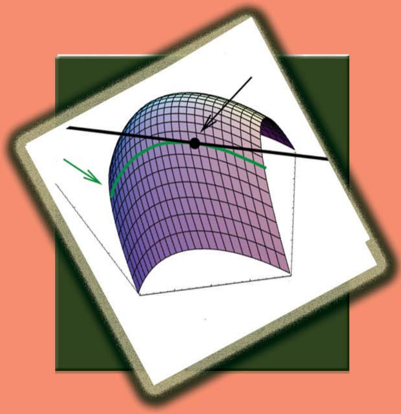
I. 1. Yes 2. Not continuous at 0

II. 1. Continuous at 2 2. Continuous at 3

4. Not continuous at -2 5. $k = 2$ or -1

III. 1. Continuous at none of the points 0, 1 and 2

2. $a = 0,\ b = -2$



Chapter 9

Differentiation

"Mathematics is the lantern by which what before was dimly visible now looms up in firm, bold outlines"

- Irving Fisher

Introduction

We shall discuss, in this chapter, the derivative which forms a basis for the fundamental concepts like velocity, acceleration and the slope of a tangent to a curve and so on. The credit goes to the great English mathematician *Sir Isaac Newton* (1642 – 1727) and the noted German mathematician *Gottfried Wilhelm Leibnitz* (1646 – 1716) who independently conceived this idea simultaneously. *Sir Isaac Newton* was the most distinguished student of his distinguished teacher *Isaac Barrow*.

Suppose $f : \mathrm{I} \to \mathbf{R}$ is a function, I being an interval. We usually denote it by the equation $y = f(x)$, where x is the independent variable and y is the dependent variable. Let c be a point in I. Let $c + h$ also be a point in I lying either to the left of c or to the right of c ($c + h < c$ if $h < 0$; $c < c + h$ if $h > 0$). Then $f(c + h) - f(c)$ denotes the change in

Isaac Barrow

(1630 – 1677)

***Barrow** was an English scholar and mathematician who is generally given credit for his role in giving impetus to the development of modern Calculus; in particular, for his work regarding the tangent. Isaac Newton was a student of him.*

$f(x)$ corresponding to a change h in x at c. The ratio $\dfrac{f(c+h)-f(c)}{h}$ is called the average change in $f(x)$ corresponding to a change h in x at c. If this ratio tends to a finite limit as h approaches zero, then the limit is called the derivative (or the rate of change) of f at c.

9.1 Derivative of a function

We begin with the definition of the derivative of a function and later prove certain important elementary properties of the derivatives.

9.1.1 Definition

Let I be an interval in $\mathbf{R}$, $f : \mathrm{I} \to \mathbf{R}$, $a \in \mathrm{I}$ *and let* $|h| \neq 0$ *be sufficiently small such that* $a+h \in \mathrm{I}$. *If* $\lim\limits_{h\to 0} \dfrac{f(a+h)-f(a)}{h}$ *exists, then* f *is said to be differentiable at a and the limit is called the* ***derivative*** *of* f *at a (or the differential coefficient of* f *at a). The derivative of* f *at a is denoted by any one of the forms*

$$f'(a),\ \left(\frac{dy}{dx}\right)_{x=a} \quad \text{or} \quad y'(a) \quad \text{where} \quad y = f(x).$$

This definition of derivative is also called 'the first principle of derivative'.

Observe that if a is not an end point of the interval I, then $f'(a)$ exists if and only if $\lim\limits_{h\to 0^+} \dfrac{f(a+h)-f(a)}{h}$ and $\lim\limits_{h\to 0^-} \dfrac{f(a+h)-f(a)}{h}$ both exist and are equal. The limits $\lim\limits_{h\to 0^+} \dfrac{f(a+h)-f(a)}{h}$ and $\lim\limits_{h\to 0^-} \dfrac{f(a+h)-f(a)}{h}$, if exist, are denoted by $f'(a^+)$ and $f'(a^-)$ respectively and are called the right and left hand derivatives of f at a.

If $f : [c,d] \to \mathbf{R}$, then f is said to be differentiable

(i) at c if $f'(c^+)$ exists

(ii) at d if $f'(d^-)$ exists.

Suppose $\mathrm{A} \subseteq \mathrm{I}$ and f is differentiable at every $x \in \mathrm{A}$. Then the function that assigns $f'(x)$ to each $x \in \mathrm{A}$ is called the derived function or the derivative of f and is denoted by f'. The process of finding the derivative of a function is called **differentiation**.

9.1.2 Note : If we denote the change h in a by Δx and the change in y by Δy, then $\Delta y = f(a+\Delta x) - f(a)$. Moreover

$$\boxed{f'(a) = \lim_{\Delta x\to 0} \frac{f(a+\Delta x)-f(a)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}}$$

We also note that

$$\boxed{f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.}$$

9.1.3 Solved Problems

1. Problem : *If* $f(x) = x^2 (x \in \mathbf{R})$, *prove that* f *is differentiable on* $\mathbf{R}$ *and find its derivative.*

Solution : Given that $f(x) = x^2 (x \in \mathbf{R})$.

For $x,\ h \in \mathbf{R}$ we have $f(x + h) - f(x) = (x + h)^2 - x^2 = 2xh + h^2 = h(2x + h)$.

Hence for $h \neq 0$, $\dfrac{f(x+h) - f(x)}{h} = 2x + h$.

Therefore $\displaystyle\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = 2x$.

Therefore f is differentiable on $\mathbf{R}$ and $f'(x) = 2x$ for each $x \in \mathbf{R}$.

2. Problem : *Suppose* $f(x) = \sqrt{x}\,(x > 0)$. *Prove that* f *is differentiable on* $(0, \infty)$ *and find* $f'(x)$.

Solution : Let $x \in (0, \infty)$, $h \neq 0$ and $|h| < x$.

$$\text{Then} \quad \frac{f(x+h) - f(x)}{h} = \frac{\sqrt{x+h} - \sqrt{x}}{h} = \frac{\left(\sqrt{x+h} - \sqrt{x}\right)\left(\sqrt{x+h} + \sqrt{x}\right)}{h\left(\sqrt{x+h} + \sqrt{x}\right)}$$

$$= \frac{1}{\sqrt{x+h} + \sqrt{x}}.$$

$$\text{Therefore} \quad \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \frac{1}{2\sqrt{x}}.$$

Hence f is differentiable at x and $f'(x) = \dfrac{1}{2\sqrt{x}}$ for each $x \in (0, \infty)$.

3. Problem : *If* $f(x) = \dfrac{1}{x^2 + 1} (x \in \mathbf{R})$, *prove that* f *is differentiable on* $\mathbf{R}$ *and find* $f'(x)$.

Solution: Let $x \in \mathbf{R}$. Then for $h \neq 0$, we have

$$\frac{f(x+h) - f(x)}{h} = \frac{1}{h}\left[\frac{1}{(x+h)^2 + 1} - \frac{1}{x^2 + 1}\right] = \frac{-h(2x+h)}{h(x^2+1)\left[(x+h)^2 + 1\right]}$$

$$= \frac{-(2x+h)}{(x^2+1)\left[(x+h)^2 + 1\right]}.$$

$$\text{Therefore} \quad \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \frac{-2x}{(x^2+1)^2}.$$

Hence f is differentiable at x and $f'(x) = \dfrac{-2x}{\left(x^2 + 1\right)^2}$ for each $x \in \mathbf{R}$.

4. Problem : *If* $f(x) = \sin x$ ($x \in$ **R**) , *then show that* f *is differentiable on* **R** *and* $f'(x) = \cos x$.

Solution : Let $x \in \mathbf{R}$. Then for $h \neq 0$, we have

$$\frac{f(x+h)-f(x)}{h} = \frac{\sin(x+h)-\sin x}{h} = \frac{2\sin\left(\frac{h}{2}\right)\cos\left(x+\frac{h}{2}\right)}{h}.$$

Therefore $$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\left\{\cos\left(x+\frac{h}{2}\right).\left[\frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}\right]\right\} = \cos x .$$

Hence f is differentiable on **R** and $f'(x) = \cos x$ for each $x \in \mathbf{R}$.

5. Problem : *Show that* $f(x) = |x|$ $(x \in \mathbf{R})$ *is not differentiable at zero and is differentiable at any* $x \neq 0$.

Solution : We have to show that $\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$ does not exist.

Given that $f(x) = |x|$, $f(h) = \begin{cases} h & \text{if} \quad h \geq 0 \\ -h & \text{if} \quad h < 0 \end{cases}$.

Thus, for $h \neq 0$,

$$\frac{f(0+h)-f(0)}{h} = \frac{f(h)}{h} = \begin{cases} 1 & \text{if} \quad h > 0 \\ -1 & \text{if} \quad h < 0. \end{cases}$$

Therefore $f'(0^+) = 1$ and $f'(0^-) = -1$.

Hence f is not differentiable at zero. It can be easily proved that f is differentiable at any $x \neq 0$ and that $f'(x) = \begin{cases} 1 & \text{if} \quad x > 0 \\ -1 & \text{if} \quad x < 0 \end{cases}$.

6. Problem : *Check whether the following function is differentiable at zero*

$$f(x) = \begin{cases} 3+x & \text{if} \quad x \geq 0 \\ 3-x & \text{if} \quad x < 0. \end{cases}$$

Solution : We show that f has the left and the right hand derivatives at zero and find them. First we observe that, for $h \neq 0$, $f(h) = \begin{cases} 3+h & \text{if} \quad h \geq 0 \\ 3-h & \text{if} \quad h < 0 \end{cases}$ and $f(0) = 3$.

Therefore, for $h > 0$, we have $\dfrac{f(0+h)-f(0)}{h} = \dfrac{f(h)-3}{h} = \dfrac{3+h-3}{h} = 1$.

Hence, $\lim\limits_{h\to 0^+} \dfrac{f(0+h)-f(0)}{h} = 1$. Thus f has the right hand derivative at zero and $f'(0^+) = 1$.

Similarly, $\lim\limits_{h\to 0^-} \dfrac{f(0+h)-f(0)}{h} = -1$, so that f has the left hand derivative at zero and $f'(0^-) = -1$.

Therefore $f'(0^+) \neq f'(0^-)$.

Hence f is not differentiable at zero.

Note that the function in the problem can be rewritten as $f(x) = 3 + |x|$, which is not differentiable at zero. (see problem 5 above)

7. Problem : *Show that the derivative of a constant function on an interval is zero.*

Solution : Let f be a constant function on an interval I. Then $f(x) = c$ for all $x \in$ I for some constant c. Let $a \in$ I. Then, for $h \neq 0$, $\dfrac{f(a+h)-f(a)}{h} = \dfrac{c-c}{h} = 0$ for sufficiently small $|h|$.

Hence $\lim\limits_{h\to 0} \dfrac{f(a+h)-f(a)}{h} = 0$.

Hence f is differentiable at a and $f'(a) = 0$. Thus,

The derivative of a constant function is zero.

8. Problem : *Suppose that for all $x, y \in \mathbf{R}$, $f(x+y) = f(x).f(y)$ and $f'(0)$ exists. Then show that $f'(x)$ exists and equals to $f(x) . f'(0)$ for all $x \in \mathbf{R}$.*

Solution : Let $x \in \mathbf{R}$. Then, for $h \neq 0$, we have

$$\frac{f(x+h)-f(x)}{h} = \frac{f(x)f(h)-f(x)}{h} = f(x)\frac{[f(h)-1]}{h}. \qquad \ldots (1)$$

Moreover $f(0) = f(0+0) = f(0)f(0)$ implies that $f(0)(1 - f(0)) = 0$, so that $f(0) = 0$ or 1.

Case (i)

Suppose $f(0) = 0$. Then $f(x) = f(x+0) = f(x)f(0) = 0$. Hence f is a constant function in this case and $f'(x) = 0$ for all $x \in \mathbf{R}$.

Thus $f'(x) = 0 = f(x) . f'(0)$, showing that the result is true in this case.

Case (ii)

Suppose now $f(0) = 1$. Then from (1)

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} f(x)\left(\frac{f(h)-1}{h}\right)$$

$$= \lim_{h\to 0} f(x)\left(\frac{f(h)-f(0)}{h}\right) \qquad \text{(since } f(0)=1\text{)}$$

$$= f(x)\,f'(0).$$

Hence f is differentiable at x and $f'(x) = f(x)\,f'(0)$.

9.2 Elementary properties

Now we shall prove certain important elementary properties of derivatives of functions.

9.2.1 Theorem : *Let* I *be an interval in* $\mathbf{R}$, $f : \mathrm{I} \to \mathbf{R}$ *and* $a \in \mathrm{I}$. *If* f *is differentiable at a, then f is continuous at* a .

Proof Suppose that f is differentiable at a. Then we have

$$\lim_{x\to a} \frac{f(x)-f(a)}{x-a} = f'(a)$$

Now $\qquad f(x) - f(a) = \dfrac{f(x)-f(a)}{(x-a)}.(x-a) \quad (x \neq a)$ and

therefore $\qquad \lim_{x\to a} [f(x) - f(a)] = f'(a).0 = 0.$

That is $\qquad \lim_{x\to a} f(x) = f(a)$ proving f is continuous at a.

> If f is differentiable at a, then f is continuous at a.

9.2.2 Note

The converse of the above theorem is not true. That is

> If f *is continuous at* a, then f *need not be differentiable at* a.

For example, $f(x) = |x|\ (x \in \mathbf{R})$ is continuous at zero but not differentiable at zero (see Problem 5 of 9.1.3)

9.2.3 Theorem (The derivative of the sum and difference of two functions)

Let I be an interval in $\boldsymbol{R}$, u *and* v *be real valued functions on* I *and* $x \in \mathrm{I}$. *Suppose that u and v are differentiable at* x. *Then* $u + v$ *is also differentiable at x and* $(u+v)'(x) = u'(x) + v'(x)$.

Proof : Let $f = u + v$. Then for sufficiently small non-zero values of $|h|$, we have

$$\frac{f(x+h)-f(x)}{h} = \frac{u(x+h)+v(x+h)-u(x)-v(x)}{h}$$

$$= \left[\frac{u(x+h)-u(x)}{h}\right] + \left[\frac{v(x+h)-v(x)}{h}\right]$$

which tends to $u'(x)+v'(x)$ as $h \to 0$. Hence f is differentiable at x and $f'(x)=u'(x)+v'(x)$.

$$\boxed{(u+v)'(x)=u'(x)+v'(x)}$$

We may similarly prove that

$$\boxed{(u-v)'(x)=u'(x)-v'(x)}$$

More generally, we have

9.2.4 Corollary

If $u_1, u_2, \ldots, u_n$ are real valued functions on an interval I and are differentiable at $x \in \mathrm{I}$, then $v = u_1 + u_2 + \ldots\ldots + u_n$ is also differentiable at x and

$$v'(x)=u_1'(x)+u_2'(x)+\ldots\ldots\ldots+u_n'(x).$$

(Proof is easy)

9.2.5 Theorem (The derivative of the product of two functions)

Let I *be an interval,* u *and* v *be real-valued functions on* I *and* $x \in \mathrm{I}$. *Suppose that* u *and* v *are differentiable at* x. *Then* uv *is differentiable at* x *and* $(uv)'(x) = u(x)v'(x) + u'(x)v(x)$

Proof : Let $f = u.v$. Then for sufficiently small non-zero values of $|h|$, we have

$$\frac{f(x+h)-f(x)}{h}=\frac{u(x+h)v(x+h)-u(x)v(x)}{h}$$

$$= u\,(x+h)\left(\frac{v(x+h)-v(x)}{h}\right)+v(x)\left(\frac{u(x+h)-u(x)}{h}\right).$$

Since u is differentiable at x, it is continuous at x so that $\lim_{h\to 0} u\left(x+h\right)=u(x)$. Hence

$\dfrac{f(x+h)-f(x)}{h} \to u(x)v'(x)+v(x)u'(x)$ *as* $h \to 0$. Thus $f = u.v$ is differentiable at x and

$$\boxed{(uv)'(x) = u(x)v'(x) + v(x)u'(x).}$$

As a consequence of mathematical induction and Theorem 9.2.5 the following result follows.

9.2.6 Corollary

If $u_1, u_2, \ldots\ldots, u_n$ are real-valued functions on an interval I and are differentiable at $x \in \mathrm{I}$, then $u = u_1 . u_2 . u_3 . \ldots\ldots\ldots\ldots . u_n$ is also differentiable at x and

$$u'(x) = \sum_{j=1}^{n} (u_1\ u_2 \ldots\ldots\ldots u_{j-1}\ u_{j+1} \ldots\ldots\ldots u_n)\,(x)\,u_j'\,(x)\,.$$

9.2.7 Corollary

If u, v are real valued functions on an interval I and are differentiable at $x \in \mathrm{I}$ and α, β are any constants, then $\alpha u + \beta v$ is also differentiable at x and

$$\boxed{(\alpha u + \beta v)' = \alpha u' + \beta v'}$$

9.2.8 Note

If u is a real valued function on an interval I and is differentiable at $x \in \mathrm{I}$, then

$$\boxed{v = u^n \ (n \in \mathbf{N}) \text{ is differentiable at } x \text{ and } v'(x) = nu^{n-1}(x).u'(x)}$$

For, take $u_1 = u_2 = ... = u_n = u$ in Corollary 9.2.6.

9.2.9 Theorem (The derivative of the reciprocal of a function)

Let f be a function defined on an interval I such that $f(t) \neq 0$ for any $t \in \mathrm{I}$ and f be differentiable at $x \in \mathrm{I}$. Then $\frac{1}{f}$ is differentiable at x and $\left(\frac{1}{f}\right)'(x) = -\frac{f'(x)}{[f(x)]^2}$.

Proof : Since f is differentiable at x, it is continuous at x. Given that $f(x) \neq 0$.

Now write $g = \frac{1}{f}$. Then for sufficiently small non-zero values of $|h|$, we have

$$\frac{g(x+h) - g(x)}{h} = \frac{\frac{1}{f(x+h)} - \frac{1}{f(x)}}{h} = \frac{f(x) - f(x+h)}{h\, f(x)\, f(x+h)}$$

$$= -\left[\frac{f(x+h) - f(x)}{h}\right] \cdot \frac{1}{f(x)\, f(x+h)} \qquad \text{... (1)}$$

From the hypothesis, we have

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = f'(x) \quad \text{and} \quad \lim_{h \to 0} f(x+h) = f(x).$$

Hence from (1) it follows that $\frac{g(x+h) - g(x)}{h} \to \frac{-f'(x)}{(f(x))^2}$ as $h \to 0$.

Therefore g is differentiable at x and $g'(x) = -\frac{f'(x)}{[f(x)]^2}$.

$$\boxed{\left(\frac{1}{f}\right)'(x) = -\frac{f'(x)}{[f(x)]^2}}$$

9.2.10 Theorem (The derivative of the quotient of two functions)

Let u and v be real valued functions on an interval I such that v is never zero on I and let u and v be differentiable at $x \in \mathrm{I}$. Then $\frac{u}{v}$ is differentiable at x and

$$\left(\frac{u}{v}\right)'(x) = \frac{1}{[v(x)]^2}[v(x)\,u'(x) - u(x)\,v'(x)].$$

Proof : From Theorem 9.2.9 it follows that $\frac{1}{v}$ is differentiable at x and $\left(\frac{1}{v}\right)'(x) = -\frac{v'(x)}{[v(x)]^2}$.

From Theorem 9.2.5 it follows that $u.\frac{1}{v}$ is differentiable at x and

$$\left(\frac{u}{v}\right)'(x) = \left(u.\frac{1}{v}\right)'(x)$$

$$= u(x)\left(\frac{1}{v}\right)'(x) + u'(x).\left(\frac{1}{v}\right)(x)$$

$$= u(x)\left(\frac{-v'(x)}{(v(x))^2}\right) + \frac{u'(x)}{v(x)}$$

$$= \frac{1}{(v(x))^2}[v(x)u'(x) - u(x)v'(x)].$$

$$\boxed{\left(\frac{u}{v}\right)'(x) = \frac{v(x)u'(x) - u(x)v'(x)}{[v(x)]^2}}$$

9.2.11 Theorem (The derivative of a composite function)

Let I be an interval, $g : \mathrm{I} \to \mathbf{R}$ and f be a real valued function on an interval containing g (I). Suppose that g is differentiable at x and f is differentiable at $g(x)$. Let F = (fog) (so that F(x) = $f(g(x))$. Then F is differentiable at x and $\mathrm{F}'(x) = f'(g(x))\, g'(x)$.

*(This is also known as **chain rule** for differentiation).*

Proof : Write $y = g(x)$.

Let us define a function ϕ in a neighbourhood of zero as follows

$$\phi(k) = \begin{cases} \dfrac{f(y+k) - f(y)}{k} - f'(y) & \text{if } k \neq 0 \\ 0 & \text{if } k = 0. \end{cases}$$

Since f is differentiable at $y = g(x)$, we have $\lim_{k\to 0} \dfrac{f(y+k)-f(y)}{k} = f'(y)$

Hence $\lim_{k\to 0} \phi(k) = 0$.

Moreover $f(y+k) - f(y) = k\,f'(y) + k\phi(k)$ for $k \neq 0$ ... (1)

Write $\psi(h) = g(x+h) - g(x)$ for $h \neq 0$.

Then $$\frac{F(x+h)-F(x)}{h} = \frac{f(g(x+h))-f(g(x))}{h}$$

$$= \frac{f(y+\psi(h))-f(y)}{h} = \frac{\psi(h)}{h} f'(y) + \frac{\psi(h)}{h}\phi(\psi(h)), \text{ (by (1))}$$

$$= \left[\frac{g(x+h)-g(x)}{h}\right] f'(y) + \left[\frac{g(x+h)-g(x)}{h}\right]\phi(\psi(h))$$

$$\to g'(x)\,f'(y) + g'(x)\,.\,0 \text{ as } h \to 0,$$

since g is differentiable at x, $\psi(h)\to 0$ as $h\to 0$ and $\varphi(k)\to 0$ as $k\to 0$.

Hence $\lim_{h\to 0} \dfrac{F(x+h)-F(x)}{h}$ exists and is equal to $f'(y)\,g'(x)$.

i.e., F is differentiable at x and $F'(x) = f'(g(x)).g'(x)$.

Thus $\boxed{(fog)'(x) = f'(g(x)).g'(x)}$

9.2.12 Note

If we write $z = f(y)$, $y = g(x)$ in the above theorem,

we get $\boxed{\dfrac{dz}{dx} = \dfrac{dz}{dy}.\dfrac{dy}{dx}}$.

The derivative of the inverse of a function is given in the following theorem.

9.2.13 Theorem (The derivative of the inverse of a function)

Let $f : [a, b] \to [c, d]$ be a bijection and g denote the inverse of f. Suppose that f is differentiable at $x \in (a, b)$, $f'(x) \neq 0$ and g is continuous at $f(x)$. Then g is differentiable at $f(x)$ and $g'(f(x)) = \dfrac{1}{f'(x)}$.

Proof : Let $y = f(x)$. Let k be a non-zero real number such that $y + k \in [c, d]$.

Let $g(y+k) - g(y) = h$. Since g is one-to-one, $h \neq 0$,
we have $g(y+k) = g(y) + h = x + h$.

Hence, $f(x+h) = y + k$. Hence $k = (y+k) - y = f(x+h) - f(x)$, since g is continuous at y, $g(y+k) \to g(y)$ as $k\to 0$. Hence $h\to 0$ as $k\to 0$. Since f is differentiable at x

$$\frac{f(x+h)-f(x)}{h} \to f'(x) \quad \text{as} \quad h \to 0.$$

since $\quad f'(x) \neq 0, \quad \dfrac{h}{f(x+h)-f(x)} \to \dfrac{1}{f'(x)} \quad \text{as} \quad h \to 0$

We have $\quad \dfrac{g(y+k)-g(y)}{k} = \dfrac{h}{f(x+h)-f(x)}.$

Hence $\quad \dfrac{g(y+k)-g(y)}{k} \to \dfrac{1}{f'(x)} \quad \text{as} \quad k \to 0.$

Therefore g is differentiable at $f(x)$ and $g'(f(x)) = \dfrac{1}{f'(x)}$.

9.2.14 Note

$$\boxed{\text{If} \quad y = f(x) \quad \text{then} \quad x = f^{-1}(y) \quad \text{and} \quad \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}}$$

We shall now find the derivatives of some standard functions.

9.2.15 Example : If $f(x) = e^x \ (x \in \mathbf{R})$, then show that $f'(x) = e^x$ by first principle.

Solution : From $f(x) = e^x$, we have for $h \neq 0$

$$\frac{f(x+h)-f(x)}{h} = \frac{e^{x+h}-e^x}{h} = e^x \frac{(e^h-1)}{h}$$

Therefore $\displaystyle \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = e^x . \lim_{h\to 0} \frac{(e^h-1)}{h} = e^x.1 = e^x$

Therefore $f'(x) = e^x$ for each $x \in \mathbf{R}$.

$$\boxed{\frac{d}{dx}(e^x) = e^x}$$

9.2.16 Example: If $f(x) = \log x \ (x > 0)$, then show that $f'(x) = \dfrac{1}{x}$ by first principle.

Solution : Now for $h \neq 0$

$$\frac{f(x+h)-f(x)}{h} = \frac{\log(x+h)-\log x}{h} = \frac{1}{h}\log\left(1+\frac{h}{x}\right) = \frac{1}{h} \cdot \frac{h}{x}\log\left(1+\frac{h}{x}\right)^{\frac{x}{h}}.$$

$$= \frac{1}{x}\log\left(1+\frac{h}{x}\right)^{\frac{x}{h}}.$$

Now, putting $\frac{h}{x} = z$, we get that $z \to 0$ as $h \to 0$.

Therefore $\log\left(1+\frac{h}{x}\right)^{\frac{x}{h}} = \log\,(1+z)^{\frac{1}{z}} \to \log e = 1$ as $z \to 0$.

Hence $\frac{f(x+h)-f(x)}{h} \to \frac{1}{x}$ as $h \to 0$.

Thus $f'(x) = \frac{1}{x}$ for each $x > 0$.

$$\boxed{\frac{d}{dx}(\log\, x) = \frac{1}{x}}$$

9.2.17 Example : If $f(x) = a^x\,(x \in \mathbf{R})\,(a > 0)$, then show that $f'(x) = a^x \log a$ by first principle.

Solution : For $h \neq 0$

$$\frac{f(x+h)-f(x)}{h} = \frac{a^{x+h}-a^x}{h} = a^x\left(\frac{a^h-1}{h}\right)$$

We know that $\frac{a^h-1}{h} \to \log a$ as $h \to 0$.

Hence $f'(x) = a^x \,.\, \log a$.

$$\boxed{\frac{d}{dx}(a^x) = a^x \log a}$$

9.2.18 Solved Problems

1. Problem : *If* $f(x) = (ax+b)^n, \left(x > \frac{-b}{a}\right)$, *then find* $f'(x)$.

Solution : Write $u = ax + b$ so that $f(x) = u^n$. Then

$$f'(x) = \frac{d}{du}(u^n).\frac{du}{dx}, \quad \text{by Note 9.2.12}$$

$$= nu^{n-1}.a = an(ax+b)^{n-1}.$$

2. Problem : *Find the derivative of* $f(x) = e^x\,(x^2+1)$.

Solution : Write $u = e^x$, $v = x^2 + 1$, so that $f(x) = u(x)\,v(x)$ and

$$f'(x) = u(x)\,v'(x) + u'(x)\,v(x), \quad \text{by Theorem 9.2.5}$$

Now $u'(x) = e^x$ and $v'(x) = 2x$ imply that

$$f'(x) = e^x\,(2x) + (x^2+1)\,e^x = (x+1)^2\,e^x.$$

3. Problem : *If* $y = \frac{a-x}{a+x}(x \neq -a)$, *find* $\frac{dy}{dx}$.

Solution : Write $u(x) = a - x$, $v(x) = a + x$, so that $y = \frac{u}{v}$

$$u'(x) = -1 \quad \text{and} \quad v'(x) = 1$$

Therefore $$\frac{dy}{dx} = \frac{1}{[v(x)]^2}[v(x)\ u'(x) - v'(x)\ u(x)]$$

$$= \frac{1}{(a+x)^2}[(a+x)(-1)-(a-x)(1)] = \frac{-2a}{(a+x)^2}.$$

4. Problem : *If* $f(x) = e^{2x}. \log x\ (x > 0)$, *then find* $f'(x)$.

Solution: Write $u(x) = e^{2x}$, $v(x) = \log x$, so that

$$f(x) = u(x)\ v(x),\ u'(x) = 2\ e^{2x},\ v'(x) = \frac{1}{x}.$$

Therefore $$f'(x) = u(x)\ v'(x) + u'(x)\ v(x)$$

$$= e^{2x}.\frac{1}{x} + 2\ e^{2x} \log x.$$

$$= e^{2x}\left(\frac{1}{x} + 2\log x\right).$$

5. Problem: *If* $f(x) = \sqrt{\frac{1+x^2}{1-x^2}}$ $(\,|\,x\,| < 1)$, *then find* $f'(x)$.

Solution: Write $u(x) = \frac{1+x^2}{1-x^2}$ and $y = f(x)$. Then $y = f(x) = u^{1/2}$

Now by the chain rule, we get $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$ where in

$$\frac{dy}{du} = \frac{1}{2}u^{\frac{1}{2}-1} = \frac{1}{2}u^{-\frac{1}{2}} = \frac{1}{2\sqrt{u}}$$

and $$\frac{du}{dx} = \frac{(1-x^2)(2x)-(1+x^2)(-2x)}{(1-x^2)^2} = \frac{4x}{(1-x^2)^2}.$$

Therefore $$f'(x) = \frac{dy}{dx} = \frac{1}{2\sqrt{u}}.\frac{4x}{(1-x^2)^2} = \frac{2x}{(1-x^2)\sqrt{1-x^4}}.$$

6. Problem : *If* $f(x) = x^2\ 2^x \log x\ (x > 0)$, *find* $f'(x)$.

Solution : Write $u(x) = x^2$, $v(x) = 2^x$ and $w(x) = \log x$ so that $f(x) = (uvw)(x)$.

Then $f'(x) = u'(x)\, v(x)\, w(x) + u(x)\, v'(x)\, w(x) + u(x)\, v(x)\, w'(x)$

$$= 2x\,(2^x \log x) + (2^x \log 2)\,(x^2 \log x) + x^2\, 2^x . \frac{1}{x}.$$

$$= x\, 2^x \left[\log x^2 + x \log x \log 2 + 1\right].$$

7. Problem : *If* $y = \begin{vmatrix} f(x) & g(x) \\ \varphi(x) & \psi(x) \end{vmatrix}$ *then show that*

$$\frac{dy}{dx} = \begin{vmatrix} f'(x) & g'(x) \\ \varphi(x) & \psi(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) \\ \varphi'(x) & \psi'(x) \end{vmatrix}.$$

Solution : Given that $y = \begin{vmatrix} f(x) & g(x) \\ \varphi(x) & \psi(x) \end{vmatrix} = f(x)\psi(x) - \varphi(x)\, g(x)$

Therefore $\dfrac{dy}{dx} = \left[f(x)\psi'(x) + f'(x)\,\psi(x)\right] - \left[\varphi(x) g'(x) + \varphi'(x)\, g(x)\right]$

$$= \left[f(x)\, \psi'(x) - g(x)\varphi'(x)\right] + \left[f'(x)\psi(x) - g'(x)\varphi(x)\right]$$

$$= \begin{vmatrix} f(x) & g(x) \\ \varphi'(x) & \psi'(x) \end{vmatrix} + \begin{vmatrix} f'(x) & g'(x) \\ \varphi(x) & \psi(x) \end{vmatrix}.$$

8. Problem : *If* $f(x) = 7^{x^3+3x}\,(x > 0)$, *then find* $f'(x)$.

Solution : Write $u(x) = x^3 + 3x$, so that $\dfrac{du}{dx} = 3x^2 + 3$ and $f(x) = 7^u$.

Therefore, by the chain rule, we get $f'(x) = \dfrac{df}{du} \cdot \dfrac{du}{dx}$

$$= (7^u \log 7)\,(3x^2 + 3)$$

$$= 3(x^2 + 1)\; 7^{x^3+3x} \log 7 .$$

9. Problem : *If* $f(x) = x\, e^x \sin x$, *then find* $f'(x)$.

Solution : Write $u(x) = x$, $v(x) = e^x$, $w(x) = \sin x$ and $y = f(x)$.

Then $y = uvw$ and

$$f'(x) = \frac{dy}{dx} = uv\frac{dw}{dx} + uw\frac{dv}{dx} + vw\frac{du}{dx}$$

But $\dfrac{du}{dx} = 1$, $\dfrac{dv}{dx} = e^x$, $\dfrac{dw}{dx} = \cos x$.

Therefore $f'(x) = \dfrac{dy}{dx} = xe^x \cos x + xe^x \sin x + e^x \sin x$.

10. Problem : *If* $f(x) = \sin(\log x), (x > 0)$ *find* $f'(x)$.

Solution : Write $u = \log x$, $y = f(x)$ so that $y = \sin u$ and $\dfrac{dy}{dx} = f'(x) = \dfrac{dy}{du} \times \dfrac{du}{dx}$

But $\dfrac{dy}{du} = \cos u, \ \dfrac{du}{dx} = \dfrac{1}{x}$

Therefore, $f'(x) = \dfrac{1}{x}\cos(\log x)$.

11. Problem : *If* $f(x) = (x^3 + 6x^2 + 12x - 13)^{100}$, *find* $f'(x)$.

Solution : Write $u(x) = x^3 + 6x^2 + 12x - 13$ and $y = f(x)$

so that $y = u^{100}$

Therefore $f'(x) = \dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$

$= 100u^{99} \cdot (3x^2 + 12x + 12)$

$= 300(x+2)^2(x^3 + 6x^2 + 12x - 13)^{99}$.

Exercise 9(a)

I. **1.** Find the derivatives of the following functions $f(x)$.

(i) $\sqrt{x} + 2x^{\frac{3}{4}} + 3x^{\frac{5}{6}} \ (x > 0)$

(ii) $\sqrt{2x-3} + \sqrt{7-3x}$

(iii) $(x^2 - 3)(4x^3 + 1)$

(iv) $\left(\sqrt{x} - 3x\right)\left(x + \dfrac{1}{x}\right)$

(v) $\left(\sqrt{x} + 1\right)(x^2 - 4x + 2) \ (x > 0)$

(vi) $(ax + b)^n (cx + d)^m$

(vii) $5 \sin x + e^x \log x$

(viii) $5^x + \log x + x^3 e^x$

(ix) $e^x + \sin x \cos x$

(x) $\dfrac{px^2 + qx + r}{ax + b} \ (|a| + |b| \neq 0)$

(xi) $\log_7(\log x) \ (x > 0)$

(xii) $\dfrac{1}{ax^2 + bx + c} \ (|a| + |b| + |c| \neq 0)$

(xiii) $e^{2x}\log(3x+4)\left(x > \dfrac{-4}{3}\right)$

(xiv) $(4 + x^2)\, e^{2x}$

(xv) $\dfrac{ax+b}{cx+d} \ (|c| + |d| \neq 0)$

(xvi) $a^x \cdot e^{x^2}$

2. If $f(x) = 1 + x + x^2 + ... + x^{100}$ then find $f'(1)$.

3. If $f(x) = 2x^2 + 3x - 5$ then prove that $f'(0) + 3f'(-1) = 0$.

II. **1.** Find the derivatives of the following functions from the first principles.

(i) x^3 (ii) $x^4 + 4$

(iii) $ax^2 + bx + c$ (iv) $\sqrt{x+1}$

(v) $\sin 2x$ (vi) $\cos ax$

(vii) $\tan 2x$ (viii) $\cot x$

(ix) $\sec 3x$ (x) $x \sin x$

(xi) $\cos^2 x$

2. Find the derivatives of the following functions.

(i) $\dfrac{1-x\sqrt{x}}{1+x\sqrt{x}}\ (x>0)$ (ii) $x^n n^x \log(nx)\quad (x>0,\ n \in \mathbf{N})$

(iii) $ax^{2n} \log x + bx^n e^{-x}$ (iv) $\left(\dfrac{1}{x} - x\right)^3 e^x$

3. Show that the function $f(x) = |x| + |x-1|,\ \ x \in \mathbf{R}$, is differentiable for all real numbers except for 0 and 1.

4. Verify whether the following function is differentiable at 1 and 3.

$$f(x) = \begin{cases} x & \text{if} \quad x < 1 \\ 3 - x & \text{if} \quad 1 \le x \le 3 \\ x^2 - 4x + 3 & \text{if} \quad x > 3 \end{cases}$$

5. Is the following function f derivable at 2? Justify.

$$f(x) = \begin{cases} x & \text{if} \quad 0 \le x \le 2 \\ 2 & \text{if} \quad x \ge 2 \end{cases}$$

9.3 Trigonometric, Inverse Trigonometric, Hyperbolic, Inverse Hyperbolic Functions - Derivatives

In this section we find the derivatives of trigonometric and hyperbolic functions and also of their inverses.

9.3.1 Derivatives of trigonometric functions

(i) $\boxed{\dfrac{d}{dx}(\sin x) = \cos x}$

We have already proved this result.

(ii) $\dfrac{d}{dx}(\cos x) = -\sin x$

$$\frac{d}{dx}(\cos x) = \frac{d}{dx}\left[\sin\left(\frac{\pi}{2} - x\right)\right]$$

$$= \cos\left(\frac{\pi}{2} - x\right) . \frac{d}{dx}\left(\frac{\pi}{2} - x\right) = -\sin x$$

This result can be obtained from the first principles also.

$$\boxed{\frac{d}{dx}(\cos x) = -\sin x}$$

(iii) If $y = \tan x,\ x \in \mathbf{R} \setminus \left\{(2n+1)\frac{\pi}{2} : n \in \mathbf{Z}\right\}$, then $\frac{dy}{dx} = \sec^2 x$

Now $y = \tan x = \dfrac{\sin x}{\cos x}$ implies

$$\frac{dy}{dx} = \frac{1}{\cos^2 x}\left[\cos x \frac{d}{dx}(\sin x) - \sin x \frac{d}{dx}(\cos x)\right]$$

$$= \frac{1}{\cos^2 x}\left[\cos^2 x + \sin^2 x\right] = \sec^2 x .$$

$$\boxed{\frac{d}{dx}(\tan x) = \sec^2 x}$$

Similarly

(iv) If $y = \cot x,\ x \in \mathbf{R} \setminus \{n\pi : n \in \mathbf{Z}\}$, then $\frac{dy}{dx} = -\operatorname{cosec}^2 x$.

$$\boxed{\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x}$$

(v) If $y = \sec x,\ x \in \mathbf{R} \setminus \left\{(2n+1)\frac{\pi}{2} : n \in \mathbf{Z}\right\}$, then

$y = \dfrac{1}{\cos x}$ and $\dfrac{dy}{dx} = \dfrac{-1}{\cos^2 x} . \dfrac{d}{dx}(\cos x) = \dfrac{\sin x}{\cos^2 x} = \tan x . \sec x$.

$$\boxed{\frac{d}{dx}(\sec x) = \sec x \tan x}$$

Similarly

(vi) If $y = \operatorname{cosec} x,\ x \in \mathbf{R} \setminus \{n\pi;\ n \in \mathbf{Z}\}$, then $\frac{dy}{dx} = -\operatorname{cosec} x \cot x$.

$$\boxed{\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x}$$

9.3.2 Derivatives of inverse trigonometric functions

Let us recall that, if f and g are functions such that $f(g(x)) = x$ and $g(f(y)) = y$ for any x and y and $f'(y) \neq 0$, then $g'(x) = \dfrac{1}{f'(y)}$, where $y = g(x)$. (See 9.2.13).

Hence, we have $y = g(x) \Leftrightarrow x = f(y)$ and $\dfrac{dy}{dx} = \left(\dfrac{dx}{dy}\right)^{-1}$.

(i) If $y = \mathrm{Sin}^{-1}x,\ x \in [-1,\ 1]$ then its range is $\left[\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right]$

$$y = \mathrm{Sin}^{-1} x \quad \Leftrightarrow \quad x = \sin y \text{ and } \frac{dx}{dy} = \cos y\,.$$

If $-1 < x < 1$ then $\dfrac{-\pi}{2} < y < \dfrac{\pi}{2}$.

Hence $\dfrac{dx}{dy} = \cos y > 0$. This implies

$$\frac{dy}{dx} = \left(\frac{dx}{dy}\right)^{-1} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}.$$

$$\boxed{\frac{d}{dx}\left(\mathrm{Sin}^{-1} x\right) = \frac{1}{\sqrt{1-x^2}}}$$

(ii) If $y = \mathrm{Cos}^{-1} x,\quad x \in [-1, 1]$, then we have $y \in [0,\ \pi]$

$$y = \mathrm{Cos}^{-1} x \quad \Leftrightarrow \quad x = \cos y$$

$$x = \cos y \quad \Rightarrow \quad \frac{dx}{dy} = -\sin y\,.$$

Hence $\dfrac{dy}{dx} = \dfrac{1}{-\sin y} = \dfrac{-1}{\sqrt{1-\cos^2 y}} = \dfrac{-1}{\sqrt{1-x^2}}$.

$$\boxed{\frac{d}{dx}\left(\mathrm{Cos}^{-1} x\right) = \frac{-1}{\sqrt{1-x^2}}}$$

(iii) If $y = \text{Tan}^{-1} x$, $x \in \mathbf{R}$, then we know that $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

$$x = \tan y \Rightarrow \frac{dx}{dy} = \sec^2 y = 1 + \tan^2 y = 1 + x^2 > 0.$$

Therefore $\frac{dy}{dx} = \left(\frac{dx}{dy}\right)^{-1} = \frac{1}{1+x^2}$.

$$\boxed{\frac{d}{dx}\left(\text{Tan}^{-1} x\right) = \frac{1}{1+x^2}}$$

Similarly

(iv) $\frac{d}{dx}(\text{Cot}^{-1} x) = \frac{-1}{1+x^2}$ (here $x \in \mathbf{R}$, $\text{Cot}^{-1} x \in (0, \pi)$)

$$\boxed{\frac{d}{dx}\left(\text{Cot}^{-1} x\right) = \frac{-1}{1+x^2}}$$

(v) If $y = \text{Sec}^{-1} x$, $x \in \mathbf{R} \setminus [-1, 1]$, then $y \in \left(0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right)$.

$$x = \sec y \Rightarrow \frac{dx}{dy} = \sec y \tan y$$

$$|x| > 1 \Rightarrow y \in \left(0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right)$$

$$\Rightarrow \frac{dx}{dy} = \sec y \tan y = \sec^2 y \sin y > 0 \left(y \neq \frac{\pi}{2}\right) \quad \text{...(1)}$$

Now

$$x < -1 \Rightarrow \sec y < -1$$

$$\Rightarrow \tan y < 0 \quad \text{(by (1))}$$

and $\tan^2 y = \sec^2 y - 1$

so that $\tan y = -\sqrt{\sec^2 y - 1}$ (since $\tan y < 0$)

$$= -\sqrt{x^2 - 1}$$

$$\text{Therefore } \frac{dy}{dx} = \left(\frac{dx}{dy}\right)^{-1} = \frac{1}{\sec y \tan y} = \begin{cases} \dfrac{1}{x\sqrt{x^2-1}} & \text{if } x > 1 \\ \dfrac{1}{-x\sqrt{x^2-1}} & \text{if } x < -1 \end{cases}$$

$$= \frac{1}{|x|\sqrt{x^2-1}}.$$

$$\boxed{\frac{d}{dx}\left(\text{Sec}^{-1} x\right) = \frac{1}{|x|\sqrt{x^2-1}}}$$

Similarly

(vi) If $y = \text{Cosec}^{-1} x$, then $\dfrac{dy}{dx} = \dfrac{-1}{|x|\sqrt{x^2-1}}$.

$$\left[x \in \mathbf{R} \setminus [-1, 1],\ \ y \in \left(-\frac{\pi}{2},\ 0\right) \cup \left(0,\ \frac{\pi}{2}\right)\right]$$

$$\boxed{\frac{d}{dx}\left(\text{Cosec}^{-1} x\right) = \frac{-1}{|x|\sqrt{x^2-1}}}$$

9.3.3 Derivatives of hyperbolic functions

(i) If $y = \sinh x\ (x \in \mathbf{R})$ then $\dfrac{dy}{dx} = \cosh x$.

For $y = \sinh x = \dfrac{e^x - e^{-x}}{2}$ implies $\dfrac{dy}{dx} = \dfrac{e^x + e^{-x}}{2} = \cosh x$.

$$\boxed{\frac{d}{dx}(\sinh x) = \cosh x}$$

(ii) If $y = \cosh x\ (x \in \mathbf{R})$ then $\dfrac{dy}{dx} = \sinh x$.

For $y = \dfrac{e^x + e^{-x}}{2}$ implies $\dfrac{dy}{dx} = \dfrac{e^x - e^{-x}}{2} = \sinh x$.

$$\boxed{\frac{d}{dx}(\cosh x) = \sinh x}$$

(iii) If $y = \tanh x\ (x \in \mathbf{R})$ then $\dfrac{dy}{dx} = \text{sech}^2 x$.

$y = \tanh x = \dfrac{\sinh x}{\cosh x}$ implies

$$\frac{dy}{dx} = \frac{1}{\cosh^2 x}\left[(\cosh x)\frac{d}{dx}(\sinh x) - (\sinh x)\frac{d}{dx}(\cosh x)\right]$$

$$= \frac{1}{\cosh^2 x}\left(\cosh^2 x - \sinh^2 x\right) = \frac{1}{\cosh^2 x} = \text{sech}^2 x.$$

$$\boxed{\frac{d}{dx}(\tanh x) = \text{sech}^2 x}$$

(iv) If $y = \text{sech } x \,(x \in \mathbf{R})$, then $\dfrac{dy}{dx} = -\text{sech } x \tanh x$

$$y = \text{sech}\, x = \frac{1}{\cosh x} \text{ implies that}$$

$$\frac{dy}{dx} = \frac{-1}{\cosh^2 x} \cdot \frac{d}{dx}(\cosh x) = -\frac{\sinh x}{\cosh^2 x} = -\text{sech } x \tanh x .$$

$$\boxed{\frac{d}{dx}(\text{sech } x) = -\text{sech } x \tanh x}$$

(v) If $y = \text{cosech } x \,(x \in \mathbf{R} \setminus \{0\})$ then $\dfrac{dy}{dx} = -\text{cosech } x . \coth x$

$$y = \text{cosech } x = \frac{1}{\sinh x} \text{ implies that}$$

$$\frac{dy}{dx} = \frac{(-1)}{\sinh^2 x} \cdot \frac{d}{dx}(\sinh x) = \frac{-\cosh x}{\sinh^2 x} = -\text{cosech } x \coth x .$$

$$\boxed{\frac{d}{dx}(\text{cosech } x) = -\text{cosech } x \coth x}$$

(vi) If $y = \coth x \,(x \in \mathbf{R} \setminus \{0\})$ then $\dfrac{dy}{dx} = -\text{cosech}^2 x$

$$y = \coth x = \frac{1}{\tanh x} \text{ implies that}$$

$$\frac{dy}{dx} = \frac{(-1)}{\tanh^2 x} \cdot \frac{d}{dx}(\tanh x) = \frac{-\text{sech}^2 x}{\tanh^2 x} = -\text{cosech}^2 x .$$

$$\boxed{\frac{d}{dx}(\coth x) = -\text{cosech}^2 x}$$

9.3.4 Derivatives of inverse hyperbolic functions

(i) If $y = \text{Sinh}^{-1} x \,(x \in \mathbf{R})$ then $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{1+x^2}}$.

$$y = \text{Sinh}^{-1} x \Rightarrow x = \sinh y$$

Hence $$\frac{dy}{dx} = \frac{1}{\cosh y} = \frac{1}{\sqrt{1+\sinh^2 y}} = \frac{1}{\sqrt{1+x^2}}.$$

$$\boxed{\frac{d}{dx}\left(\text{Sinh}^{-1} x\right) = \frac{1}{\sqrt{1+x^2}}}$$

(ii) If $y = \mathrm{Cosh}^{-1}x \ (x \in (1, \infty))$, then $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{x^2-1}}$.

For $x > 1$, $x = \cosh y$ implies $\dfrac{dx}{dy} = \sinh y > 0$.

Therefore for $x > 1$, we have

$$\frac{dy}{dx} = \left(\frac{dx}{dy}\right)^{-1} = \frac{1}{\sinh y} = \frac{1}{\sqrt{\cosh^2 y - 1}} = \frac{1}{\sqrt{x^2-1}}.$$

$$\boxed{\frac{d}{dx}\left(\mathrm{Cosh}^{-1} x\right) = \frac{1}{\sqrt{x^2-1}}}$$

(iii) If $y = \mathrm{Tanh}^{-1} x \ (x \in (-1, 1))$, then $\dfrac{dy}{dx} = \dfrac{1}{1-x^2}$.

For $x = \tanh y$, $\dfrac{dx}{dy} = \mathrm{sech}^2 y = 1 - \tanh^2 y = 1 - x^2 > 0$.

Therefore $\dfrac{dy}{dx} = \left(\dfrac{dx}{dy}\right)^{-1} = \dfrac{1}{1-x^2}$.

$$\boxed{\frac{d}{dx}\left(\mathrm{Tanh}^{-1} x\right) = \frac{1}{1-x^2}}$$

(iv) If $y = \mathrm{Sech}^{-1} x \ \left(x \in (0, 1)\right)$ then $\dfrac{dy}{dx} = \dfrac{-1}{x\sqrt{1-x^2}}$.

For $x \in (0, 1)$, $y = \mathrm{Sech}^{-1} x = \mathrm{Cosh}^{-1}\left(\dfrac{1}{x}\right)$.

Hence $$\frac{dy}{dx} = \frac{1}{\sqrt{\left(\frac{1}{x}\right)^2 - 1}} \times \frac{-1}{x^2} = \frac{-1}{x\sqrt{1-x^2}}.$$

$$\boxed{\frac{d}{dx}\left(\mathrm{Sech}^{-1} x\right) = \frac{-1}{x\sqrt{1-x^2}}}$$

(v) If $y = \mathrm{Cosech}^{-1} x \ (x \in \mathbf{R} \setminus \{0\})$ then $\dfrac{dy}{dx} = \dfrac{-1}{|x|\sqrt{1+x^2}}$.

$y = \mathrm{Cosech}^{-1} x = \mathrm{Sinh}^{-1}\left(\dfrac{1}{x}\right)$

Hence $$\frac{dy}{dx} = \frac{1}{\sqrt{1+\left(\frac{1}{x}\right)^2}} \times \frac{-1}{x^2} = \frac{-1}{|x|\sqrt{1+x^2}}.$$

$$\boxed{\frac{d}{dx}\left(\text{Cosech}^{-1}\,x\right) = \frac{-1}{|x|\sqrt{1+x^2}}}$$

(vi) If $y = \text{Coth}^{-1}x \;\left(x\in(-\infty,-1)\cup(1,\infty)\right)$ then $\dfrac{dy}{dx}=\dfrac{1}{1-x^2}$.

$y = \text{Coth}^{-1}\,x = \text{Tanh}^{-1}\left(\dfrac{1}{x}\right)$ implies that

$$\frac{dy}{dx} = \frac{1}{\left(1-\frac{1}{x^2}\right)}\times\left(\frac{-1}{x^2}\right)=\frac{1}{1-x^2}.$$

$$\boxed{\frac{d}{dx}\left(\text{Coth}^{-1}\,x\right) = \frac{1}{1-x^2}}$$

Observe that though the formulae for the derivatives of $\text{Tanh}^{-1}\,x$, $\text{Coth}^{-1}\,x$ are the same, their domains are disjoint.

9.3.5 Note

The formulae mentioned under (i), (ii), (iii) above can also be obtained by using the following identities.

$$\text{Sinh}^{-1}\,x = \log\left(x+\sqrt{1+x^2}\right).$$

$$\text{Cosh}^{-1}\,x = \log\left(x+\sqrt{x^2-1}\right)\quad (x\ge 1).$$

$$\text{Tanh}^{-1}\,x = \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)\quad (1-x^2>0).$$

9.3.6 Note

Hereafter, in order to find the derivative of a function, even though its domain is not explicitly mentioned, we mean that in its appropriate domain the derivative exists and we have to find the same.

9.3.7 Solved Problems

1. Problem : *Find the derivative of* $f(x)=\dfrac{x\cos x}{\sqrt{1+x^2}}$.

Solution : Write $u(x) = x\cos x$, $v(x)=\sqrt{1+x^2}$ so that

$$f(x) = \left(\frac{u}{v}\right)(x) \quad\text{and}\quad f'(x) = \frac{1}{[v(x)]^2}\left[v(x)\;u'(x)-v'(x)\;u(x)\right].$$

Here $\quad u'(x) = \dfrac{d}{dx}(x \cos x) = \cos x - x \sin x$

and $\quad v'(x) = \dfrac{d}{dx}\left(\sqrt{1+x^2}\right) = \dfrac{2x}{2\sqrt{1+x^2}} = \dfrac{x}{\sqrt{1+x^2}}$.

Therefore the derivative of $f(x)$ is

$$f'(x) = \frac{1}{1+x^2}\left[\sqrt{1+x^2}(\cos x - x \sin x) - \frac{x^2 \cos x}{\sqrt{1+x^2}}\right]$$

$$= (1 + x^2)^{-3/2} [\cos x - x(1 + x^2) \sin x].$$

2. Problem : If $f(x) = \log(\sec x + \tan x)$, find $f'(x)$.

Solution : Write $u(x) = \sec x + \tan x$, $y = f(x)$ so that

$$y = \log u \quad \text{and} \quad \frac{dy}{dx} = f'(x) = \frac{dy}{du} \times \frac{du}{dx}.$$

Now $\quad \dfrac{dy}{du} = \dfrac{1}{u}$, $\quad \dfrac{du}{dx} = \sec x \tan x + \sec^2 x$.

Therefore $f'(x) = \dfrac{1}{u}\left(\sec x \tan x + \sec^2 x\right) = \sec x$.

3. Problem : *If* $y = \mathrm{Sin}^{-1}\sqrt{x}$, *find* $\dfrac{dy}{dx}$.

Solution : Write $u(x) = \sqrt{x}$, then $y = \mathrm{Sin}^{-1} u$ and $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$.

Hence $\quad \dfrac{dy}{dx} = \dfrac{1}{2\sqrt{x}} \times \dfrac{1}{\sqrt{1-u^2}} = \dfrac{1}{2\sqrt{x - x^2}}$.

4. Problem : *If* $y = \sec\left(\sqrt{\tan x}\right)$, *find* $\dfrac{dy}{dx}$.

Solution : Write $u = \sqrt{\tan x}$, $v = \tan x$. Then $y = \sec u$, $u = \sqrt{v}$, $v = \tan x$

imply that $\quad \dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dv} \times \dfrac{dv}{dx}$

Now $\quad \dfrac{dy}{du} = \sec u \tan u$, $\dfrac{du}{dv} = \dfrac{1}{2\sqrt{v}}$ and $\dfrac{dv}{dx} = \sec^2 x$.

Therefore $\quad \dfrac{dy}{dx} = \dfrac{\sec^2 x}{2\sqrt{\tan x}} \cdot \sec\left(\sqrt{\tan x}\right) \cdot \tan\left(\sqrt{\tan x}\right)$.

5. Problem: *If* $y = \dfrac{x \sin^{-1} x}{\sqrt{1-x^2}}$, *find* $\dfrac{dy}{dx}$.

Solution : Write $u = x \operatorname{Sin}^{-1} x,\ v = \sqrt{1-x^2}$ so that $y = \dfrac{u}{v}$

Now $\dfrac{du}{dx} = \dfrac{x}{\sqrt{1-x^2}} + \operatorname{Sin}^{-1} x$ and $\dfrac{dv}{dx} = \dfrac{-2x}{2\sqrt{1-x^2}} = \dfrac{-x}{\sqrt{1-x^2}}$

$$\frac{dy}{dx} = \frac{1}{v^2}[vu' - v'u]$$

$$= \frac{1}{(1-x^2)}\left[\sqrt{1-x^2}\left(\frac{x}{\sqrt{1-x^2}} + \operatorname{Sin}^{-1} x\right) + \frac{x^2 \operatorname{Sin}^{-1} x}{\sqrt{1-x^2}}\right]$$

$$= \frac{1}{(1-x^2)^{3/2}}\left[x\sqrt{1-x^2} + \operatorname{Sin}^{-1} x\right].$$

6. Problem : *If* $y = \log(\cosh 2x)$, *find* $\dfrac{dy}{dx}$.

Solution : Let $u = \cosh 2x$, so that $y = \log u$.

Then $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$

Here $\dfrac{dy}{du} = \dfrac{1}{u}$ and $\dfrac{du}{dx} = 2\sinh 2x$.

Hence $\dfrac{dy}{dx} = \dfrac{2}{u}\sinh 2x = \dfrac{2\sinh 2x}{\cosh 2x} = 2\tanh 2x.$

7. Problem : *If* $y = \log(\sin(\log x))$, *find* $\dfrac{dy}{dx}$.

Solution : Write $v = \log x,\ u = \sin v$ so that $y = \log u$

Therefore $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dv} \times \dfrac{dv}{dx}$

$$= \frac{1}{u} \times \cos v \times \frac{1}{x} = \frac{\cos(\log x)}{x \sin(\log x)} = \frac{1}{x}.\cot(\log x).$$

8. Problem : *If* $y=\left(\mathrm{Cot}^{-1}x^3\right)^2$, *find* $\dfrac{dy}{dx}$.

Solution : Put $u=\mathrm{Cot}^{-1}x^3$ so that $y=u^2$

Then $\dfrac{dy}{dx}=\dfrac{dy}{du}\times\dfrac{du}{dx}=2u\times\dfrac{-1}{\left(1+x^6\right)}.3x^2=\dfrac{-6x^2\,\mathrm{Cot}^{-1}(x^3)}{1+x^6}$.

9. Problem : *If* $y=\mathrm{Cosec}^{-1}\left(e^{2x+1}\right)$, *find* $\dfrac{dy}{dx}$.

Solution : Let $u=e^{2x+1}$ then $y=\mathrm{Cosec}^{-1}u$ and $\dfrac{dy}{dx}=\dfrac{dy}{du}\times\dfrac{du}{dx}$

$$=\frac{-1}{u\sqrt{u^2-1}}.\,2e^{2x+1}=\frac{-2}{\sqrt{e^{4x+2}-1}}.$$

Exercise 9(b)

I. 1. Find the derivatives of the following functions.

(i) $\cot^n x$ (ii) $\mathrm{cosec}^4 x$

(iii) $\tan(e^x)$ (iv) $\dfrac{1-\cos 2x}{1+\cos 2x}$

(v) $\sin^m x\cos^n x$ (vi) $\sin mx.\cos nx$

(vii) $x\,\mathrm{Tan}^{-1}x$ (viii) $\mathrm{Sin}^{-1}(\cos x)$

(ix) $\log(\tan 5x)$ (x) $\mathrm{Sinh}^{-1}\left(\dfrac{3x}{4}\right)$

(xi) $\mathrm{Tan}^{-1}(\log x)$ (xii) $\log\left(\dfrac{x^2+x+2}{x^2-x+2}\right)$

(xiii) $\log(\mathrm{Sin}^{-1}(e^x))$ (xiv) $(\sin x)^2(\mathrm{Sin}^{-1}x)^2$

(xv) $\dfrac{\cos x}{\sin x+\cos x}$ (xvi) $\dfrac{x(1+x^2)}{\sqrt{1-x^2}}$

(xvii) $e^{\mathrm{Sin}^{-1}x}$ (xviii) $\cos(\log x+e^x)$

(xix) $\dfrac{\sin(x+a)}{\cos x}$ (xx) $\mathrm{Cot}^{-1}(\mathrm{cosec}\,3x)$

2. Find the derivatives of the following functions.

(i) $x=\sinh^2 y$ (ii) $x=\tanh^2 y$ (iii) $x=e^{\sinh y}$

(iv) $x=\tan(e^{-y})$ (v) $x=\log(1+\sin^2 y)$ (vi) $x=\log\left(1+\sqrt{y}\right)$

II. Find the derivatives of the following functions.

(i) $\cos(\log(\cot x))$ (ii) $\text{Sinh}^{-1}\left(\dfrac{1-x}{1+x}\right)$ (iii) $\log(\cot(1-x^2))$

(iv) $\sin(\cos(x^2))$ (v) $\sin(\text{Tan}^{-1}(e^x))$ (vi) $\dfrac{\sin(ax+b)}{\cos(cx+d)}$

(vii) $\text{Tan}^{-1}\left(\tanh\left(\dfrac{x}{2}\right)\right)$ (viii) $\sin x \,.\, (\text{Tan}^{-1} x)^2$

III. Find the derivatives of the following functions.

1. $\text{Sin}^{-1}\left(\dfrac{b+a\sin x}{a+b\sin x}\right)\ (a>0, b>0)$ **2.** $\text{Cos}^{-1}\left(\dfrac{b+a\cos x}{a+b\cos x}\right)\ (a>0, b>0)$

3. $\text{Tan}^{-1}\left(\dfrac{\cos x}{1+\cos x}\right)$

9.4 Methods of differentiation

Some times the formulae obtained so far, may prove to be difficult in finding the derivatives of some typical functions. There are some special methods of differentiation to deal with such situations. Our main aim in this section is to discuss such methods.

9.4.1 Substitution methods

Let $y = fog$. If we are able to find a function h such that $goh = f^{-1}$, then the substitution $x = h(u)$ may give the derivative of y with respect to x easily (Here f is a bijection defined on an interval).

The method is well illustrated in the following examples.

Examples

1. If $y = \text{Tan}^{-1}\sqrt{\dfrac{1-x}{1+x}}\ (|x|<1)$, we shall find $\dfrac{dy}{dx}$.

substituting $x = \cos u\ (u \in (0, \pi))$ in y, we get

$$\frac{1-x}{1+x} = \frac{1-\cos u}{1+\cos u} = \frac{2\sin^2(u/2)}{2\cos^2(u/2)} = \tan^2(u/2)$$

so that $$\sqrt{\frac{1-x}{1+x}} = \tan\left(\frac{u}{2}\right)$$

and $\quad y = \mathrm{Tan}^{-1}\left(\tan\left(\frac{u}{2}\right)\right) = \frac{u}{2}.$

Therefore, $\quad \frac{dy}{du} = \frac{dy}{dx}\cdot\frac{dx}{du}$ implies that $\frac{1}{2} = \frac{dy}{dx}(-\sin u)$.

Hence $\quad \frac{dy}{dx} = \frac{-1}{2\sin u} = \frac{-1}{2\sqrt{1-x^2}}.$

Observe that $\mathrm{Tan}^{-1}\, x$, $\sqrt{\frac{1-x}{1+x}}$ and $\cos u$ are the functions that stand for $f(x)$, $g(x)$ and $h(u)$ respectively, mentioned in the method.

2. If $y = \mathrm{Tan}^{-1}\left(\frac{2x}{1-x^2}\right) (|x|<1)$ then we shall find $\frac{dy}{dx}$.

Substituting $\quad x = \tan u$

we get $\quad \frac{2x}{1-x^2} = \frac{2\tan u}{1-\tan^2 u} = \tan 2u$

and $\quad y = \mathrm{Tan}^{-1}(\tan 2u) = 2u.$

Therefore from $\quad \frac{dy}{du} = \frac{dy}{dx}.\frac{dx}{du}$

we get that $\quad 2 = \frac{dy}{dx}\cdot\sec^2 u.$

Therefore $\quad \frac{dy}{dx} = 2\cos^2 u = \frac{2}{1+\tan^2 u} = \frac{2}{1+x^2}.$

9.4.2 Logarithmic Differentiation

Use of the logarithms will be of great help in finding the derivatives of functions of the form $y = f(x)^{g(x)}$, $f : \mathrm{A} \to (0, \infty)$, $g : \mathrm{A} \to \mathbf{R}$ (A an interval).

Write $y = h(x) = f(x)^{g(x)}$. Then $\log h(x) = g(x) \log f(x)$. Differentiating both sides with respect to x we get $\frac{h'(x)}{h(x)} = g'(x) \log f(x) + g(x)\frac{f'(x)}{f(x)}$.

Therefore $\quad h'(x) = h(x)\left[g'(x) \log f(x) + g(x)\frac{f'(x)}{f(x)}\right].$

$$\boxed{h(x) = f(x)^{g(x)} \Rightarrow h'(x) = f(x)^{g(x)}\left[g'(x) \log f(x) + g(x)\frac{f'(x)}{f(x)}\right]}$$

This method is well illustrated in the following examples.

Examples

1. If $y = x^x\ (x > 0)$, we shall find $\frac{dy}{dx}$.

Taking logarithms on both the sides of $y = x^x$, we obtain $\log y = x \log x$.

Differentiating with respect to x, we get $\frac{y'}{y} = x.\frac{1}{x} + \log x = 1 + \log x$.

$$\text{Therefore} \quad \frac{dy}{dx} = y' = y(1 + \log x) = x^x(1 + \log x).$$

2. If $y = (\tan x)^{\sin x} \left(0 < x < \frac{\pi}{2}\right)$ compute $\frac{dy}{dx}$.

Taking logarithms on both sides of $y = (\tan x)^{\sin x}$, we get

$\log y = \sin x \cdot \log(\tan x)$.

Differentiating with respect to x, we get $\frac{y'}{y} = \frac{\sin x}{\tan x} \cdot \sec^2 x + \cos x \cdot \log(\tan x)$

$$= \sec x + \cos x \cdot \log(\tan x).$$

Hence $\frac{dy}{dx} = (\tan x)^{\sin x}[\sec x + \cos x \ \log(\tan x)]$.

9.4.3 Parametric Differentiation

Let A, B, C be intervals, $f : A \to B$, $g : A \to C$, f a bijection, f^{-1}, g be differentiable.

Then, writing $x = f(t)$, $y = g(t)$ we get $y = \left(gof^{-1}\right)(x) = \varphi(x)$.

$x = f(t), y = g(t)$ are called the parametric equations of the function $y = \varphi(x)$.

$$y = g\left(f^{-1}(x)\right) \Rightarrow \frac{dy}{dx} = g'(f^{-1}(x))(f^{-1}(x))' = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}.$$

Hence $\frac{dy}{dx} = \frac{g'(t)}{f'(t)}$. This process is called the parametric differentiation.

$$\boxed{\text{If} \quad x = f(t), \quad y = g(t) \quad \text{then} \quad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{g'(t)}{f'(t)}}$$

The following examples illustrate parametric differentiation.

Examples

1. If $x = a\cos^3 t$, $y = a\sin^3 t$, find $\frac{dy}{dx}$.

Here $\frac{dx}{dt} = 3\,a\cos^2 t(-\sin t)$ and $\frac{dy}{dt} = 3\,a\sin^2 t.\cos t$.

Therefore $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = -\tan t$.

2. If $y = e^t + \cos t$, $x = \log t + \sin t$ find $\frac{dy}{dx}$.

Here $\frac{dy}{dt} = e^t - \sin t$ and $\frac{dx}{dt} = \frac{1}{t} + \cos t$.

Therefore $\frac{dy}{dx} = \frac{t(e^t - \sin t)}{(1 + t\cos t)}$.

3. To find the derivative of $f(x) = x^{\text{Sin}^{-1}x}$ with respect to $g(x) = \text{Sin}^{-1}x$,

we have to compute $\frac{df}{dg}$.

Now $f(x) = x^{\text{Sin}^{-1}x}$ implies that

$\log f(x) = \text{Sin}^{-1}x \,.\, \log x$ so that

$$\frac{f'(x)}{f(x)} = \left[\frac{1}{x}\text{Sin}^{-1}x + \frac{\log x}{\sqrt{1-x^2}}\right] \Rightarrow f'(x) = x^{\text{Sin}^{-1}x}\left[\frac{\text{Sin}^{-1}x}{x} + \frac{\log x}{\sqrt{1-x^2}}\right].$$

$$g(x) = \text{Sin}^{-1}x \Rightarrow g'(x) = \frac{1}{\sqrt{1-x^2}}.$$

Therefore $\frac{df}{dg} = \frac{f'(x)}{g'(x)} = \sqrt{1-x^2}\,.\,x^{\text{Sin}^{-1}x}\left[\frac{\text{Sin}^{-1}x}{x} + \frac{\log x}{\sqrt{1-x^2}}\right].$

9.4.4 Differentiation of implict functions

A function defined on a set $A(\subseteq \mathbf{R})$ is usually denoted by $y = f(x)$. A function which can be put in this form, is said to be in explicit form. Some times, such a form of a function may not be possible. But f can be defined in terms of a function F which is defined on $\mathbf{R}^2$ by the equation of the form $F(x, y) = 0$. For example, $y = f(x)$ defined by $x^2 - 6xy + y^2 = 0$ is a function which can't be given in explicit form.

A function $y = f(x)$, defined by $F(x, y) = 0$ is called an implicit function. In order to differentiate such functions, we differentiate F with respect to x (treating y as a function of x) and equate it to zero and thereby we get $\frac{dy}{dx}$.

The following examples illustrate this process.

Examples

1. If $x^3 + y^3 - 3axy = 0$, find $\frac{dy}{dx}$.

Let the given equation define the function

$y = f(x)$ that is $x^3 + (f(x))^3 - 3ax\, f(x) = 0$

Differentiating both sides of this equation with respect to x, we get

$$3x^2 + 3(f(x))^2 f'(x) - [3a \cdot f(x) + 3axf'(x)] = 0.$$

Hence $3x^2 + 3y^2 f'(x) - [3ay + 3ax\, f'(x)] = 0$.

Therefore $f'(x) = \frac{dy}{dx} = \frac{ay - x^2}{y^2 - ax}$.

2. If $2x^2 - 3xy + y^2 + x + 2y - 8 = 0$, find $\frac{dy}{dx}$.

Treating y as a function of x and then differentiating with respect to x, we get $4x - 3y - 3x\,y' + 2y\,y' + 1 + 2\,y' = 0$

Therefore $\frac{dy}{dx} = y' = \frac{3y - 4x - 1}{2y - 3x + 2}$.

9.4.5 Solved Problems

1. Problem : *If* $y = \mathrm{Tan}^{-1}\left(\cos\sqrt{x}\right)$ *find* $\frac{dy}{dx}$.

Solution : Substitute $t = \sqrt{x},\ u = \cos\sqrt{x}$.

Then $y = \mathrm{Tan}^{-1} u$ and $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dt} \times \frac{dt}{dx}$

$$= \frac{1}{1+u^2} \times -\sin t \times \frac{1}{2\sqrt{x}}$$

$$= -\frac{\sin\sqrt{x}}{2\sqrt{x}\left(1+\cos^2\sqrt{x}\right)}.$$

2. Problem : *If* $y = \mathrm{Tan}^{-1}\left[\dfrac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right]$ *for* $0 < |x| < 1$, *find* $\dfrac{dy}{dx}$.

Solution : Substituting $x^2 = \cos 2\theta$, we get

$$y = \mathrm{Tan}^{-1}\left(\frac{\sqrt{1+\cos 2\theta}+\sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta}-\sqrt{1-\cos 2\theta}}\right)$$

$$= \mathrm{Tan}^{-1}\left(\frac{\sqrt{2\cos^2\theta}+\sqrt{2\sin^2\theta}}{\sqrt{2\cos^2\theta}-\sqrt{2\sin^2\theta}}\right)$$

$$= \mathrm{Tan}^{-1}\left(\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right) = \mathrm{Tan}^{-1}\left(\frac{1+\tan\theta}{1-\tan\theta}\right)$$

$$= \mathrm{Tan}^{-1}\left(\tan\left(\frac{\pi}{4}+\theta\right)\right) = \frac{\pi}{4}+\theta .$$

Therefore $\quad y = \dfrac{\pi}{4}+\dfrac{1}{2}\mathrm{Cos}^{-1}(x^2).$

Hence $\quad \dfrac{dy}{dx} = \dfrac{1}{2}\dfrac{(-1)}{\sqrt{1-x^4}}\times 2x = \dfrac{-x}{\sqrt{1-x^4}}.$

3. Problem : *If* $y = x^2 e^x \sin x$, *find* $\dfrac{dy}{dx}$.

Solution : Let us use logarithmic differentiation to find the derivative. Taking logarithms on both sides of $y = x^2 e^x \sin x$, we get $\log y = 2\log x + \log e^x + \log \sin x$.

Differentiating both sides of the above equation with respect to x, we get

$$\frac{y'}{y} = \left(\frac{2}{x}+1+\cot x\right).$$

Therefore $\quad \dfrac{dy}{dx} = y' = y\left(\dfrac{2}{x}+1+\cot x\right)$

$$= x^2 e^x \sin x\left(\frac{2}{x}+1+\cot x\right).$$

The derivative can also be found by using the product rule.

4. Problem : *If* $y = x^{\tan x} + (\sin x)^{\cos x}$, *find* $\dfrac{dy}{dx}$.

Solution: Write $\quad u = x^{\tan x}$, $v = (\sin x)^{\cos x}$.

Then $\quad y = u + v$ and $\dfrac{dy}{dx} = \dfrac{du}{dx}+\dfrac{dv}{dx}$.

Now $\log u = (\tan x)(\log x)$ so that

$$\frac{u'}{u} = \frac{\tan x}{x} + \sec^2 x.\log x$$

Hence $$\frac{du}{dx} = u' = x^{\tan x}\left[\frac{\tan x}{x} + \sec^2 x.\log x\right]. \quad \text{... (1)}$$

Similarly, $\log v = \cos x.(\log \sin x)$ and

$$\frac{v'}{v} = \left[-\sin x \cdot \log(\sin x) + \frac{\cos^2 x}{\sin x}\right]$$

Hence $$\frac{dv}{dx} = v' = (\sin x)^{\cos x}\left[-\sin x.\log(\sin x) + \frac{\cos^2 x}{\sin x}\right]. \quad \text{... (2)}$$

Therefore $$\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$$

$$= x^{\tan x}\left[\frac{\tan x}{x} + \sec^2 x \ \log x\right] + (\sin x)^{\cos x}\left[-\sin x.\log(\sin x) + \frac{\cos^2 x}{\sin x}\right].$$

5. Problem : *If* $x = a\left[\cos t + \log\tan\left(\frac{t}{2}\right)\right]$, $y = a\sin t$, *find* $\frac{dy}{dx}$.

Soltuion : Here, $\frac{dx}{dt} = a\left[-\sin t + \frac{1}{\tan(t/2)} \times \sec^2\left(\frac{t}{2}\right) \times \frac{1}{2}\right] = \frac{a\cos^2 t}{\sin t}$ and

$$\frac{dy}{dt} = a\cos t, \quad \text{so that} \quad \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \tan t.$$

6. Problem : *If* $x^y = e^{x-y}$, *then show that* $\frac{dy}{dx} = \frac{\log x}{(1+\log x)^2}$.

Solution : Taking logarithms on both sides of $x^y = e^{x-y}$ we get $y\log x = x - y$.

That is, $y = \frac{x}{1+\log x}$.

Therefore $$\frac{dy}{dx} = \frac{(1+\log x).1 - x.\frac{1}{x}}{(1+\log x)^2} = \frac{\log x}{(1+\log x)^2}.$$

7. Problem : *If* $\sin y = x\sin(a+y)$, *then show that* $\frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$.

(a is not a multiple of π *).*

Solution : $\sin y = x\sin(a+y)$ implies that $x = \frac{\sin y}{\sin(a+y)}$.

Differentiating both the sides with respect to x, we get

$$1 = \frac{\sin(a+y).\cos y - \sin y.\cos(a+y)}{\sin^2(a+y)} \cdot \frac{dy}{dx}$$

Hence $$\frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin(a+y-y)} = \frac{\sin^2(a+y)}{\sin a}.$$

Exercise 9(c)

I. **1.** Find the derivatives of the following functions.

(i) $\text{Sin}^{-1}(3x-4x^3)$ (ii) $\text{Cos}^{-1}\left(4x^3-3x\right)$

(iii) $\text{Sin}^{-1}\left(\frac{2x}{1+x^2}\right)$ (iv) $\text{Tan}^{-1}\left(\frac{a-x}{1+ax}\right)$

(v) $\text{Tan}^{-1}\sqrt{\frac{1-\cos x}{1+\cos x}}$ (vi) $\sin(\cos(x^2))$

(vii) $\text{Sec}^{-1}\left(\frac{1}{2x^2-1}\right)\left(0 < x < \frac{1}{\sqrt{2}}\right)$ (viii) $\sin\left(\text{Tan}^{-1}(e^{-x})\right)$

2. Differentiate $f(x)$ with respect to $g(x)$ for the following.

(i) $f(x) = e^x$, $g(x) = \sqrt{x}$ (ii) $f(x) = e^{\sin x}$, $g(x) = \sin x$

(iii) $f(x) = \text{Tan}^{-1}\left(\frac{2x}{1-x^2}\right)$, $g(x) = \text{Sin}^{-1}\left(\frac{2x}{1+x^2}\right)$

3. If $y = e^{a\text{Sin}^{-1}x}$ then prove that $\frac{dy}{dx} = \frac{ay}{\sqrt{1-x^2}}$

II. **1.** Find the derivatives of the following functions.

(i) $\text{Tan}^{-1}\left[\frac{3a^2x-x^3}{a(a^2-3x^2)}\right]$ (ii) $\text{Tan}^{-1}(\sec x + \tan x)$

(iii) $\text{Tan}^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ (iv) $(\log x)^{\tan x}$

(v) $\left(x^x\right)^x$ (vi) $20^{\log(\tan x)}$

(vii) $x^x + e^{e^x}$ (viii) $x \,.\, \log x \,.\, \log(\log x)$

(ix) $e^{-ax^2}\sin(x\log x)$ (x) $\text{Sin}^{-1}\left(\frac{2^{x+1}}{1+4^x}\right)$ (put $2^x = \tan\theta$)

2. Find $\dfrac{dy}{dx}$ for the following functions.

(i) $x = 3\cos t - 2\cos^3 t,\ y = 3\sin t - 2\sin^3 t$

(ii) $x = \dfrac{3at}{1+t^3},\quad y = \dfrac{3at^2}{1+t^3}$

(iii) $x = a(\cos t + t\sin t),\ y = a(\sin t - t\cos t)$

(iv) $x = a\left(\dfrac{1-t^2}{1+t^2}\right),\quad y = \dfrac{2bt}{1+t^2}$

3. Differentiate $f(x)$ with respect to $g(x)$ for the following.

(i) $f(x) = \log_a x,\ g(x) = a^x$

(ii) $f(x) = \mathrm{Sec}^{-1}\left(\dfrac{1}{2x^2-1}\right),\quad g(x) = \sqrt{1-x^2}$

(iii) $f(x) = \mathrm{Tan}^{-1}\left(\dfrac{\sqrt{1+x^2}-1}{x}\right),\quad g(x) = \mathrm{Tan}^{-1}x$

4. Find the derivative of the function y defined implicitly by each of the following equations.

(i) $x^4 + y^4 - a^2xy = 0$ (ii) $y = x^y$ (iii) $y^x = x^{\sin y}$

5. Establish the following

(i) If $\sqrt{1-x^2} + \sqrt{1-y^2} = a(x-y)$ then $\dfrac{dy}{dx} = \sqrt{\dfrac{1-y^2}{1-x^2}}$.

(ii) If $y = x\sqrt{a^2+x^2} + a^2\log\left(x + \sqrt{a^2+x^2}\right)$ then $\dfrac{dy}{dx} = 2\sqrt{a^2+x^2}$.

(iii) If $x^{\log y} = \log x$ then $\dfrac{dy}{dx} = \dfrac{y}{x}\left[\dfrac{1-\log x\ \log y}{\log^2 x}\right]$.

(iv) If $y = \mathrm{Tan}^{-1}\left(\dfrac{2x}{1-x^2}\right) + \mathrm{Tan}^{-1}\left(\dfrac{3x-x^3}{1-3x^2}\right) - \mathrm{Tan}^{-1}\left(\dfrac{4x-4x^3}{1-6x^2+x^4}\right)$.

Then $\dfrac{dy}{dx} = \dfrac{1}{1+x^2}$.

(v) If $x^y = y^x$ then $\dfrac{dy}{dx} = \dfrac{y(x\log y - y)}{x(y\log x - x)}$.

(vi) If $x^{2/3} + y^{2/3} = a^{2/3}$ then $\dfrac{dy}{dx} = -\sqrt[3]{\dfrac{y}{x}}$.

6. Find $\frac{dy}{dx}$ of each of the following functions.

(i) $y = \frac{(1-2x)^{2/3}(1+3x)^{-3/4}}{(1-6x)^{5/6}(1+7x)^{-6/7}}$ (ii) $y = \frac{x^4 . \sqrt[3]{x^2+4}}{\sqrt{4x^2-7}}$

(iii) $y = \frac{(a-x)^2(b-x)^3}{(c-2x)^3}$ (iv) $y = \frac{x^3 . \sqrt{2+3x}}{(2+x)(1-x)}$

(v) $y = \sqrt{\frac{(x-3)(x^2+4)}{3x^2+4x+5}}$

III. Find the derivatives of the following functions.

1. (i) $(\sin x)^{\log x} + x^{\sin x}$ (ii) x^{x^x} (iii) $(\sin x)^x + x^{\sin x}$

(iv) $x^x + (\cot x)^x$

2. Establish the following

(i) If $x^y + y^x = a^b$ then $\frac{dy}{dx} = -\left[\frac{yx^{y-1} + y^x \log y}{x^y \log x + xy^{x-1}}\right]$.

(ii) If $f(x) = \mathrm{Sin}^{-1}\sqrt{\frac{x-\beta}{\alpha-\beta}}$ and $g(x) = \mathrm{Tan}^{-1}\sqrt{\frac{x-\beta}{\alpha-x}}$ then

$f'(x) = g'(x) (\beta < x < \alpha)$.

(iii) If $a > b > 0$ and $0 < x < \pi$; $f(x) = (a^2 - b^2)^{-1/2} . \mathrm{Cos}^{-1}\left(\frac{a\cos x + b}{a + b\cos x}\right)$

then $f'(x) = (a + b\cos x)^{-1}$.

3. Differentiate $(x^2 - 5x + 8)(x^3 + 7x + 9)$ by

(i) Using product rule

(ii) Obtaining a single polynomial expanding the product

(iii) Logarithmic differentiation

Do they all give the same answer?

9.5 Second Order Derivatives

If f is a function of x, then its derivative f' (if exists) is also a function of x. If f' is differentiable, then the derivative of f' denoted by f'' is called the **second order** derivative of f.

$$\boxed{f''(x) = \lim_{h\to 0} \frac{f'(x+h) - f'(x)}{h}}$$

$f''(x)$ is also denoted by the symbols $f^{(2)}(x)$, $\dfrac{d^2}{dx^2} f(x)$, $D^2 f(x)$, $\dfrac{d^2 y}{dx^2}$, y'' (here $y = f(x)$) etc.

Similarly third, fourth, ... higher order dervatives can also be defined. In general for $n \geq 1$, $f^{(n)}(x)$ is defined by

$$f^{(n)}(x) = \lim_{h\to 0}\left[\frac{f^{(n-1)}(x+h) - f^{(n-1)}(x)}{h}\right] \qquad \text{(Here } f^{(0)}(x) = f(x)\text{)}.$$

The process of finding higher order derivatives of a function is called **successive differentiaton**.

In this section we confine our discussion to the derivatives upto the second order only.

9.5.1 Solved problems

1. Problem : *If $y = x^4 + \tan x$ then find y''.*

Solution: Given that $y = x^4 + \tan x$

implies $y' = 4x^3 + \sec^2 x$.

Hence $y'' = 4.3x^2 + 2.\sec x.\sec x \tan x$

$= 12x^2 + 2\sec^2 x \tan x$.

2. Problem : *If $f(x) = \sin x \sin 2x \sin 3x$, find $f''(x)$.*

Solution : $f(x) = \dfrac{1}{2}\sin 2x(2\sin 3x \sin x)$

$= \dfrac{1}{2}\sin 2x(\cos 2x - \cos 4x)$

$= \dfrac{1}{4}[2\sin 2x\cos 2x - 2\sin 2x\cos 4x]$

$= \dfrac{1}{4}[\sin 2x + \sin 4x - \sin 6x]$

Therefore $f'(x)=\frac{1}{4}[2\cos 2x+4\cos 4x-6\cos 6x]$.

Hence $f''(x)=\frac{1}{4}[-4\sin 2x-16\sin 4x+36\sin 6x]$

$$= 9\sin 6x-4\sin 4x-\sin 2x.$$

3. Problem : *Show that* $y = x + \tan x$ *satisfies* $\cos^2 x\dfrac{d^2y}{dx^2}+2x=2y$.

Solution : $y = x + \tan x$ implies that $y' = 1 + \sec^2 x$

that is, $y'\cos^2 x = 1 + \cos^2 x$.

Differentiating both sides of the above equation we get

$$y''\cos^2 x + y' .\ 2\cos x\,(-\sin x) = 2\cos x\,(-\sin x).$$

Hence $y''\cos^2 x = 2(y'-1)\sin x\ \cos x$

$$= 2\sec^2 x\ \sin x\ \cos x = 2\tan x = 2(y-x).$$

This proves the result.

4. Problem : *If* $x = a(t-\sin t),\ y = a(1+\cos t)$ *find* $\dfrac{d^2y}{dx^2}$.

Solution : Here $\dfrac{dx}{dt}=a(1-\cos t),\ \dfrac{dy}{dt}=-a\sin t$.

Therefore $$\frac{dy}{dx}=\frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}=\frac{-a\sin t}{a(1-\cos t)}$$

$$=\frac{-2\sin\left(\frac{t}{2}\right)\cos\left(\frac{t}{2}\right)}{2\sin^2\left(\frac{t}{2}\right)}=-\cot\left(\frac{t}{2}\right).$$

Hence $$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dt}\left(\frac{dy}{dx}\right).\frac{dt}{dx}$$

$$=\frac{1}{2}\operatorname{cosec}^2\left(\frac{t}{2}\right)\times\frac{1}{a(1-\cos t)}=\frac{1}{4a\sin^4\left(\frac{t}{2}\right)}.$$

5. Problem : *Find the second order derivative of* $y = \operatorname{Tan}^{-1}\left(\dfrac{2x}{1-x^2}\right)$.

Solution : Put $x = \tan\theta$. Then

$$y = \operatorname{Tan}^{-1}\left(\frac{2\tan\theta}{1-\tan^2\theta}\right) = \operatorname{Tan}^{-1}(\tan 2\theta) = 2\theta = 2\operatorname{Tan}^{-1}x.$$

Therefore $\dfrac{dy}{dx} = \dfrac{2}{1+x^2}$ and $\dfrac{d^2y}{dx^2} = \dfrac{-4x}{(1+x^2)^2}$.

6. Problem : *If* $y = \sin(\sin x)$, *show that* $y'' + (\tan x)\,y' + y\cos^2 x = 0$.

Solution : $y = \sin(\sin x)$ implies that $y' = \cos x \cdot \cos(\sin x)$

and $y'' = -\cos^2 x\,\sin(\sin x) - \sin x\,\cos(\sin x)$

$$= -y\cos^2 x - \sin x\left(\frac{y'}{\cos x}\right)$$

$$= -y\cos^2 x - y'\tan x.$$

Therefore $y'' + (\tan x)\,y' + y\cos^2 x = 0$.

Exercise 9(d)

I. 1. If $y = \dfrac{2x+3}{4x+5}$ then find y''.

2. If $y = ae^{nx} + be^{-nx}$ then prove that $y'' = n^2 y$.

II. 1. Find the second order derivatives of the following functions $f(x)$.

(i) $\cos^3 x$ (ii) $\sin^4 x$

(iii) $\log(4x^2 - 9)$ (iv) $e^{-2x}\sin^3 x$

(v) $e^x \sin x \cos 2x$ (vi) $\operatorname{Tan}^{-1}\left(\dfrac{1+x}{1-x}\right)$ (vii) $\operatorname{Tan}^{-1}\left(\dfrac{3x-x^3}{1-3x^2}\right)$

2. Prove the following

(i) If $y = ax^{n+1} + bx^{-n}$ then $x^2 y'' = n(n+1)\,y$.

(ii) If $y = a\cos x + (b+2x)\sin x$ then $y'' + y = 4\cos x$.

(iii) If $y = 6(x+1) + (a+bx)e^{3x}$ then $y'' - 6y' + 9y = 54x + 18$.

(iv) If $ay^4 = (x+b)^5$ then $5y\,y'' = (y')^2$.

(v) If $y = a\cos(\sin x) + b\sin(\sin x)$ then $y'' + (\tan x)\,y' + y\cos^2 x = 0$.

III. 1. (i) If $y = 128\sin^3 x\cos^4 x$ then find y''.

(ii) If $y = \sin 2x\ \sin 3x\ \sin 4x$ then find y''.

(iii) If $ax^2 + 2hxy + by^2 = 1$ then prove that $\dfrac{d^2y}{dx^2} = \dfrac{h^2 - ab}{(hx+by)^3}$.

(iv) If $y = ae^{-bx}\cos(cx+d)$ then prove that $y'' + 2b\,y' + (b^2 + c^2)\,y = 0$.

(v) If $y = e^{\frac{-k}{2}x}(a\cos nx + b\sin nx)$ prove that then $y'' + k\,y' + \left(n^2 + \dfrac{k^2}{4}\right) y = 0$.

Key Concepts

- The derivative of a function f at $x = a$ is defined by

$$f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h} = \lim_{x\to a}\frac{f(x)-f(a)}{x-a}.$$

- Every differentiable function is continuous, but the converse need not be true.
- Let u, v be functions of x whose derivatives exist. Then

(i) $\dfrac{d}{dx}(c) = 0$ (ii) $\dfrac{d}{dx}(ku) = k\dfrac{du}{dx}$ (iii) $\dfrac{d}{dx}(u \pm v) = \dfrac{du}{dx} \pm \dfrac{dv}{dx}$

(iv) $\dfrac{d}{dx}(uv) = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$ (v) $\dfrac{d}{dx}\left(\dfrac{1}{u}\right) = \dfrac{-1}{u^2}\cdot\dfrac{du}{dx}$

(vi) $\dfrac{d}{dx}\left(\dfrac{u}{v}\right) = \dfrac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$

- Chain rule is used to differentiate composites of functions.

(i) If $t = u(x)$, $f = v(t)$ so that $f = vou$ and if both $\dfrac{dt}{dx}$ and $\dfrac{dv}{dt}$ exist then $\dfrac{df}{dx} = \dfrac{dv}{dt}\cdot\dfrac{dt}{dx}$.

(ii) If $u = f(y)$ and y is a function of x then $\dfrac{du}{dx} = \dfrac{du}{dy}\cdot\dfrac{dy}{dx} = f'(y)\dfrac{dy}{dx}$

(iii) $\frac{dy}{dx} \cdot \frac{dx}{dy} = 1$ or $\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$.

❖ If $y = [f(x)]^{g(x)}$ then $\frac{dy}{dx} = [f(x)]^{g(x)} \left[g(x) \frac{f'(x)}{f(x)} + \log f(x).g'(x) \right]$.

❖ If $x = f(t)$, $y = g(t)$ then $\frac{dy}{dx} = \frac{\left(\frac{dy}{dt}\right)}{\left(\frac{dx}{dt}\right)}$.

❖ Differentiation of implicit function: If $f(x, y) = c$. Differentiate each term w.r.t. x and note that $\frac{d}{dx}(\varphi(y)) = \frac{d}{dy}(\varphi(y)).\frac{dy}{dx}$.

❖ $f''(x) = \lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h}$.

Some of the standard derivatives (in appropriate domains)

S.No.	The Function y	The derivative $\frac{dy}{dx}$
1.	constant function	0
2.	x^n	nx^{n-1}
3.	e^x	e^x
4.	$\log x$	$\frac{1}{x}$
5.	a^x	$a^x \log a$
6.	$\sin x$	$\cos x$
7.	$\cos x$	$-\sin x$
8.	$\tan x$	$\sec^2 x$
9.	$\cot x$	$-\operatorname{cosec}^2 x$
10.	$\sec x$	$\sec x \tan x$

11.	cosec x	$-\text{cosec } x. \cot x$
12.	$\text{Sin}^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$
13.	$\text{Cos}^{-1} x$	$\frac{-1}{\sqrt{1-x^2}}$
14.	$\text{Tan}^{-1} x$	$\frac{1}{1+x^2}$
15.	$\text{Cot}^{-1} x$	$\frac{-1}{1+x^2}$
16.	$\text{Sec}^{-1} x$	$\frac{1}{\lvert x\rvert\sqrt{x^2-1}}$
17.	$\text{Cosec}^{-1} x$	$\frac{-1}{\lvert x\rvert\sqrt{x^2-1}}$
18.	sinh x	cosh x
19.	cosh x	sinh x
20.	tanh x	$\text{sech}^2 x$
21.	coth x	$-\text{cosech}^2 x$
22.	sech x	$-\text{sech } x. \tanh x$
23.	cosech x	$-\text{cosech } x. \coth x$
24.	$\text{Sinh}^{-1} x$	$\frac{1}{\sqrt{1+x^2}}$
25.	$\text{Cosh}^{-1} x$	$\frac{1}{\sqrt{x^2-1}}$
26.	$\text{Tanh}^{-1} x$	$\frac{1}{1-x^2}$
27.	$\text{Coth}^{-1} x$	$\frac{1}{1-x^2}$
28.	$\text{Sech}^{-1} x$	$\frac{-1}{\lvert x\rvert\sqrt{1-x^2}}$
29.	$\text{Cosech}^{-1} x$	$\frac{-1}{\lvert x\rvert\sqrt{1+x^2}}$

Historical Note

Bhaskaracharya of the 12th century solved for the first time, the seeds for the growth of the concept of calculus.

Isaac Newton (1642 – 1727) and *G.W. Leibnitz* (1646 – 1716) independently invented calculus during the seventeenth century. Later on, many mathematicians contributed for further development of calculus. *A.L. Cauchy*, *J.L. Lagrange* and *Karl Weierstrass* made the subject more rigorous. *Cauchy* used *D'Alembert's* limit concept to define the derivative of a function.

Today not only mathematics but many other subjects such as physics, chemistry, economics and biological sciences are enjoying the fruits of calculus.

Mostly, *G.W. Leibnitz* and *L. Euler* (1707 – 1783) developed the present day notation for calculus.

Answers

Exercise 9(a)

I. (i) $\frac{1}{2}\left[x^{-1/2}+3x^{-1/4}+5x^{-1/6}\right]$ (ii) $\frac{1}{\sqrt{2x-3}}-\frac{3}{2\sqrt{7-3x}}$

(iii) $20x^4 - 36x^2 + 2x$ (iv) $\frac{3}{2}\sqrt{x}-\frac{1}{2x\sqrt{x}}-6x$

(v) $\left(\sqrt{x}+1\right)(2x-4)+\frac{1}{2\sqrt{x}}\left(x^2-4x+2\right)$

(vi) $(ax+b)^n(cx+d)^m\left[\frac{na}{ax+b}+\frac{mc}{cx+d}\right]$

(vii) $5\cos x + e^x\log x + \frac{1}{x}e^x$ (viii) $5^x\log 5+\frac{1}{x}+3x^2e^x+x^3e^x$

(ix) $e^x + \cos 2x$ (x) $\frac{pax^2+2pbx+bq-ra}{(ax+b)^2}$

(xi) $\log_7 e \times \frac{1}{x \log_e x}$ (xii) $\frac{-(2ax+b)}{(ax^2+bx+c)^2}$

(xiii) $e^{2x}\left[2\log(3x+4)+\frac{3}{3x+4}\right]$ (xiv) $2e^{2x}(x^2+x+4)$

(xv) $\frac{ad-bc}{(cx+d)^2}$ (xvi) $a^x e^{x^2}(\log a + 2x)$

2. 5050

II. 1. (i) $3x^2$ (ii) $4x^3$

(iii) $2ax + b$ (iv) $\frac{1}{2\sqrt{x+1}}$

(v) $2 \cos 2x$ (vi) $-a \sin a x$

(vii) $2 \sec^2 2x$ (viii) $-\operatorname{cosec}^2 x$

(ix) $3 \sec 3x \tan 3x$ (x) $x \cos x + \sin x$

(xi) $-\sin 2x$

2. (i) $\frac{-3\sqrt{x}}{\left(1+x\sqrt{x}\right)^2}$

(ii) $x^{n-1} n^x (n \log (nx) + x \log n \,.\, \log (nx) + 1)$

(iii) $ax^{2n-1} + 2an\,x^{2n-1} \log\, x - bx^n\, e^{-x} + bn\, x^{n-1} e^{-x}$

(iv) $-3\left(\frac{1}{x}-x\right)^2 e^x\left(1+\frac{1}{x^2}\right)+\left(\frac{1}{x}-x\right)^3 e^x$

4. Differentiable at neither of the points

5. Not differentiable at 2

Exercise 9(b)

I. 1. (i) $-n \cot^{n-1} x \operatorname{cosec}^2 x$ (ii) $-4 \operatorname{cosec}^4 x \cot x$

(iii) $e^x \sec^2 (e^x)$ (iv) $2 \tan x \sec^2 x$

(v) $m \cos^{n+1} x \,.\, \sin^{m-1} x - n \sin^{m+1} x \cos^{n-1} x$

(vi) $m \cos mx \,.\, \cos nx - n \sin mx \,.\, \sin nx$

(vii) $\text{Tan}^{-1} x + \dfrac{x}{1+x^2}$ (viii) -1

(ix) $10 \operatorname{cosec} 10x$ (x) $\dfrac{3}{\sqrt{9x^2+16}}$

(xi) $\dfrac{1}{x[1+(\log x)^2]}$ (xii) $\dfrac{4-2x^2}{x^4+3x^2+4}$

(xiii) $\dfrac{e^x}{\left(\text{Sin}^{-1}\left(e^x\right)\right)\sqrt{1-e^{2x}}}$ (xiv) $\sin 2x\left(\text{Sin}^{-1}x\right)^2 + \dfrac{2\sin^2 x.\ \text{Sin}^{-1}x}{\sqrt{1-x^2}}$

(xv) $\dfrac{-1}{1+\sin 2x}$ (xvi) $\dfrac{1+3x^2-2x^4}{\left(1-x^2\right)^{3/2}}$

(xvii) $\dfrac{e^{\text{Sin}^{-1}x}}{\sqrt{1-x^2}}$ (xviii) $-\sin(\log x + e^x)\left(\dfrac{1}{x}+e^x\right)$

(xix) $\dfrac{\cos a}{\cos^2 x}$ (xx) $\dfrac{3\operatorname{cosec} 3x\ \cot 3x}{1+\operatorname{cosec}^2 3x}$

2. (i) $\dfrac{1}{2\sqrt{x+x^2}}$ if $y>0$ and $\dfrac{-1}{2\sqrt{x+x^2}}$ if $y<0$

(ii) $\dfrac{1}{2\sqrt{x}(1-x)}$ if $y>0$ and $\dfrac{-1}{2\sqrt{x}(1-x)}$ if $y<0$

(iii) $\dfrac{1}{x\cosh y}$ (iv) $\dfrac{-e^y}{1+x^2}$

(v) $\dfrac{e^x}{\sin 2y}$ (vi) $2\left(y+\sqrt{y}\right)$

II. (i) $\sin\left(\log\left(\cot x\right)\right)\dfrac{\operatorname{cosec} x}{\cos x}$

(ii) $\dfrac{-\sqrt{2}}{(1+x)\sqrt{1+x^2}}(x>-1)$ and $\dfrac{\sqrt{2}}{(1+x)\sqrt{1+x^2}}\ (x<-1)$

(iii) $4x \operatorname{cosec}(2(1-x^2))$ (iv) $-2x\sin(x^2)\cos(\cos(x^2))$

(v) $\dfrac{e^x}{1+e^{2x}}.\cos\left(\mathrm{Tan}^{-1}\left(e^x\right)\right)$

(vi) $\dfrac{a\cos(ax+b)\cos(cx+d)+c\sin(ax+b)\sin(cx+d)}{\cos^2(cx+d)}$

(vii) $\dfrac{\mathrm{sech}^2\left(\frac{x}{2}\right)}{2\left(1+\tanh^2\left(\frac{x}{2}\right)\right)}$ (viii) $\cos x(\mathrm{Tan}^{-1}x)^2+\dfrac{2\sin x\,\mathrm{Tan}^{-1}x}{1+x^2}$

III. **1.** $\dfrac{\sqrt{a^2-b^2}}{a+b\sin x}$ **2.** $\dfrac{\sqrt{a^2-b^2}}{a+b\cos x}$ **3.** $\dfrac{-\sin x}{2\cos^2 x+2\cos x+1}$

Exercise 9(c)

I. **1.** (i) $\dfrac{3}{\sqrt{1-x^2}}$ (ii) $\dfrac{-3}{\sqrt{1-x^2}}$

(iii) $\dfrac{2}{1+x^2}$ (iv) $\dfrac{-1}{1+x^2}$

(v) $\dfrac{1}{2}$ if $0<x<\pi$ and $\dfrac{-1}{2}$ if $-\pi<x<0$

(vi) $-2x\sin(x^2)\cos(\cos(x^2))$

(vii) $\dfrac{-2}{\sqrt{1-x^2}}$ (viii) $\dfrac{-e^{-x}}{1+e^{-2x}}\left(\cos\left(\tan^{-1}(e^{-x})\right)\right)$

2. (i) $2\sqrt{x}e^x$ (ii) $e^{\sin x}$ (iii) 1 if $-1<x<1$

II. **1.** (i) $\dfrac{3a}{a^2+x^2}$ (ii) $\dfrac{1}{2}$

(iii) $\dfrac{1}{2(1+x^2)}$ (iv) $(\log x)^{\tan x}\left[\dfrac{\tan x}{x\log x}+\sec^2 x\log(\log x)\right]$

(v) $x^{x^2+1}.\log(ex^2)$ (vi) $2\log(20).\mathrm{cosec}(2x).20^{\log(\tan x)}$

(vii) $x^x(1+\log x)+e^x.e^{e^x}$ (viii) $\log(e\log x)+\log x.\log(\log x)$

(ix) $e^{-ax^2}[\cos(x\log x)\log(ex)-2ax\sin(x\log x)]$ (x) $\dfrac{2^{x+1}\log 2}{1+4^x}$

2. (i) $\cot t$ (ii) $\dfrac{t(2-t^3)}{1-2t^3}$

(iii) $\tan t$ (iv) $\dfrac{b(t^2-1)}{2at}$

3. (i) $\dfrac{1}{x.a^x.(\log a)^2}$ (ii) $\dfrac{2}{|x|}$ (iii) $\dfrac{1}{2}$

4. (i) $\dfrac{a^2y-4x^3}{4y^3-a^2x}$ (ii) $\dfrac{y^2}{x(1-y\log x)}$ (iii) $\dfrac{y(\sin y-x\log y)}{x(x-y\cos y(\log x))}$

6. (i) $y\left[\dfrac{5}{1-6x}+\dfrac{6}{1+7x}-\dfrac{4}{3(1-2x)}-\dfrac{9}{4(1+3x)}\right]$

(ii) $y\left[\dfrac{4}{x}+\dfrac{2x}{3(x^2+4)}+\dfrac{4x}{7-4x^2}\right]$ (iii) $y\left[\dfrac{6}{c-2x}-\dfrac{3}{b-x}-\dfrac{2}{a-x}\right]$

(iv) $y\left[\dfrac{3}{x}+\dfrac{3}{2(2+3x)}+\dfrac{1}{1-x}-\dfrac{1}{2+x}\right]$ (v) $\dfrac{y}{2}\left[\dfrac{1}{x-3}+\dfrac{2x}{x^2+4}-\dfrac{(6x+4)}{3x^2+4x+5}\right]$

III. 1. (i) $(\sin x)^{\log x}\left[\dfrac{\log(\sin x)}{x}+\cot x\log x\right]+x^{\sin x}\left[\cos x\log x+\dfrac{\sin x}{x}\right]$

(ii) $(1+x\log x\log(ex))\,x^{x^x}+x-1$

(iii) $x^{\sin x}\left[\dfrac{\sin x}{x}+\cos x.\log x\right]+(\sin x)^x\,[x\cot x+\log(\sin x)]$

(iv) $x^x(1+\log x)+(\cot x)^x\left[\log(\cot x)-\dfrac{2x}{\sin 2x}\right]$

3. Yes

Exercise 9(d)

I. **1.** $\dfrac{16}{(4x+5)^3}$

II. **1.** (i) $\dfrac{-3}{4}\left(\cos x+3\cos 3x\right)$ (ii) $2(\cos 2x-\cos 4x)$ (iii) $\dfrac{-8\left(4x^2+9\right)}{\left(4x^2-9\right)^2}$

(iv) $e^{-2x}(\sin^3 x-12\sin^2 x\cos x+6\sin x\cos^2 x)$

(v) $e^x(3\cos 3x-4\sin 3x-\cos x)$ (vi) $\dfrac{-2x}{(1+x^2)^2}$

(vii) $\dfrac{-6x}{(1+x^2)^2}$

III. **1.** (i) $128\sin x\cos^2 x\,(12\sin^4 x-31\sin^2 x\cos^2 x+6\cos^4 x)$

(ii) $\dfrac{1}{4}(81\sin 9x-25\sin 5x-9\sin 3x-\sin x)$

Chapter 10

Applications of Derivatives

"A mathematician, like a painter or poet, is a maker of patterns. If the patterns are more permanent than theirs, it is because they are made with ideas"

- G.H. Hardy

Introduction

In Chapter 9, we have studied the concept of the derivative of a function. In this chapter, we will study some applications of derivatives. In fact, the derivative plays a vital role in solving some problems such as errors and approximations, finding maxima and minima (extreme values) of a function. We shall also discuss the geometrical interpretation of the derivative and the methods of finding the equations of the tangent and the normal at a point on a given curve.

G.H.Hardy
(1877 - 1947)

G.H. Hardy was one of the finest mathematicians of Cambridge of 20th century. His interests covered many topics of pure mathematics. His long collaboration with Littlewood produced mathematics of the highest quality. Even more remarkable was his collaboration with Srinivasa Ramanujan.

10.1 Errors and approximations

The word infinitesimal is used in the sense that it is extremely small or very very small. In other words, it is so small that it can not be distinguished from zero by any available means. Roughly, we can say that an infinitesimal is close to zero but it is not equal to zero. The infinitesimal in the variable x is denoted by Δx and the infinitesimal in the variable y is denoted by Δy. The infinitesimals Δx and Δy are referred as change in x and change in y respectively.

If a dependent variable y depends on x by a functional relation $y = f(x)$ then change in y is given by

$$\Delta y = f(x + \Delta x) - f(x) \qquad \text{..... (1)}$$

where the variable x is changed from x to $x + \Delta x$.

10.1.1 Notation

$\left(\dfrac{dy}{dx}\right)_{(x_0, y_0)}$ denotes the value of the derivative of the function $f(x, y)$ at (x_0, y_0).

If the function y is in explicit form $y = f(x)$ then we write $\left(\dfrac{dy}{dx}\right)_{(x_0, y_0)}$ as $f'(x_0)$.

10.1.2 Formula for approximate value of Δy

We define $f'(x)\Delta x$ or $\dfrac{dy}{dx}\Delta x$ as differential of y and is denoted by dy, i.e.,

$$dy = f'(x)\Delta x \qquad \text{..... (1)}$$

or

$$dy = \frac{dy}{dx}\Delta x \qquad \text{..... (2)}$$

If we take $y = f(x) = x$ in (1) , we get

$$dx = \Delta x \qquad \text{..... (3)}$$

and we call dx as differential of x.

Though dx and Δx are equal, Δy and dy need not be equal. In case $y = f(x)$ represents a line then dy and Δy are equal. Geometrically dy denotes the change in y along the tangent line where as Δy is the change along the curve (Fig.10. 1).

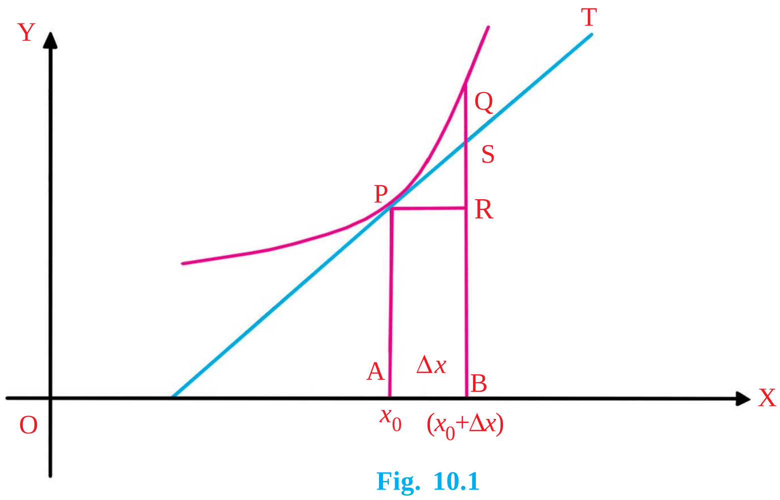


Fig. 10.1

In Fig. 10.1, PT is the tangent to the curve $y = f(x)$ at $P(x_0, f(x_0))$ and $Q(x_0 + \Delta x, f(x_0 + \Delta x))$ is a neighbouring point of P lying on the curve. The line segments PR, RS and RQ are respectively equal to Δx, dy and Δy.

If Δx is an infinitesimal then

$$\frac{\Delta y}{\Delta x} \approx f'(x) \qquad \text{.... (4)}$$

Since $\frac{\Delta y}{\Delta x}$ is approximately equal to $f'(x)$ there exists an infinitesimal ε such that

$$\frac{\Delta y}{\Delta x} = f'(x) + \varepsilon \qquad \text{.... (5)}$$

where ε depends on x and Δx. Equation (5) can be expressed as

$$\Delta y = f'(x)\,\Delta x + \varepsilon\,\Delta x \qquad \text{.... (6)}$$

Since ε and Δx are infinitesimals , their product is very very small and nearer to zero [like the product of (0.001) . (0.00000l) = 0.00000000l]. Therefore we take $f'(x)\,\Delta x$ as an approximate value of Δy. Thus,

$$\Delta y \approx f'(x)\,\Delta x \qquad \text{.... (7)}$$

or

$$\Delta y \approx f'(x)\,dx \quad \text{(by (3))} \qquad \text{.... (8)}$$

In view of (1) , the equation (7) can be written as

$$\Delta y \approx dy$$

We can also have another formula from equation (8), i.e.,

$$f(x + \Delta x) \approx f(x) + f'(x)\,dx \qquad \text{.... (9)}$$

since $\Delta y = f(x + \Delta x) - f(x)$. The equation (9) can be used to find an approximate value of y at $x = x_0 + \Delta x$.

10.1.3 Note

When x changes from x_0 to $x_0 + \Delta x$ then change in y is given by

$$\Delta y = f(x_0 + \Delta x) - f(x_0) \qquad \text{.... (10)}$$

and we call Δy given by (10) as the change in y at $x = x_0$.

10.1.4 Definition

If a number A is very close to a number B but it is not equal to B then A is called an approximate value of B. For example 3.141592 is an approximate value of $\pi = 3.14159263589$ ...

If K(e) is an exact value of a certain entity (length of a side, square root of a number) and K(a) is an approximate value of K(e) then the difference of these two is defined as an error i.e., K(e) – K(a) is the error. If Δx is considered as an error in x then the error in $y = f(x)$ is Δy. The exact error can be computed from equation (1) of 10.1.2 and the approximation of Δy can be computed from equation (8) of 10.1.2

Definition *(Absolute error, Relative error and Percentage error)*

If y is any variable then

(i) Δy is called an absolute error in y.

(ii) $\dfrac{\Delta y}{y}$ is called a relative error in y.

(iii) $\dfrac{\Delta y}{y}\times 100$ is called percentage error in y.

If $y = f(x)$ is a differentiable function and Δx is an error in x then the approximation of absolute error, relative error and percentage error in y are respectively as given below

$$\Delta y \approx f'(x)\Delta x \quad \text{.... (1)}$$

$$\frac{\Delta y}{y} \approx \left(\frac{f'(x)}{f(x)}\right)\Delta x \quad \text{.... (2)}$$

and

$$\frac{\Delta y}{y}\times 100 \approx \left(\frac{f'(x)}{f(x)}\right)\times 100 \times \Delta x. \quad \text{.... (3)}$$

10.1.5 Solved Problems.

1. Problem: *Find dy and Δy of $y = f(x) = x^2 + x$ at $x = 10$ when $\Delta x = 0.1$.*

Solution: As change in $y = f(x)$ is given by $\Delta y = f(x + \Delta x) - f(x)$, this change at $x = 10$ with $\Delta x = 0.1$ is

$$\Delta y = f(10.1) - f(10) = \{(10.1)^2 + 10.1\} - \{10^2 + 10\} = 2.11.$$

Since $dy = f'(x)\,\Delta x$, dy at $x = 10$ with $\Delta x = 0.1$ is

$$dy = \{(2)(10)+1\}0.1 = 2.1 \quad (\text{since } \frac{dy}{dx} = 2x + 1).$$

2. Problem: *Find Δy and dy for the function $y = \cos(x)$ at $x = 60°$ with $\Delta x = 1^0$.*

Solution: For the given problem Δy and dy at $x = 60^0$ with $\Delta x = 1^0$ are

$$\Delta y = \cos(60^0 + 1^0) - \cos(60^0) \quad \text{.... (1)}$$

and $dy = -\sin(60^0)\,(1^0)$ .... (2)

$\text{Cos}(60^0) = 0.5$, $\text{Cos}\,(61^0) = 0.4848$, $\text{Sin}(60^0) = 0.8660$, $1^0 = 0.0174$ radians

Therefore, $\Delta y = -0.0152$ and $dy = -0.0150$.

3. Problem: *The side of a square is increased from 3 cm to 3.01 cm. Find the approximate increase in the area of the square.*

Solution: Let x be the side of a square and A be its area. Then

$$A = x^2. \qquad .... (1)$$

Clearly A is a function of x. As the side is increased from 3 cm to 3.01 cm we can take $x = 3$ and $\Delta x = 0.01$ to compute the approximate increase in the area of square. The approximate value of change in area is

$$\Delta A \approx \frac{dA}{dx}\Delta x \qquad .... (2)$$

In veiw of equation (1), the equation (2) becomes

$$\Delta A \approx 2x\Delta x.$$

Hence the approximate increase in the area when the side is increased from 3 to 3.01 is

$$\Delta A \approx 2(3)(0.01) = 0.06.$$

4. Problem: *If the radius of a sphere is increased from 7 cm to 7.02 cm then find the approximate increase in the volume of the sphere.*

Solution: let r be the radius of a sphere and V be its volume. Then

$$V = \frac{4\pi r^3}{3}. \qquad .... (1)$$

Here V is a function of r. As the radius is increased from 7 cm to 7.02, we can take $r = 7$ cm and $\Delta r = 0.02$ cm. Now we have to find the approximate increase in the volume of the sphere.

$$\therefore\ \Delta V \approx \frac{dV}{dr}\Delta r = 4\pi r^2\ \Delta r.$$

Thus, the approximate increase in the volume of the sphere is

$$\frac{4(22)(7)(7)(0.02)}{7} = 12.32\ \text{cm}^3.$$

5. Problem: *If $y = f(x) = k\,x^n$ then show that the approximate relative error (or increase) in y is n times the relative error (or increase) in x where n and k are constants.*

Solution: The approximate relative error (or increase) in y by the equation (2) of 10.1.4 is

$$\left(\frac{f'(x)}{f(x)}\right)\Delta x = \frac{knx^{n-1}}{kx^n}\Delta x = n(\frac{\Delta x}{x} = n) \text{ relative error (or increase) in } x.$$

Hence the approximate relative error in $y = kx^n$ is n times the relative error in x.

6. Problem: *If the increase in the side of a square is 2% then find the approximate percentage of increase in its area.*

Solution: Let x be the side of a square and A be its area. Then

$$A = x^2.$$

Approximate percentage error in area A

$$= \left(\frac{\frac{dA}{dx}}{A}\right) \times 100 \times \Delta x \qquad (\text{ by (3) of 10.1.4 with } f = A)$$

$$= \frac{100(2x)\Delta x}{x^2} = \frac{200\Delta x}{x} = 2(2) = 4. \qquad \left(\because \frac{\Delta x}{x} \times 100 = 2\right)$$

7. Problem: *If an error of 0.01 cm is made in measuring the perimeter of a circle and the perimeter is measured as 44 cm then find the approximate error and relative error in its area.*

Solution: Let r, p and A be the radius, perimeter and area of the circle respectively. Given that $p = 44$ cm and $\Delta p = 0.0l$. We have to find approximation of ΔA and $\frac{\Delta A}{A}$. Note that $A = \pi r^2$ which is a function of r. As p and Δp are given we have to transform $A = \pi r^2$ into the form $A = f(p)$. This can be achieved by using the relation, perimeter $2\pi r = p$.

$$\therefore A = \pi\left(\frac{p}{2\pi}\right)^2 = \frac{p^2}{4\pi}.$$

Hence the approximate error in $A = \frac{dA}{dp}\Delta p = \frac{2p}{4\pi}\Delta p = \frac{p}{2\pi}\Delta p.$

The approximate error in A when $p = 44$ and $\Delta p = 0.01 = \frac{44}{2\pi}(0.01) = 0.07.$

$$\text{The approximate relative error } = \frac{\left(\frac{dA}{dp}\right)}{A} \cdot \Delta p = \frac{\left(\frac{p}{2\pi}\right)}{\left(\frac{p^2}{4\pi}\right)} \cdot \Delta p = 2\frac{\Delta p}{p} = \frac{2(0.01)}{44}$$

$$= 0.0004545.$$

8. Problem: *Find the approximate value of* $\sqrt[3]{999}$.

Solution: This problem can be answered by

$$f(x + \Delta x) \approx f(x) + f'(x)\,\Delta x \qquad \text{.... (1)}$$

with x =1000 and $\Delta x = -1$. The reason for taking x = 1000 is to make the calculation of $f(x)$ simpler when $f(x) = \sqrt[3]{x}$. Suppose

$$y = f(x) = \sqrt[3]{x} \quad \text{.... (2)}$$

Then equation (1) becomes

$$f(x + \Delta x) \approx f(x) + \frac{1}{3x^{\frac{2}{3}}}\Delta x$$

Hence $f(1000 - 1) \approx f(1000) + \dfrac{1}{3(1000)^{\frac{2}{3}}}(-1) = 9.9967.$

Exercise 10(a)

I. 1. Find Δy and dy for the following functions for the values of x and Δx which are shown against each of the functions.

(i) $y = x^2 + 3x + 6,\ x = 10$ and $\Delta x = 0.01$

(ii) $y = e^x + x,\ x = 5$ and $\Delta x = 0.02$

(iii) $y = 5x^2 + 6x + 6,\ \ x = 2$ and $\Delta x = 0.001$

(iv) $y = \dfrac{1}{x+2},\ x = 8$ and $\Delta x = 0.02$

(v) $y = \cos(x),\ x = 60^0$ and $\Delta x = 1^0$.

II.1. Find the approximations of the following

(i) $\sqrt{82}$ (ii) $\sqrt[3]{65}$ (iii) $\sqrt{25.001}$ (iv) $\sqrt[3]{7.8}$

(v) $\sin(62^0)$ (vi) $\cos(60^0\ 5')$ (vii) $\sqrt[4]{17}$

2. If the increase in the side of a square is 4% then find the approximate percentage of increase in the area of the square.

3. The radius of a sphere is measured as 14 cm. Later it was found that there is an error 0.02 cm in measuring the radius. Find the approximate error in surface area of the sphere.

4. The diameter of a sphere is measured to be 40 cm. If an error of 0.02 cm is made in it, then find approximate errors in volume and surface area of the sphere.

5. The time t, of a complete oscillation of a simple pendulum of length l is given by $t = 2\pi\sqrt{\dfrac{l}{g}}$ where g is gravitational constant. Find the approximate percentage of error in t when the percentage of error in l is 1 %.

10.2 Geometrical interpretation of the derivative

In this section we first recall the definition of tangent at a point to a curve. Then we give the geometrical interpretation of the derivative.

10.2.1 Definition

Let P *be a point on a curve (Fig. 10.2). Let* Q *be a neighbouring point to* P *on the curve. The line through* P *and* Q *is a secant of the curve. The limiting position of the secant* PQ *as* Q *moves nearer to* P *along the curve is called the* ***tangent*** *to the curve at the point* P.

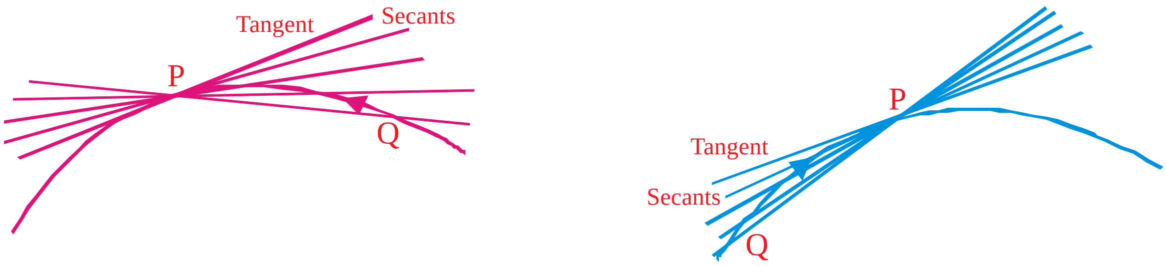


Fig. 10.2

10.2.2 Geometrical interpretation of derivative

Let APQ denote the curve $y = f(x)$ defined on an interval. Let P be a point on the curve and

P = $(c, \ f(c))$.

If we let Q to be a neighbouring point of P on the curve (see Fig. 10.3), then Q can be taken as Q = $(c + \delta c, f(c + \delta c))$. Let the tangent at P to the curve which is not parallel to X-axis in general, meet the X-axis at T and make an angle ψ with X-axis. Let the chord drawn through P, Q meet X-axis in S and make an angle θ. Let L, M be the feet of the perpendiculars drawn from P, Q respectively on the X-axis. Then PL = $f(c)$ and QM = $f(c+\delta c)$.

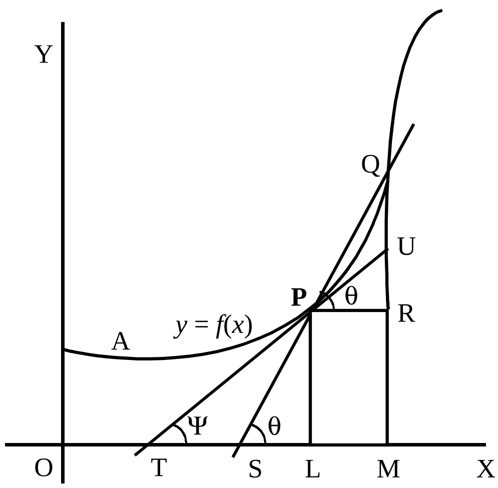


Fig. 10.3

Since, PR is parallel to OM we have $Q\hat{P}R = \theta$.

$$\text{Then, } \tan\theta = \frac{QR}{PR} = \frac{QM-RM}{OM-OL} = \frac{QM-PL}{OM-OL}$$
$$= \frac{f(c+\delta c)-f(c)}{c+\delta c-c} = \frac{f(c+\delta c)-f(c)}{\delta c}.$$

As the point Q approaches P, the limiting position of the chord PQ is the tangent PT at P. i.e., if $Q \to P$ then $QR \to 0$, $PR \to 0$, $\theta \to \psi$ and chord $\overleftrightarrow{PQ}$ approaches $\overleftrightarrow{PT}$.

$$\therefore \quad f'(c) = \lim_{\delta c \to 0} \frac{f(c+\delta c)-f(c)}{\delta c}$$
$$= \lim_{Q \to P} \frac{QR}{PR}$$
$$= \lim_{\theta \to \psi} \tan\theta = \tan\psi .$$

Observe that $\tan\psi$ is the slope of the tangent PT. Thus, the summary of the above discussion is that the derivative of $f(x)$ at c is the slope of the tangent to the curve $y = f(x)$ at the point $(c, f(c))$.

10.3 Equations of tangent and normal to a curve

In the previous section 10.2 we have seen that $\frac{dy}{dx}$ represents the slope of the tangent at a point (x, y) on the curve $y = f(x)$. Using this concept, it is easy to find the equations of tangent and normal.

10.3.1 Equation of tangent

Let $y = f(x)$ be a curve and $P(a, b)$ be a point on it. Then we know that the slope m of the tangent at P is

$$m = f'(a) \quad \text{or} \quad \left.\frac{dy}{dx}\right|_{(a,b)}$$

Therefore, the equation of the tangent to the curve at (a, b) is

$$y - b = m(x - a) \text{ or}$$
$$y - b = f'(a)\ (x - a).$$

10.3.2 Definition

*Let P be a point on a curve C. The straight line passing through P and perpendicular to the tangent to the curve at P is called the **normal** to the curve C at P. (see Fig. 10.4)*

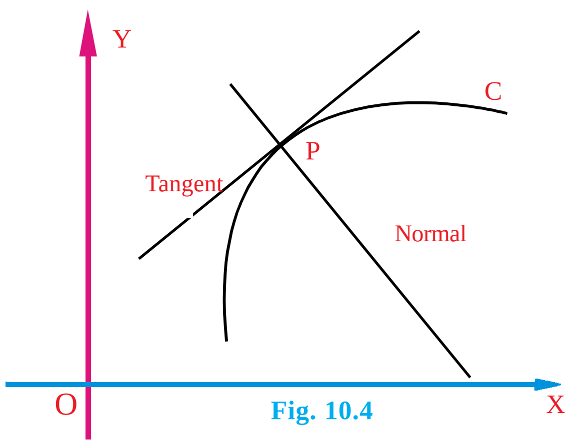


Fig. 10.4

10.3.3 Equation of normal

Since, the slope of the tangent to the curve $y = f(x)$ at $P(a, b)$ is $f'(a)$, the slope of the normal at P is $\frac{-1}{f'(a)}$, if $f'(a) \neq 0$.

If $f'(a) = 0$ the tangent to the curve at P is parallel to the X-axis and therefore the normal is parallel to the Y-axis.

Thus the equation of the normal is

$$y - b = \frac{-1}{f'(a)}(x - a) \quad \text{if} \quad f'(a) \neq 0 \quad \text{and}$$

$$x = a \quad \text{if} \quad f'(a) = 0.$$

10.3.4 Note

The curve $y = f(x)$ is said to have a

(i) horizontal tangent at a point $(a, f(a))$ on the curve when $f'(a) = 0$.

(ii) vertical tangent at a point $(a, f(a))$ on the curve when

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \infty \text{ or } -\infty.$$

10.3.5 Solved Problems

1. Problem: *Find the slope of the tangent to the following curves at the points as indicated:*

(i) $y = 5x^2$ *at* $(-1, 5)$

(ii) $y = \frac{1}{x-1}$ $(x \neq 1)$ *at* $(3, \frac{1}{2})$

(iii) $x = a \sec\theta$, $y = a \tan\theta$ *at* $\theta = \frac{\pi}{6}$

(iv) $\left(\frac{x}{a}\right)^n + \left(\frac{y}{b}\right)^n = 2$ *at* (a, b)

Solution

(i) $y = 5x^2$ then $\frac{dy}{dx} = 10x$.

$\therefore$ The slope of the tangent at the given point is $\left.\frac{dy}{dx}\right|_{(-1,5)} = -10$.

(ii) $y = \frac{1}{x-1}$ then $\frac{dy}{dx} = \frac{-1}{(x-1)^2}$

$\therefore$ Slope of the tangent at $\left(3, \frac{1}{2}\right)$ is $\left.\frac{dy}{dx}\right|_{(3,1/2)} = \frac{-1}{(3-1)^2} = -\frac{1}{4}$.

(iii) $x = a\sec\theta, \quad y = a\tan\theta$

$$\frac{dy}{dx} = \frac{\left(\dfrac{dy}{d\theta}\right)}{\left(\dfrac{dx}{d\theta}\right)} = \frac{a\sec^2\theta}{a\sec\theta\ \tan\theta} = \operatorname{cosec}\theta.$$

$\therefore$ Slope of the tangent at the point with $\theta = \dfrac{\pi}{6}$ is

$$\left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{6}} = \operatorname{cosec}\left(\frac{\pi}{6}\right) = 2.$$

(iv) $\left(\dfrac{x}{a}\right)^n + \left(\dfrac{y}{b}\right)^n = 2.$

Differentiating both sides with respect to x,

$$n\left(\frac{x}{a}\right)^{n-1}\cdot\frac{1}{a} + n\cdot\left(\frac{y}{b}\right)^{n-1}\cdot\frac{1}{b}\cdot\frac{dy}{dx} = 0$$

i.e., $$\frac{dy}{dx} = -\left(\frac{b}{a}\right)^n\left(\frac{x}{y}\right)^{n-1}.$$

$\therefore$ Slope of the tangent at $(a, b) = \left.\dfrac{dy}{dx}\right|_{(a,b)} = \dfrac{-b}{a}$.

2. Problem : *Find the equations of the tangent and the normal to the curve* $y = 5x^4$ *at the point* $(1, 5)$.

Solution : $y = 5x^4$ implies that $\dfrac{dy}{dx} = 20\,x^3$.

Slope of the tangent to the curve at $(1, 5)$ is $\left.\dfrac{dy}{dx}\right|_{(1,\,5)} = 20(1)^3 = 20.$

$\therefore$ The slope of the normal to the curve at $(1, 5)$ is $\dfrac{-1}{20}$.

$\therefore$ Equations of the tangent and normal to the curve at $(1, 5)$ are

$$y - 5 = 20(x - 1) \quad \text{and} \quad y - 5 = \frac{-1}{20}(x - 1) \quad \text{respectively.}$$

i.e., $y = 20x - 15$ and $20y = 101 - x$ respectively.

3. Problem: *Find the equations of the tangent and the normal to the curve* $y^4 = ax^3$ *at* (a, a).

Solution : Differentiating $y^4 = ax^3$ with respect to x, we get $4y^3 y_1 = 3ax^2$.

$$y_1 = \frac{3ax^2}{4y^3}.$$

$$\therefore \text{ Slope of the tangent at } (a, a) = y_1\Big|_{(a,a)} = \frac{3a.a^2}{4a^3} = \frac{3}{4}.$$

Slope of the normal at $(a, a) = \dfrac{-4}{3}$.

The equation of the tangent is $y - a = \dfrac{3}{4}(x - a)$

i.e., $4y = 3x + a$

The equation of the normal is

$$y - a = \frac{-4}{3}(x - a)$$

i.e., $3y + 4x = 7a$.

4. Problem: *Find the equations of the tangents to the curve* $y = 3x^2 - x^3$, *where it meets the X-axis.*

Solution : Putting $y = 3x^2 - x^3 = 0$, we get the points of intersection of the curve and X-axis,

i.e., $y = 0$. They are given by

$3x^2 - x^3 = 0$ or $x^2(3 - x) = 0$

i.e., $x = 0$, $x = 3$.

Thus, the curve crosses the X-axis at the points O(0, 0) and A(3, 0).

$\dfrac{dy}{dx} = 6x - 3x^2 \Rightarrow$ slope of the tangent at O(0, 0) to the curve is

$$\frac{dy}{dx}\Big|_{(0,0)} = 0.$$

$\therefore$ Tangent at O(0, 0) is $y - 0 = 0(x - 0)$.

i.e., $y = 0$.

i.e., X-axis is the tangent to the curve at (0, 0).

Now slope of the tangent at A(3, 0) to the curve is $\dfrac{dy}{dx}\Big|_{(3,0)} = 6(3) - 3(3)^2 = -9$.

$\therefore$ Tangent at (3, 0) is $y - 0 = -9(x - 3)$, $y + 9x = 27$.

5. Problem : *Find the points at which the curve $y = \sin x$ has horizontal tangents.*

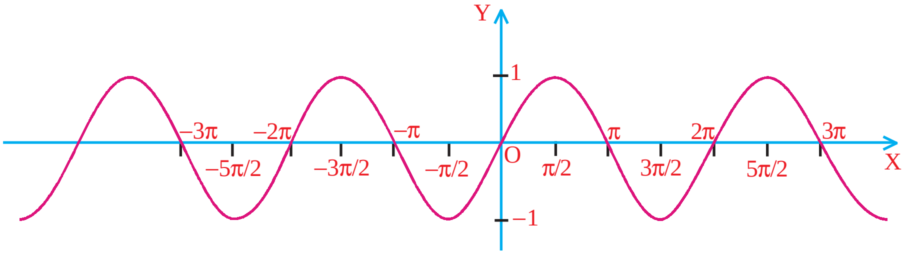
Solution

$$y = \sin x$$

$$\therefore \frac{dy}{dx} = \cos x.$$

A tangent is horizontal if and only if its slope is zero.

Therefore $\cos x = 0$



y = sin *x*

Fig. 10.5

Hence $x = (2n+1)\frac{\pi}{2}$; $n \in \mathbf{Z}$.

Hence the given curve has horizontal tangent at point (x_0, y_0)

$$\Leftrightarrow x_0 = (2n+1)\frac{\pi}{2} \text{ and } y_0 = (-1)^n \text{ for } n \in \mathbf{Z}. \text{ (see Fig. 10.5)}$$

6. Problem : *Verify whether the curve* $y = f(x) = x^{\frac{1}{3}}$ *has a vertical tangent at* $x = 0$.

Solution: For $h \neq 0$, we have

$$\lim_{h\to 0} \frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h\to 0} \frac{h^{\frac{1}{3}}}{h} = \lim_{h\to 0} \frac{1}{h^{\frac{2}{3}}}$$

$$= \lim_{h\to 0} \frac{1}{\left(h^{\frac{1}{3}}\right)^2} = \infty.$$

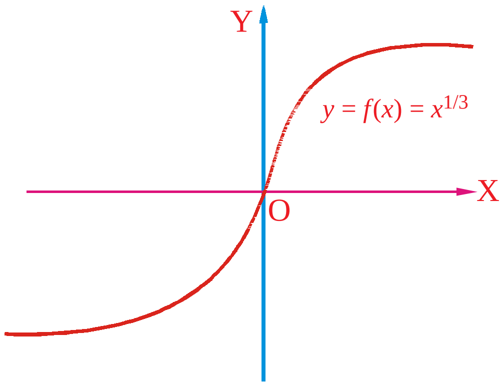


Fig. 10.6

$\therefore$ By Note 10.3.4 (ii), the function has a vertical tangent at $x = 0$ (see Fig.10.6.)

7. Problem : *Find whether the curve* $y = f(x) = x^{\frac{2}{3}}$ *has a vertical tangent at* $x = 0$.

Solution : For $h \neq 0$, we have

$$\frac{f(0+h) - f(0)}{h} = \frac{h^{\frac{2}{3}}}{h} = \frac{1}{h^{\frac{1}{3}}}.$$

The left handed limit of $\frac{1}{h^{\frac{1}{3}}}$ as $h \to 0$ is $-\infty$

while the right handed limit is ∞. Hence $\lim_{h\to 0} \frac{1}{h^{\frac{1}{3}}}$ does not exist.

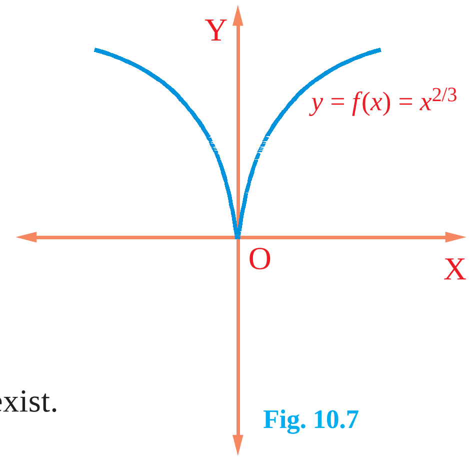


Fig. 10.7

Hence, by Note 10.3.4(ii), the vertical tangent does not exist at $x = 0$ (see Fig. 10.7).

8. Problem : *Show that the tangent at any point* θ *on the curve* $x = c\sec\theta$, $y = c\tan\theta$ *is* $y\sin\theta = x - c\cos\theta$.

Solution: Slope of the tangent at any point θ (i.e. at $(c\sec\theta,\ c\tan\theta)$) on the curve is

$$\frac{dy}{dx} = \frac{\left(\frac{dy}{d\theta}\right)}{\left(\frac{dx}{d\theta}\right)} = \frac{c\sec^2\theta}{c\sec\theta\tan\theta} = \operatorname{cosec}\theta.$$

$\therefore$ The equation of the tangent is

$$y - c\tan\theta = \operatorname{cosec}\theta\,(x - c\sec\theta)$$

i.e., $$y\sin\theta = x - c\cos\theta.$$

9. Problem : *Show that the area of the triangle formed by the tangent at any point on the curve* $xy = c$ $(c \neq 0)$, *with the coordinate axes is constant.*

Solution : Observe that $c \neq 0$. For otherwise $xy = 0$ represents the coordinate axes, which is against the hypothesis.

Let $P(x_1, y_1)$ be a point on the curve $xy = c$. Then $x_1 \neq 0$, $y_1 \neq 0$.

$$y = \frac{c}{x} \quad\Rightarrow\quad y' = -\frac{c}{x^2}.$$

$\therefore$ Equation of the tangent at (x_1, y_1) is $y - y_1 = -\dfrac{c}{x_1^2}(x - x_1)$

i.e., $$cx + x_1^2 y = cx_1 + x_1^2 y_1 = cx_1 + (x_1 y_1)x_1$$

$$= cx_1 + cx_1 = 2cx_1 \qquad (\because x_1 y_1 = c)$$

i.e., $$\frac{x}{2x_1} + \frac{y}{\left(\frac{2c}{x_1}\right)} = 1.$$

$\therefore$ The area of the triangle formed by this tangent and the coordinate axes is

$$= \frac{1}{2}(2x_1)\left(\frac{2c}{x_1}\right) = 2c, \text{ a constant.}$$

10. Problem : *Show that the equation of the tangent to the curve*

$$\left(\frac{x}{a}\right)^n+\left(\frac{y}{b}\right)^n = 2\ (a\neq 0,\ \ b\neq 0)\ \ \textit{at the point}\ \ (a,\ b)\ \ \textit{is}\ \ \frac{x}{a}+\frac{y}{b}=2.$$

Solution : Differentiating both sides of $\left(\frac{x}{a}\right)^n+\left(\frac{y}{b}\right)^n = 2$ w.r. to x, we get

$$\left(\frac{n}{a}\right)\left(\frac{x}{a}\right)^{n-1}+\left(\frac{n}{b}\right)\left(\frac{y}{b}\right)^{n-1}\frac{dy}{dx} = 0.$$

$$\therefore\ \left.\frac{dy}{dx}\right|_{(a,b)} = \left(\frac{-n}{a}\right)\left(\frac{a}{a}\right)^{n-1}\left(\frac{b}{n}\right)\left(\frac{b}{b}\right)^{n-1} = \frac{-b}{a}$$

$\therefore$ The equation of the tangent to the curve at the point (a, b) is

$$y-b = \frac{-b}{a}(x-a)$$

i.e., $ay-ab = -bx+ab$

or $bx+ay = 2ab.$ or $\frac{x}{a}+\frac{y}{b} = 2.$

Exercise 10(b)

I. 1. Find the slope of the tangent to the curve $y=3x^4-4x$ at $x=4$.

2. Find the slope of the tangent to the curve $y=\frac{x-1}{x-2}, x\neq 2$ at $x = 10$.

3. Find the slope of the tangent to the curve $y=x^3-x+1$ at the point whose x coordinate is 2.

4. Find the slope of the tangent to the curve $y=x^3-3x+2$ at the point whose x-coordinate is 3.

5. Find the slope of the normal to the curve $x=a\cos^3\theta,\ y=a\sin^3\theta$ at $\theta = \frac{\pi}{4}$.

6. Find the slope of the normal to the curve $x=1-a\sin\theta,\ y=b\cos^2\theta$ at $\theta=\frac{\pi}{2}$.

7. Find the points at which the tangent to the curve $y=x^3-3x^2-9x+7$ is parallel to the x-axis.

8. Find a point on the curve $y=(x-2)^2$ at which the tangent is parallel to the chord joining the points (2, 0) and (4, 4).

9. Find the point on the curve $y=x^3-11x+5$ at which the tangent is $y=x-11$.

10. Find the equations of all lines having slope 0 which are tangents to the curve $y=\frac{1}{x^2-2x+3}$.

II. 1. Find the equations of tangent and normal to the following curves at the points indicated against:

(i) $y=x^4-6x^3+13x^2-10x+5$ at (0, 5)

(ii) $y = x^3$ at $(1, 1)$

(iii) $y = x^2$ at $(0, 0)$

(iv) $x = \cos t,\ y = \sin t$ at $t = \frac{\pi}{4}$.

(v) $y = x^2 - 4x + 2$ at $(4, 2)$

(vi) $y = \frac{1}{1+x^2}$ at $(0, 1)$.

2. Find the equations of tangent and normal to the curve $xy = 10$ at $(2, 5)$.

3. Find the equations of tangent and normal to the curve $y = x^3 + 4x^2$ at $(-1, 3)$.

4. If the slope of the tangent to the curve $x^2 - 2xy + 4y = 0$ at a point on it is $\frac{-3}{2}$, then find the equations of tangent and normal at that point.

5. If the slope of the tangent to the curve $y = x \log x$ at a point on it is $\frac{3}{2}$, then find the equations of tangent and normal at that point.

6. Find the tangent and normal to the curve $y = 2e^{\frac{-x}{3}}$ at the point where the curve meets the Y-axis.

III. 1. Show that the tangent at $P(x_1, y_1)$ on the curve

$$\sqrt{x} + \sqrt{y} = \sqrt{a} \quad \text{is} \quad y\, y_1^{\frac{-1}{2}} + x\, x_1^{\frac{-1}{2}} = a^{\frac{1}{2}}.$$

2. At what points on the curve $x^2 - y^2 = 2$, the slopes of tangents are equal to 2 ?

3. Show that the curves $x^2 + y^2 = 2$ and $3x^2 + y^2 = 4x$ have a common tangent at the point $(1, 1)$.

4. At a point (x_1, y_1) on the curve $x^3 + y^3 = 3axy$, show that the tangent is

$$\left(x_1^2 - ay_1\right)x + \left(y_1^2 - ax_1\right)y = ax_1y_1.$$

5. Show that the tangent at the point $P(2, -2)$ on the curve $y(1 - x) = x$ makes intercepts of equal length on the coordinate axes and the normal at P passes through the origin.

6. If the tangent at any point on the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ intersects the coordinate axes in A and B, then show that the length AB is a constant.

7. If the tangent at any point P on the curve $x^m y^n = a^{m+n}$ $(mn \neq 0)$ meets the coordinate axes in A, B, then show that AP : BP is a constant.

10.4 Lengths of tangent, normal, subtangent and subnormal

In this section we define the lengths of tangent, normal, subtangent and subnormal and derive formulae to find these lengths.

10.4.1 Definition

Suppose $P = (a, f(a))$ *is a point on the curve* $y = f(x)$. *Let the tangent and normal to the curve at* P *meet the X-axis in* L *and* G *repsectively. Let* M *be the foot of the perpendicular drawn from P onto the X-axis.*

Then

(i) PL is called the *length of the tangent.*

(ii) PG is called the *length of the normal.*

(iii) LM is called the *length of the subtangent.*

(iv) MG is called the *length of the subnormal.*

If $\angle PLM = \phi$, then $\angle MPG = \phi$.

In general if, $\phi \neq 0$ and $\phi \neq \frac{\pi}{2}$,

we can find simple formulae for the above four lengths

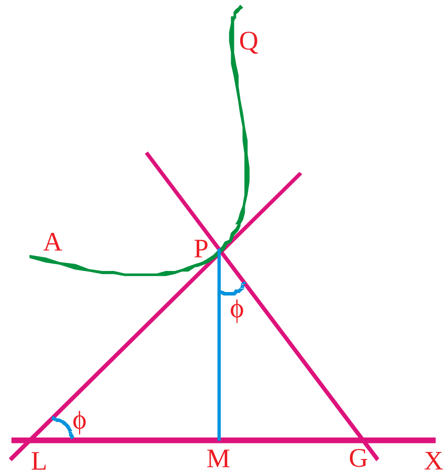


Fig. 10.8

(i) Length of the tangent $= PL = PM \operatorname{cosec} \phi$.

$$= |f(a) \operatorname{cosec}\phi| = \left|\frac{f(a)\sqrt{1+f'(a)^2}}{f'(a)}\right| \qquad (f'(a) \neq 0 \ \ as \ \ \phi \neq 0)$$

(ii) Length of the normal $= PG = PM \ |\sec \phi|$

$$= |f(a) \sec\phi| = \left|f(a)\sqrt{1+(f'(a))^2}\right|$$

(iii) Length of the subtangent $= LM = \left|\frac{f(a)}{\tan\phi}\right| = \left|\frac{f(a)}{f'(a)}\right|$

(iv) Length of the subnormal $= MG = |f(a)\tan\phi| = |f(a)f'(a)|$.

In case of implicit functions we write $\left(\frac{dy}{dx}\right)_{(a,\, f(a))}$ instead of $f'(a)$ in the above formulae.

In case of a general point (x, y) on a curve, the above formulae can be remembered as

(i) Length of tangent $= \left|\frac{y\sqrt{1+(y')^2}}{y'}\right|$

(ii) Length of normal $= \left|y\sqrt{1+(y')^2}\right|$

(iii) Length of subtangent $= \left|\frac{y}{y'}\right|$

(iv) Length of subnormal $= |y\,y'|$

10.4.2 Solved Problems

1. Problem: *Show that the length of the subnormal at any point on the curve $y^2 = 4ax$ is a constant.*

Solution : Differentiating $y^2 = 4ax$ with respect to x, we have

$$2y\,y' = 4a \quad \Rightarrow \quad y' = \frac{2a}{y}$$

i.e., $y\,y' = 2a$.

$\therefore$ The length of the subnormal at any point (x, y) on the curve

$$= |y\,y'| = |2a|, \text{ a constant.}$$

2. Problem : *Show that the length of the subtangent at any point on the curve $y = a^x$ $(a > 0)$ is a constant.*

Solution : Differentiating $y = a^x$ w.r.t. x, we have $y' = a^x \log a$.

$\therefore$ The length of the subtangent at any point (x, y) on the curve is

$$= \left|\frac{y}{y'}\right| = \left|\frac{a^x}{a^x \log a}\right| = \frac{1}{\log a} = \text{constant.}$$

3. Problem : *Show that the square of the length of subtangent at any point on the curve $by^2 = (x + a)^3$ $(b \neq 0)$ varies with the length of the subnormal at that point.*

Solution : Differentiating $by^2 = (x + a)^3$ w.r.t. x, we get

$$2\,by\,y' = 3(x + a)^2$$

$\therefore$ The length of the subnormal at any point (x, y) on the curve

$$= |y\,y'| = \left|\frac{3}{2b}(x+a)^2\right| \qquad \text{... (1)}$$

The square of the length of subtangent

$$= \left|\frac{y}{y'}\right|^2 = \frac{y^2}{y'^2}$$

$$= \frac{(x+a)^3}{b\left[\frac{3(x+a)^2}{2by}\right]^2} = \frac{(x+a)^3}{b} \times \frac{4 \times b^2 \times y^2}{9(x+a)^4}$$

$$= \frac{(x+a)^3}{b} \times \frac{4}{9} \times b^2 \times \frac{(x+a)^3}{b} \times \frac{1}{(x+a)^4} \qquad (\because by^2 = (x+a)^3)$$

$$= \frac{4}{9}(x+a)^2 \qquad \ldots (2)$$

$\therefore \quad \dfrac{(\text{length of subtangent})^2}{(\text{length of the subnormal})} = \dfrac{\frac{4}{9}(x+a)^2}{\frac{3}{2b}(x+a)^2} = \dfrac{8b}{27}$, which is a constant.

$\therefore$ (length of subtangent)$^2 \propto$ (length of subnormal).

4. Problem : *Find the value of k, so that the length of the subnormal at any point on the curve* $y = a^{1-k} x^k$ *is a constant.*

Solution : Differentiating $y = a^{1-k} x^k$ with respect to x, we get $y' = ka^{1-k} x^{k-1}$.

Length of subnormal at any point P(x, y) on the curve

$$= |y\,y'| = |y\,k\,a^{1-k}\,x^{k-1}|$$

$$= |k\,a^{1-k}\,x^k\,a^{1-k}\,x^{k-1}|$$

$$= |k\,a^{2-2k}\,x^{2k-1}|$$

In order to make these values a constant, we should have $2k - 1 = 0$ i.e., $k = \dfrac{1}{2}$.

Exercise 10(c)

I. 1. Find the lengths of subtangent and subnormal at a point on the curve $y = b\sin\dfrac{x}{a}$.

2. Show that the length of the subnormal at any point on the curve $xy = a^2$ varies as the cube of the ordinate of the point.

3. Show that at any point (x, y) on the curve $y = b\,e^{\frac{x}{a}}$, the length of the subtangent is a constant and the length of the subnormal is $\dfrac{y^2}{a}$.

II.1. Find the value of k so that the length of the subnormal at any point on the curve $xy^k = a^{k+1}$ is a constant.

2. At any point t on the curve $x = a(t + \sin t)$, $y = a(1 - \cos t)$, find the lengths of tangent, normal, subtangent and subnormal.

3. Find the lengths of normal and subnormal at a point on the curve $y = \frac{a}{2}\left(e^{\frac{x}{a}} + e^{\frac{-x}{a}}\right)$.

4. Find the lengths of subtangent, subnormal at a point t on the curve

$x = a(\cos t + t \sin t)$,

$y = a(\sin t - t \cos t)$.

10.5 Angle between two curves and condition for orthogonality of curves

If two curves C_1 and C_2 intersect at a point P, then the angle between the tangents to the curves at P is called the angle between the curves at P (see Fig. 10.9).

In general, there are two angles between these two tangents; if both of these angles are not equal, then one is an acute angle and the other obtuse.

It is customary to consider the acute angle to be the angle between the curves.

Let $y = f(x)$, $y = g(x)$ denote the curves C_1, C_2 and let these two curves intersect at the point $P(x_0, y_0)$.

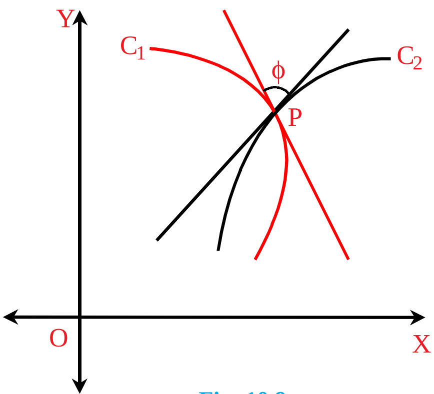


Fig. 10.9

Let $m_1 = f'(x)|_P$ and $m_2 = g'(x)|_P$ be the slopes of tangents at P to curves C_1 and C_2 respectively.

(i) In case $m_1 = m_2$, the curves have a common tangent at P. Then the angle between the curves is zero. In this case we say that the curves **touch each other** at P. This includes $m_1 = m_2 = 0$ also.

(ii) If $m_1 m_2 = -1$, then the tangents at P to the curves are perpendicular. In this case the curves are said to cut each other *orthogonally* at P.

(iii) If $m_1 \neq m_2$, $m_1 m_2 \neq -1$ and ϕ is the acute angle between the curves at P, then

$$\tan \phi = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right|$$

(iv) If either of m_1 and m_2, say $m_2 = 0$, then the angle between the curves is $\phi = \mathrm{Tan}^{-1}(m_1)$.

10.5.1 Solved Problems

1. Problem : *Find the angle between the curves* $xy = 2$ *and* $x^2 + 4y = 0$.

Solution : Let us first find the points of intersection of $xy = 2$ and $x^2 + 4y = 0$.

Putting $y = \dfrac{-x^2}{4}$ in $xy = 2$, we get

$$x^3 = -8 \text{ i.e., } x = -2.$$

$$x = -2 \Rightarrow y = \frac{-x^2}{4} = -1.$$

Therefore, the point of intersection of the curves is P (–2, –1).

$$xy = 2 \Rightarrow y' = \frac{-2}{x^2}$$

$$x^2 + 4y = 0 \Rightarrow y' = \frac{-x}{2}$$

Slope of the tangent to the curve $xy = 2$ at P is

$$m_1 = y'\Big|_{(-2,-1)} = \frac{-2}{(-2)^2} = \frac{-1}{2}.$$

Slope of the tangent to the curve $x^2 + 4y = 0$ at P is

$$m_2 = \frac{-x}{2}\Big|_{(-2,\,-1)} = 1.$$

Let ϕ be the angle between the curves at P. Then

$$\tan\phi = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \left|\frac{-\frac{1}{2} - 1}{1 + \left(\frac{-1}{2} \times 1\right)}\right| = 3.$$

$$\therefore \quad \phi = \mathrm{Tan}^{-1}\, 3.$$

2. Problem : *Find the angle between the curve* $2y = e^{\frac{-x}{2}}$ *and Y-axis.*

Solution : Equation of Y-axis is $x = 0$. The point of intersection of the curve

$2y = e^{\frac{-x}{2}}$ and $x = 0$ is $P\left(0, \frac{1}{2}\right)$.

The angle ψ made by the tangent to the curve $2y = e^{\frac{-x}{2}}$ at P with X-axis is given by

$$\tan\psi = \frac{dy}{dx}\Big|_{\left(0,\frac{1}{2}\right)} = \frac{-1}{4} e^{\frac{-x}{2}}\Big|_{\left(0,\,\frac{1}{2}\right)} = \frac{-1}{4}.$$

Further, if ϕ is the angle between the Y-axis and tangent at P to the curve $2y = e^{\frac{-x}{2}}$, then we have

$$\tan\phi = \left|\tan\left(\frac{\pi}{2} - \psi\right)\right| = |\cot\psi| = 4.$$

$\therefore$ The angle between the curve and the Y-axis is $\mathrm{Tan}^{-1} 4$.

3. Problem : *Show that the condition for the orthogonality of the curves* $ax^2 + by^2 = 1$ *and* $a_1x^2 + b_1y^2 = 1$ is $\frac{1}{a} - \frac{1}{b} = \frac{1}{a_1} - \frac{1}{b_1}$.

Solution: Let the curves $ax^2 + by^2 = 1$ and $a_1x^2 + b_1y^2 = 1$ intersect at $P(x_1, y_1)$ so that

$$a x_1^2 + b y_1^2 = 1 \text{ and } a_1 x_1^2 + b_1 y_1^2 = 1,$$

from which we get, (by cross multiplication rule),

$$\frac{x_1^2}{b_1 - b} = \frac{y_1^2}{a - a_1} = \frac{1}{ab_1 - a_1b} \qquad \text{... (1)}$$

Differentiating $ax^2+by^2 = 1$ with respect to x we get

$$\frac{dy}{dx} = \frac{-ax}{by}.$$

Hence, if m_1 is the slope of the tangent at $P(x_1, y_1)$ to the curve

$$ax^2 + by^2 = 1, \text{ then } m_1 = \frac{-ax_1}{by_1}.$$

Similarly, the slope (m_2) of the tangent at P to $a_1x^2+b_1y^2 = 1$ is given by

$$m_2 = \frac{-a_1x_1}{b_1y_1}.$$

Since the curves cut orthogonally, we have $m_1m_2 = -1$,

i.e., $$\frac{aa_1x_1^2}{bb_1y_1^2} = -1$$

or $$\frac{x_1^2}{y_1^2} = \frac{-bb_1}{aa_1} \qquad \text{... (2)}$$

Now from (1) and (2), the condition for the orthogonality of the given curves is

$$\frac{b_1 - b}{a - a_1} = \frac{-bb_1}{aa_1}$$

or $$(b - a)\, a_1b_1 = (b_1 - a_1)\, ab$$

or $$\frac{1}{a} - \frac{1}{b} = \frac{1}{a_1} - \frac{1}{b_1}.$$

4. Problem : *Show that the curves* $y^2 = 4(x + 1)$ *and* $y^2 = 36\ (9 - x)$ *intersect orthogonally.*

Solution: Solving $y^2 = 4(x + 1)$ and $y^2 = 36\ (9 - x)$ for the points of intersection, we get

$$4(x + 1) = 36\ (9 - x)$$

i.e., $10x = 80$ or $x = 8.$

$$y^2 = 4(x + 1) \Rightarrow y^2 = 4(9) = 36 \Rightarrow y = \pm 6.$$

$\therefore$ The points of intersection of the two curves are P(8, 6), Q(8, –6).

$$y^2 = 4(x + 1) \Rightarrow \frac{dy}{dx} = \frac{2}{y} \quad \text{(first curve)}$$

$$y^2 = 36(9 - x) \Rightarrow \frac{dy}{dx} = \frac{-18}{y} \quad \text{(second curve)}$$

Slope of the tangent to the curve $y^2 = 4\ (x + 1)$ at P is

$$m_1 = \frac{2}{6} = \frac{1}{3}.$$

Slope of the tangent to the curve $y^2 = 36\ (9 - x)$ at P is

$$m_2 = \frac{-18}{6} = -3.$$

$$m_1 m_2 = \frac{1}{3} \times -3 = -1 \Rightarrow \text{the curves intersect orthogonally at P.}$$

We can prove, similarly, that the curves intersect orthogonally at Q also.

Exercise 10(d)

I.. Find the angle between the curves given below:

1. $x + y + 2 = 0; \quad x^2 + y^2 - 10y = 0$

2. $y^2 = 4x\ ; \quad x^2 + y^2 = 5$

3. $x^2 + 3y = 3; \quad x^2 - y^2 + 25 = 0$

4. $x^2 = 2(y + 1); \quad y = \dfrac{8}{x^2 + 4}$

5. $2y^2 - 9x = 0\ ; \quad 3x^2 + 4y = 0$ (in the 4^{th} quadrant)

6. $y^2 = 8x\ ; \quad 4x^2 + y^2 = 32$

7. $x^2y = 4\ ; \quad y\ (x^2 + 4) = 8$

8. Show that the curves $6x^2 - 5x + 2y = 0$ and $4x^2 + 8y^2 = 3$ touch each other at $\left(\frac{1}{2}, \frac{1}{2}\right)$.

10.6 Derivative as a rate of change

In this section we learn how the derivative can be used to determine the rate of change of a variable. We also discuss their application to the physics and social sciences.

10.6.1 Average rate of change

If $y = f(x)$ then the average rate of change in y between $x = x_1$ and $x = x_2$ is defined as

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} \qquad \text{.... (1)}$$

Geometrically the average rate of change of y w.r.t. x is the slope of secant line joining $(x_1, f(x_1))$ and $(x_2, f(x_2))$ which are the points lying on the graph of $y = f(x)$.

The units of average rate of change of a function are the units of y per unit of the variable x.

10.6.2 Instantaneous rate of change of a function f at $x = x_0$

If $y = f(x)$ then instantaneous rate of change of a function f at $x = x_0$ is defined as

$$\lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

which is equal to $f'(x_0)$.

i.e., Instantaneous rate of change of f at $x = x_0$ is $f'(x_0)$.

10.6.3 Note

Instantaneous rate of change of the function f at x is $f'(x)$.

10.6.4 Rectilinear motion.

The motion of a particle in a line is called rectilinear motion. It is customary to represent the line of motion by a coordinate axis. We choose a reference point (origin), a positive direction (to the right of origin) and a unit of distance on the line.

The rectilinear motion is described by $s = f(t)$ where $f(t)$ is the rule connecting s and t. Here s is the coordinate of the particle for the amount of time t that elapsed since the motion began.

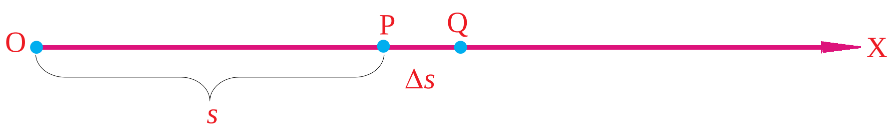


Fig. 10.10

If a particle moves according to the rule $s = f(t)$ where s is the displacement of the particle at time t, then $\frac{\Delta s}{\Delta t}$ is the average rate of change in s between t and $t + \Delta t$

i.e., $$\frac{\Delta s}{\Delta t} = \frac{f(t+\Delta t) - f(t)}{\Delta t}$$

Since the rate of change of displacement is the velocity, we call $\frac{\Delta s}{\Delta t}$ as the average velocity of the function $s = f(t)$ between the time t and $t + \Delta t$.

10.6.5 Note

If $s = f(t)$ then the average velocity between $t = t_1$ and $t = t_2$ is $\frac{f(t_2) - f(t_1)}{t_2 - t_1}$.

10.6.6 Instantaneous velocity

Suppose a taxi-car travelled 400 kms in 8 hours. Then its average velocity in 8 hours is 50 km/hr. The average velocity 50 km/hr of the taxi-car does not imply that the car at each point of its travelled path has the velocity 50 km/hr. The velocity of taxi-car at a given instant during movement of the car is shown on its speedometer.

Expression for velocity.

If $s = f(t)$ then instantaneous rate of change of function f or s at $t = t_0$ is

$$\lim_{\Delta t \to 0} \frac{f(t_0 + \Delta t) - f(t)}{\Delta t}$$

and it is equal to $f'(t_0)$ or $\left(\frac{ds}{dt}\right)_{t=t_0}$

The rate of change of displacement in a unit time is the velocity. Therefore $f'(t_0)$ [or $\left(\frac{ds}{dt}\right)_{t=t_0}$] represents the instantaneous velocity of the particle at time $t = t_0$. Further $f'(t)$ (or $\frac{ds}{dt}$) represents the instantaneous velocity at any time t.

10.6.7 Note

The acceleration of a particle at time $t = t_0$, moving with $s = f(t)$ is given by $\left(\frac{d^2 s}{dt^2}\right)$ at $t = t_0$ since the acceleration is the rate of change of velocity.

10.6.8 Note

1. The acceleration of a particle at any time t, moving with $s = f(t)$ is given by $\dfrac{d^2s}{dt^2}$.

2. If $y = f(x)$ and x, y are functions of t then $\dfrac{dy}{dt} = f'(x)\dfrac{dx}{dt}$.

10.6.9 Solved problems

1. Problem: *Find the average rate of change of $s = f(t) = 2t^2 + 3$ between $t = 2$ and $t = 4$.*

Solution: The average rate of change of s between $t = 2$ and $t = 4$ is

$$\frac{f(4) - f(2)}{4 - 2} = \frac{35 - 11}{4 - 2} = 12.$$

2. Problem: *Find the rate of change of area of a circle w.r.t. radius when $r = 5$ cm.*

Solution: Let A be the area of the circle with radius r. Then $A = \pi r^2$. Now, the rate of change of area A w.r. t. r is given by $\dfrac{dA}{dr} = 2\pi r$. When $r = 5$ cm, $\dfrac{dA}{dr} = 10\pi$.

Thus, the area of the circle is changing at the rate of 10π cm^2/cm.

3. Problem: *The volume of a cube is increasing at a rate of 9 cubic centimeters per second. How fast is the surface area increasing when the length of the edge is 10 centimeters?*

Solution: Let x be the length of the edge of the cube, V be its volume and S be its surface area. Then, $V = x^3$ and $S = 6x^2$. Given that rate of change of volume is 9 cm^3 /sec.

Therefore, $\dfrac{dV}{dt} = 9$ cm^3 /sec.

Now differentiating V w.r.t. t, we get,

$$\frac{dV}{dt} = 3x^2\frac{dx}{dt} \Rightarrow 9 = 3x^2\frac{dx}{dt}$$

i.e., $$\frac{dx}{dt} = \frac{3}{x^2}.$$

Differentiating S w.r.t. t, we get

$$\frac{dS}{dt} = 12x \times \frac{dx}{dt}$$

$$= 12x \times \frac{3}{x^2} = \frac{36}{x}.$$

Hence, when $x = 10$ cm, $\dfrac{dS}{dt} = \dfrac{36}{10} = 3.6$ cm^2/sec.

4. Problem: *A particle is moving in a straight line so that after t seconds its distance is s (in cms) from a fixed point on the line is given by $s = f(t) = 8t + t^3$. Find (i) the velocity at time $t = 2$ sec (ii) the initial velocity (iii) acceleration at $t = 2$ sec.*

Solution: The distance s and time t are connected by the relation

$$s = f(t) = 8t + t^3 \qquad \text{.... (1)}$$

$$\therefore \quad \text{velocity} \qquad v = 8 + 3t^2 \qquad \text{.... (2)}$$

and the acceleration is given by

$$a = \frac{d^2s}{dt^2} = 6t \qquad \text{.... (3)}$$

(i) The velocity at $t = 2$ is $8 + 3(4) = 20$ cm/sec.

(ii) The initial velocity ($t = 0$) is 8 cm/sec.

(iii) The acceleration at $t = 2$ is $6(2) = 12$ cm/sec^2 .

5. Problem: *A container in the shape of an inverted cone has height 12 cm and radius 6 cm at the top. If it is filled with water at the rate of 12 cm^3/sec., what is the rate of change in the height of water level when the tank is filled 8 cm?*

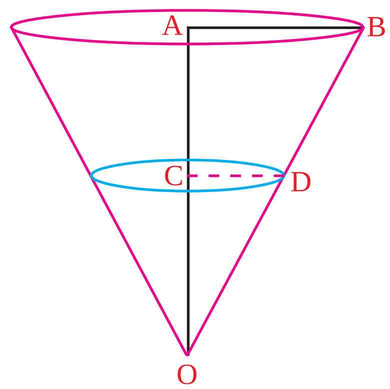


Fig. 10.11

Solution: Let OC be height of water level at t sec (Fig. 10.11). The triangles OAB and OCD are similar triangles. Therefore

$$\frac{CD}{AB} = \frac{OC}{OA}$$

Let $OC = h$ and $CD = r$. Given that $AB = 6$ cm, $OA = 12$ cm.

$$\frac{r}{6} = \frac{h}{12}.$$

$$\text{i.e.,} \qquad r = \frac{h}{2} \qquad \text{.... (1)}$$

Volume of the cone V is given by

$$V = \frac{\pi r^2 h}{3} \qquad \text{.... (2)}$$

Using (1), we have

$$V = \frac{\pi h^3}{12} \qquad \text{.... (3)}$$

Differentiating (3) w.r.t. t, we get

$$\frac{dV}{dt} = \frac{\pi h^2}{4} \cdot \frac{dh}{dt}$$

Hence

$$\frac{dh}{dt} = \frac{4}{\pi h^2} \frac{dV}{dt}.$$

When $h = 8$ cm, the rate of rise of the water level (height) is $\left(\frac{dh}{dt}\right)_{h=8}$.

i.e., $\left(\frac{1}{\pi}\right) \frac{4}{8^2}(12) = \frac{3}{4\pi}$ cm/sec.

Hence, the rate of change of water level is $\frac{3}{4\pi}$ cm/sec when the water level of the tank is 8 cm.

6. Problem: *A particle is moving along a line according to $s = f(t) = 4t^3 - 3t^2 + 5t - 1$ where s is measured in meters and t is measured in seconds. Find the velocity and acceleration at time t. At what time the acceleration is zero.*

Solution: Since $f(t) = 4t^3 - 3t^2 + 5t - 1$, the velocity at time t is

$$v = \frac{ds}{dt} = 12t^2 - 6t + 5$$

and the acceleration at time t is $a = \frac{d^2s}{dt^2} = 24t - 6.$

The acceleration is 0 if $24t - 6 = 0$

i.e., $t = \frac{1}{4}.$

The acceleration of the particle is zero at $t = \frac{1}{4}$ sec.

7. Problem : *The quantity (in mg) of a drug in the blood at time t (sec) is given by $q = 3(0.4)^t$. Find the instantaneous rate of change at $t = 2$ sec.*

Solution: Given that $q = 3(0.4)^t$. Therefore,

$$\frac{dq}{dt} = 3(0.4)^t \log_e(0.4)$$

is the instantaneous rate of change in q. Hence the instantaneous rate of q at time $t = 2$ sec. is given by

$$\left(\frac{dq}{dt}\right)_{t=2} = 3(0.4)^2 \log_e(0.4).$$

8. Problem: *Let a kind of bacteria grow by t^3 (t in sec). At what time the rate of growth of the bacteria is 300 bacteria per sec?*

Solution: Let g be the amount of growth of bacteria at t sec. Then

$$g(t) = t^3 \quad \text{.... (1)}$$

The growth rate at time t is given by

$$g'(t) = 3t^2 \quad \text{.... (2)}$$

$$300 = 3t^2 \text{ (given that growth rate is 300)}$$

$$t = 10 \text{ sec.}$$

$\therefore$ After $t = 10$ sec, the growth rate of bacteria should be 300 bacteria / sec.

9. Problem: *The total cost $C(x)$ in rupees associated with production of x units of an item is given by $C(x) = 0.005\,x^3 - 0.02x^2 + 30x + 500$. Find the marginal cost when 3 units are produced (marginal cost is the rate of change of total cost).*

Solution: Let M represent the marginal cost. Then

$$M = \frac{dC}{dx}.$$

Hence,

$$M = \frac{d}{dx}(0.005x^3 - 0.02x^2 + 30x + 500)$$

$$= 0.005(3x^2) - 0.02(2x) + 30$$

$\therefore$ The Marginal cost at $x = 3$ is

$$(M)_{x=3} = 0.005\ (27) - 0.02\ (6) + 30 = 30.015.$$

Hence the required marginal cost is Rs. 30.02 to produce 3 units.

10. Problem: *The total revenue in rupees received from the sale of x units of a product is given by* $R(x) = 3x^2 + 36x + 5$. *Find the marginal revenue when x=5(marginal revenue is the rate of change of total revenue).*

Solution: Let m denote the marginal revenue. Then

$$m = \frac{dR}{dx} \text{(since the total revenue is R}(x))$$

Given that $R(x) = 3x^2 + 36x + 5$

$$\therefore \quad m = 6x + 36$$

The marginal revenue at $x = 5$ is

$$\left[m = \frac{dR}{dx}\right]_{x=5} = 30 + 36 = 66.$$

Hence the required marginal revenue is Rs.66.

Exercise 10(e)

I. 1. At time t, the distance s of a particle moving in a straight line is given by $s = -4t^2 + 2t$. Find the average velocity between $t = 2$sec and $t = 8$ sec.

2. If $y = x^4$ then find the average rate of change of y between $x = 2$ and $x = 4$.

3. A particle moving along a straight line has the relation $s = t^3 + 2t + 3$, connecting the distance s described by the particle in time t. Find the velocity and acceleration of the particle at $t = 4$ seconds.

4. The distance-time formula for the motion of a particle along a straight line is

$s = t^3 - 9t^2 + 24t - 18$. Find when and where the velocity is zero.

5. The displacement s of a particle travelling in a straight line in t seconds is given by $s = 45t + 11t^2 - t^3$. Find the time when the particle comes to rest.

II.1. The volume of a cube is increasing at the rate of 8 cm^3/sec. How fast is the surface area increasing when the length of an edge is 12 cm?

2. A stone is dropped into a quiet lake and ripples move in circles at the speed of 5 cm/sec. At the instant when the radius of circular ripple is 8 cm., how fast is the enclosed area increases?

3. The radius of a circle is increasing at the rate of 0.7 cm/sec. What is the rate of increase of its circumference?

4. A balloon, which always remains spherical on inflation, is being inflated by pumping in 900 cubic centimeters of gas per second. Find the rate at which the radius of balloon increases when the radius is 15 cm.

5. The radius of an air bubble is increasing at the rate of $\frac{1}{2}$ cm/sec. At what rate is the volume of the bubble increasing when the radius is 1 cm?

6. Assume that an object is launched upward at 980 m/sec. Its position would be given by $s = -4.9\, t^2 + 980\, t$. Find the maximum height attained by the object.

7. Let a kind of bacteria grow in such a way that at time t sec. there are $t^{(3/2)}$ bacteria. Find the rate of growth at time t = 4 hours.

8. Suppose we have a rectangular aquarium with dimensions of length 8 m, width 4 m and height 3 m. Suppose we are filling the tank with water at the rate of 0.4 m^3/sec. How fast is the height of water changing when the water level is 2.5m?

9. A container is in the shape of an inverted cone has height 8 m and radius 6 m at the top. If it is filled with water at the rate of 2 m^3/minute, how fast is the height of water changing when the level is 4 m?

10. The total cost C(x) in rupees associated with the production of x units of an item is given by $C(x) = 0.007x^3 - 0.003x^2 + 15x + 4000$. Find the marginal cost when 17 units are produced.

11. The total revenue in rupees received from the sale of x units of a produce is given by $R(x) = 13x^2 + 26x + 15$. Find the marginal revenue when $x = 7$.

12. A point P is moving on the curve $y = 2x^2$. The x coordinate of P is increasing at the rate of 4 units per second. Find the rate at which the y coordinate is increasing when the point is at (2, 8).

10.7 Rolle's Theorem and Lagrange's Mean Value Theorem

In this section, we give statements of Rolle's and Lagrange's mean value theorems (without proofs) and their geometrical interpretation.

10.7.1 Theorem : (Rolle's theorem)

Suppose a, b ($a < b$) are two real numbers. Let $f : [a, b] \rightarrow \mathbf{R}$ be a function satisfying the following conditions :

(i) *f is continuous on $[a, b]$*

(ii) *f is differentiable on (a, b) and*

(iii) $f(a) = f(b)$.

Then $\exists$ *at least one* $c \in (a, b)$ *such that* $f'(c) = 0$.

Note that there can be more than one point at which the derivative is zero.

10.7.2 Example

Let $f : [-3, 8] \to \mathbf{R}$ be defined by $f(x) = x^2 - 5x + 6$. This function f satisfies the following conditions.

(i) f is continuous on $[-3, 8]$ as every polynomial is continuous on any closed interval.

(ii) f is differentiable on $(-3, 8)$ as every polynomial is differentiable on every interval.

(iii) $f(-3) = f(8)$ because $f(-3) = 30$ and $f(8) = 30$.

Thus, f is continuous on $[-3, 8]$, differentiable in $[-3, 8]$ and $f(-3) = f(8)$. Thus f satisfies all the conditions of Rolle's theorem. Also $f'(x) = 2x - 5 = 0$ at $x = \frac{5}{2}$. Let $c = \frac{5}{2}$. Clearly $c = \frac{5}{2} \in (-3, 8)$.

10.7.3 Geometrical interpretation of Rolle's theorem

Let f be a function satisfying the conditions of Rolle's theorem. The existence of a point $c \in (a, b)$ such that $f'(c) = 0$, on the curve $y = f(x)$ shows that there exists at least one point where the tangent is parallel to X-axis (Fig. 10.12).

Note that we may have more than one point having this property at different points whose abscissae lie between a and b (Fig. 10.13).

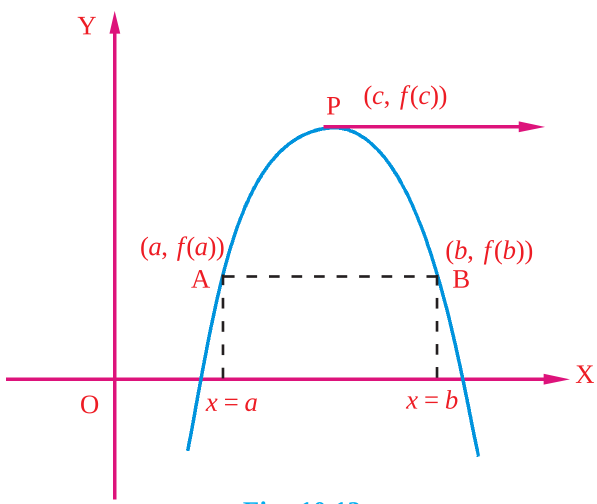


Fig. 10.12

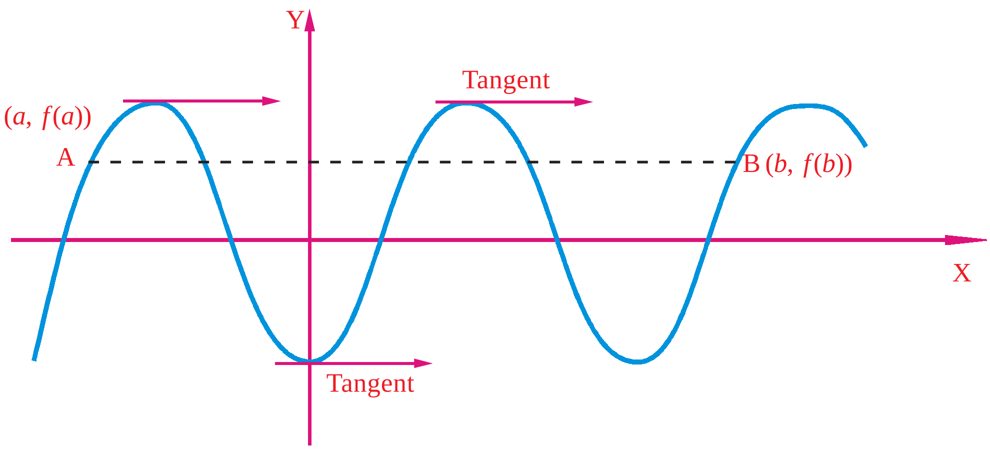


Fig. 10.13

10.7.4 Solved Problems

1. Problem : *Verify Rolle's theorem for the function* $y = f(x) = x^2 + 4$ *in* $[-3, 3]$.

Solution: Here $f(x) = x^2 + 4$. f is continuous on $[-3, 3]$ as $x^2 + 4$ is a polynomial which is continuous on any closed interval. Further $f(3) = f(-3) = 13$ and f is differentiable on $[-3, 3]$.

$\therefore$ By Rolle's theorem $\exists\ c \in (-3, 3)$ such that $f'(c) = 0$

$$f'(x) = 2x = 0 \text{ for } x = 0$$

The point $c = 0 \in (-3, 3)$. Thus Rolle's theorem is verified.

2. Problem : *Verify Rolle's theorem for the function* $f(x) = x(x + 3)e^{-x/2}$ *in* $[-3, 0]$.

Solution : Here $f(-3) = 0$ and $f(0) = 0$.

We have

$$f'(x) = \frac{(-x^2 + x + 6)}{2} e^{\frac{-x}{2}}$$

$f'(x) = 0 \Leftrightarrow -x^2 + x + 6 = 0 \Leftrightarrow x = -2$ or 3. Of these two values -2 is in the open interval $(-3, 0)$ which satisfies the conclusion of Rolle's theorem.

3. Problem : *Let* $f(x) = (x - 1)(x - 2)(x - 3)$. *Prove that there is more than one 'c' in* $(1, 3)$ *such that* $f'(c) = 0$.

Solution : Observe that f is continuous on $[1, 3]$, differentiable in $(1, 3)$ and $f(1) = f(3) = 0$.

$$\begin{aligned} f'(x) &= (x-1)(x-2) + (x-1)(x-3) + (x-2)(x-3) \\ &= 3x^2 - 12x + 11. \end{aligned}$$

The roots of $f'(x) = 0$ are $\dfrac{12 \pm \sqrt{144-132}}{6} = 2 \pm \dfrac{1}{\sqrt{3}}$.

Both these roots lie in the open interval (1, 3) and are such that the derivative vanishes at these points.

10.7.5 Lagrange's Mean Value Theorem

Suppose a, b $(a < b)$ are two real numbers. Let $f : [a, b] \to \mathbf{R}$ be a function satisfying the following conditions

(i) f is continuous on $[a, b]$

(ii) f is differentiable in (a, b)

Then there exists at least one point $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}. \qquad \text{.... (1)}$$

Note that there may be more than one point satisfying (1).

10.7.6 Example

Let $f(x) = x^2$ be defined on [2, 4]. The function f is continuous on [2, 4] and differentiable in (2, 4) as $f(x) = x^2$ is a polynomial function. Now $f'(x) = 2x$. By Lagrange's mean value theorem there exists $c \in (2, 4)$ such that

$$f'(c) = \frac{f(4) - f(2)}{4 - 2}$$

i.e., $\quad 2c = \dfrac{16 - 4}{4 - 2}$

i.e., $\quad c = 3.$

Clearly $\quad 3 \in (2, 4).$

10.7.7 Geometrical Interpretation of Lagranges Mean Value Theorem.

If A$(a, f(a))$ and B$(b, f(b))$ are two points on the curve $y = f(x)$ where $f(x)$ satisfies the hypothesis of Lagrange's mean value theorem, then geometrically Lagrange's mean value theorem indicates that there exists at least one point c in (a, b) such that the tangent at $(c, f(c))$ to the curve $y = f(x)$ is parallel to the chord AB. (Fig. 10.14)

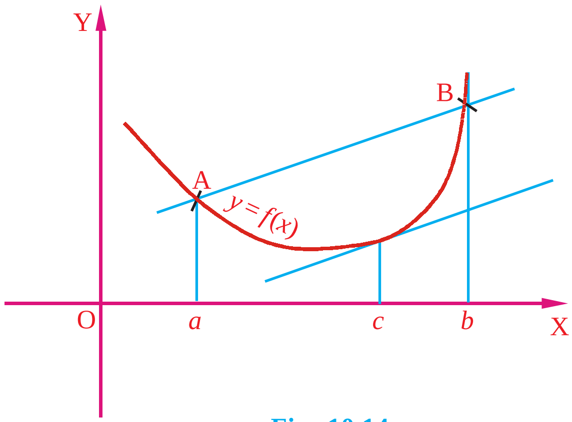


Fig. 10.14

10.7.8 Solved Problem

Problem : *On the curve $y = x^2$, find a point at which the tangent is parallel to the chord joining (0, 0) and (1, 1).*

Solution: The slope of the chord is $\dfrac{1-0}{1-0} = 1.$

The derivative is $\dfrac{dy}{dx} = 2x.$

We want x such that $2x = 1$

i.e., $x = \dfrac{1}{2}.$

We note that $\dfrac{1}{2}$ is in the open interval (0, 1), as required in the Lagrange's mean value theorem.

The corresponding point on the curve is $\left(\dfrac{1}{2}, \dfrac{1}{4}\right).$

Exercise 10(f)

I. 1. Verify Rolle's theorem for the following functions :

(i) $x^2 - 1$ on $[-1, 1]$

(ii) $\sin x - \sin 2x$ on $[0, \pi]$

(iii) $\log (x^2 + 2) - \log 3$ on $[-1, 1]$

2. It is given that Rolle's theorem holds for the function $f(x) = x^3 + bx^2 + ax$ on [1, 3] with $c = 2 + \dfrac{1}{\sqrt{3}}$. Find the values of a and b.

3. Show that there is no real number k, for which the equation $x^2 - 3x + k = 0$ has two distinct roots in [0, 1].

4. Find a point on the graph of the curve $y = (x - 3)^2$, where the tangent is parallel to the chord joining (3, 0) and (4, 1).

5. Find a point on the graph of the curve $y = x^3$, where the tangent is parallel to the chord joining (1, 1) and (3, 27).

6. Find 'c', so that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$ in the following cases :

(i) $f(x) = x^2 - 3x - 1;\ a = \dfrac{-11}{7}, b = \dfrac{13}{7}$ (ii) $f(x) = e^x;\ a = 0,\ b = 1.$

7. Verify the Rolle's theorem for the function $(x^2 - 1)(x - 2)$ on $[-1, 2]$. Find the point in the interval where the derivate vanishes.

8. Verify the conditions of the Lagrange's mean value theorem for the following functions. In each case find a point 'c' in the interval as stated by the theorem.

(i) $x^2 - 1$ on $[2, 3]$ (ii) $\sin x - \sin 2x$ on $[0, \pi]$

(iii) $\log x$ on $[1, 2]$

10.8 Increasing and Decreasing functions

10.8.1 Definitions of increasing and decreasing functions on an interval

Let f be a real function on an interval I. Then f is said to be

(i) an increasing function on I if

$$x_1 < x_2 \quad \Rightarrow \quad f(x_1) \leq f(x_2) \ \forall x_1, x_2 \in \text{I}.$$

(ii) a decreasing function on I if

$$x_1 < x_2 \quad \Rightarrow \quad f(x_1) \geq f(x_2) \ \forall x_1, x_2 \in \text{I}.$$

The graphs of increasing and decreasing functions are shown in Fig. 10.15 and Fig. 10.16.

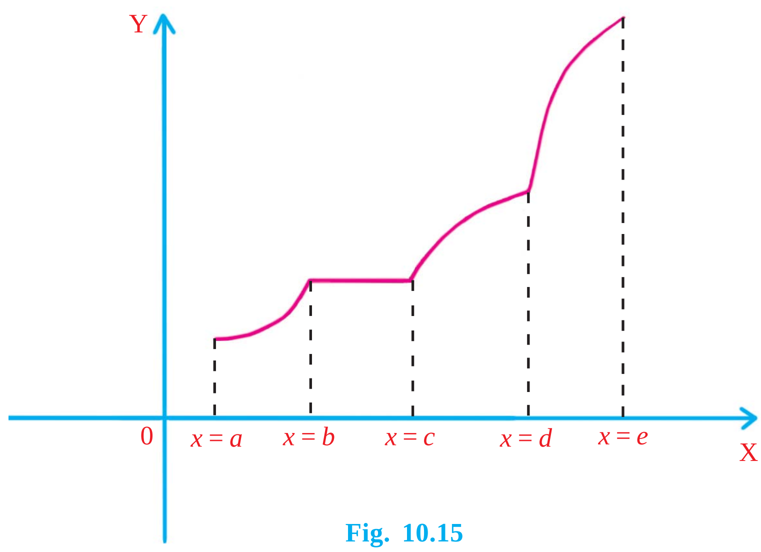


Fig. 10.15

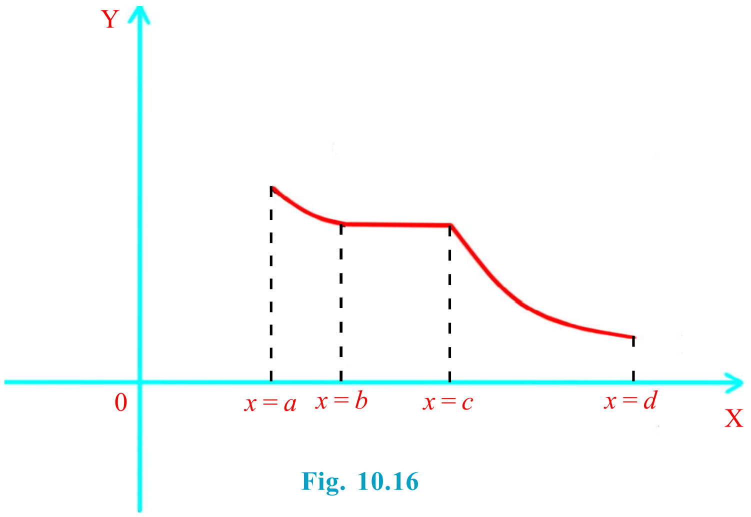


Fig. 10.16

10.8.2 Definitions

Let f be a real function on an interval I. Then f is said to be a

(i) Strictly increasing function on I if $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)\ \forall x_1, x_2 \in I$
(f (x) increases as x increases)

(ii) Strictly decreasing function on I if $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)\ \forall x_1, x_2 \in I$
(f (x) decreases as x increases)

Graphically, a function f is

(i) Strictly increasing on $[a, b]$ if as x moves to right hand side values from $x = a$ to $x = b$ lying in the interval $[a, b]$, its graph moves upwards (Fig. 10.17) .

(ii) Strictly decreasing on $[a, b]$ if as x moves to the right hand side values from $x = a$ to $x = b$ lying in the interval $[a, b]$, its graph moves downwards (Fig. 10.18).

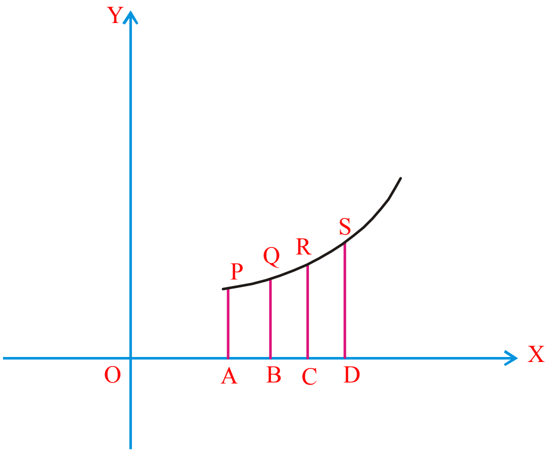


Fig.10.17

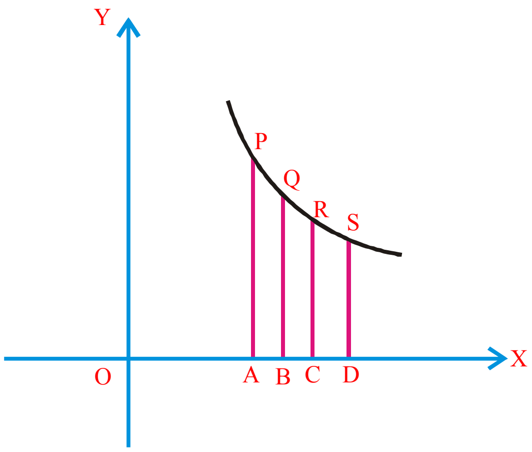


Fig.10.18

10.8.3 Note

The slopes of tangents to the curve $y = f(x)$ at the points lying in $[a, b]$ on which f is strictly increasing are positive (Fig. l0.19).

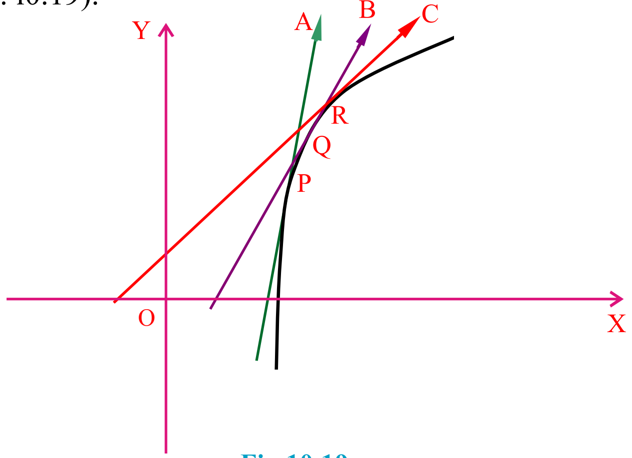


Fig.10.19

10.8.4 Note

The slopes of the tangents to the curve $y = f(x)$ at the points lying in $[a\ b]$ on which f is strictly decreasing are negative (Fig. 10.20).

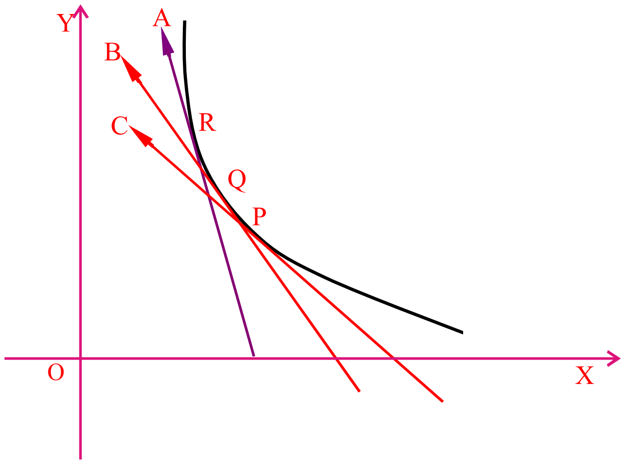


Fig. 10.20

10.8.5 Note

The slope of the tangent to the curve $y = f(x)$ at $x = c$ is zero if $f'(c) = 0$.

10.8.6 Definition

Let f be a real function defined on an interval I. Then f is said to be strictly monotonic on I if it is either strictly increasing on I or strictly decreasing on I.

10.8.7 Solved problems

1. Problem: Show that $f(x) = 8x + 2$ *is a strictly increasing function on* $\boldsymbol{R}$ *without using the graph of* $y = f(x)$.

Solution: Let $x_1, x_2 \in \mathbf{R}$ with $x_1 < x_2$. Then $8x_1 < 8x_2$. Adding 2 to both sides of this inequality, we have $8x_1 + 2 < 8x_2 + 2$. i.e., $f(x_1) < f(x_2)$.

Thus

$$x_1 < x_2 \quad \Rightarrow \quad f(x_1) < f(x_2) \quad \forall x_1, x_2 \in \mathbf{R}.$$

Therefore, the given function f is strictly increasing on $\mathbf{R}$.

2.Problem: *Show that* $f(x) = e^x$ *is strictly increasing on* $\boldsymbol{R}$ (*without graph*).

Solution: Let $x_1, x_2 \in \mathbf{R}$ such that $x_1 < x_2$. We know that if $a > b$ then $e^a > e^b$

$$\therefore \quad x_1 < x_2 \quad \Rightarrow \quad e^{x_1} < e^{x_2}$$

i.e., $f(x_1) < f(x_2)$.

Hence the given function f is a strictly increasing function.

3.Problem : *Show that* $f(x) = -x + 2$ *is strictly decreasing on* $\mathbf{R}$.

Solution: Let $x_1, x_2 \in \mathbf{R}$ and $x_1 < x_2$.

$$\text{Then} \quad x_1 < x_2$$

$$\Rightarrow \quad -x_1 > -x_2$$

$$\Rightarrow \quad -x_1 + 2 > -x_2 + 2$$

$$\Rightarrow \quad f(x_1) > f(x_2).$$

Therefore the given function f is strictly decreasing on $\mathbf{R}$.

10.8.8 Note

If $f(x)$ is strictly increasing (strictly decreasing) then, $-f(x)$ is strictly decreasing (strictly increasing). Similarly, if $f(x)$ is increasing(decreasing) then $-f(x)$ is decreasing (increasing)

10.8.9 Note

If $f(x)$ is strictly increasing (strictly decreasing) then f is also increasing(decreasing).

10.8.10 Note

If $f(x)$ is an increasing (decreasing) function then $f(x)$ need not be strictly increasing (decreasing). For example $f(x) = [x]$ on $\mathbf{R}$ is increasing but it is not strictly increasing.

10.8.11 Note

A function is said to be monotonic on an interval I if it is either an increasing function on I or a decreasing function on I.

10.8.12 Strictly increasing (decreasing) and increasing(decreasing) functions at a point

Definitions

Let f be a real function defined an interval I and $c \in I$.

A function f is said to be strictly increasing (decreasing) and increasing (decreasing) at $x = c$ respectively if $f(x)$ is strictly increasing (decreasing) and increasing (decreasing) in neighbourhood of c.

1. Example: If a real function f is defined on **R** by $f(x) = x^2$ then $f(x)$ is strictly increasing at $x = 3$. Choose $\delta = 0.1$ (this can be usually any positive number so that $3 + \delta$ is close to 3). In this case $(3 - \delta, 3 + \delta)$ becomes (2.9, 3.1). Let

$x_1, x_2 \in (2.9, 3.1)$ and $x_1 < x_2$. Then $x_1^2 < x_2^2$

$$\text{i.e.,} \quad f(x_1) < f(x_2).$$

$\therefore$ f is strictly increasing at $x = 3$.

2. Example. If a real function f is defined on **R** by $f(x) = x^2$ then f is a strictly decreasing function at $x = -2$. Choose $\delta = 0.1$. Let

$x_1, x_2 \in (-2.1, -1.9)$ and $x_1 < x_2$. Then

$$x_1^2 > x_2^2$$

$$\text{i.e.,} \quad f(x_1) > f(x_2).$$

$\therefore$ f is a strictly decreasing function at $x = -2$.

We need not do the work what we did in Example 1 and Example 2 to know the increasing and decreasing nature of a function. There is a tool to know where a function is increasing and decreasing. The tool is the consequence of the following theorem.

10.8.13 Theorem

Let $f(x)$ be a real function defined on $I = (a, b)$ or $[a, b)$ or $(a, b]$ or $[a, b]$. Suppose f is continuous on I and differentiable in (a, b). If

(i) $f'(c) > 0 \ \forall c \in (a, b)$ *then f is strictly increasing on I.*

(ii) $f'(c) < 0 \ \forall c \in (a, b)$ *then f is strictly decreasing on I.*

(iii) $f'(c) \geq 0 \ \forall c \in (a, b)$ *then f is increasing on I.*

(iv) $f'(c) \leq 0 \ \forall c \in (a, b)$ *then f is decreasing on I.*

10.8.14 Remark

Converse of the above theorem is not true. For example, a function may increase on an interval without having a derivative at one or more points of that interval.

Example: Let $f(x) = x^{1/3}$ be defined on $[0, \infty)$. Then $f'(0)$ does not exist. However the function is increasing on any interval $[0, c]$ for any $c > 0$ (Fig. 10.21).

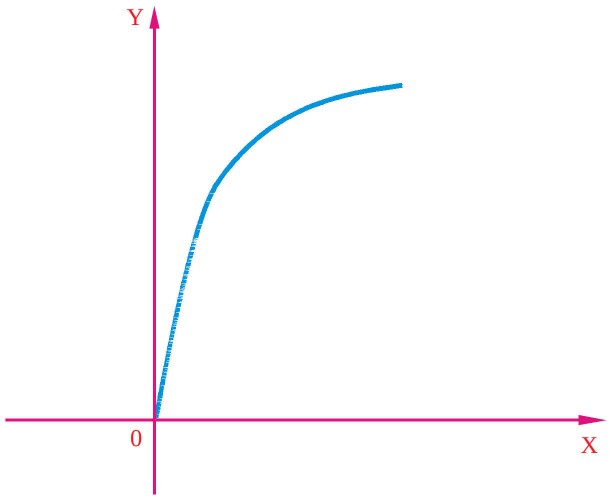


Fig. 10.21

10.8.15 Definition (Critical point)

A point $x = c$ in the domain of the function f is said to be a critical point of the function f if either $f'(c) = 0$ or $f'(c)$ does not exist.

Example: *For the function $f(x) = x^2 - 4x + 6$, $x = 2$ is a critical point.*

10.8.16 Note

The critical point $x = c$ of a function f seperates the increasing and decreasing parts of the function if $f''(c) \neq 0$.

10.8.17 Definition (Stationary point)

A point $x = c$ in the domain of the function is said to be a stationary point of $y = f(x)$ if $f'(c) = 0$.

Example : *The stationary points of $f(x) = \sin x$ are $\pm\pi/2, \pm 3\pi/2, \pm 5\pi/2 \ldots\ldots$.*

10.8.18 Solved problems

1. Problem: *Find the intervals on which $f(x) = x^2 - 3x + 8$ is increasing or decreasing?*

Solution: Given function is $f(x) = x^2 - 3x + 8$. Differentiating it w.r.t. x, we get $f'(x) = 2x - 3$. $f'(x) = 0$ for $x = 3/2$.

Values of x	Sign of $f'(x)$
$x < 3/2$	negative
$x = 3/2$	zero
$x > 3/2$	positive

Since $f'(x) < 0$ in $(-\infty, 3/2)$, the function $f(x)$ is strictly deceasing on $(-\infty, \frac{3}{2})$.

Further since $f'(x) > 0$ in $(\frac{3}{2}, \infty)$, the function $f(x)$ is a strictly increasing function on $(\frac{3}{2}, \infty)$.

2. Problem: *Show that $f(x) = |x|$ is strictly decreasing on $(-\infty, 0)$ and strictly increasing on $(0, \infty)$.*

Solution: The given function is $f(x) = |x|$ i.e.,

$$f(x) = \begin{cases} x \text{ if } x \geq 0 \\ -x \text{ if } x < 0 \end{cases}$$

Thus $f'(c) = 1$ if $c > 0$, $f'(c) = -1$ if $c < 0$.

Since $f'(c) > 0$ on $(0, \infty)$, the function $f(x)$ is strictly increasing on $(0, \infty)$. Since $f'(c) < 0$ on $(-\infty, 0)$, the function $f(x)$ is strictly decreasing on $(-\infty, 0)$.

3. Problem: *Find the intervals on which the function $f(x) = x^3 + 5x^2 - 8x + 1$ is a strictly increasing function.*

Solution : Given that $f(x) = x^3 + 5x^2 - 8x + 1$.

$$\therefore \quad f'(x) = 3x^2 + 10x - 8 = (3x - 2)(x + 4) = 3\left(x - \frac{2}{3}\right)(x - (-4)).$$

$f'(x)$ is negative in $(-4, \frac{2}{3})$ and positive in $(-\infty, -4) \cup (\frac{2}{3}, \infty)$.

$\therefore$ The function is strictly deceasing in $(-4, \frac{2}{3})$ and is strictly decreasing in

$(-\infty, -4)$ and $(\frac{2}{3}, \infty)$.

4. Problem: *Find the intervals on which $f(x) = x^x$ $(x > 0)$ is increasing and decreasing.*

Solution: Taking logarithms on both sides of $f(x) = x^x$, we get

$\log(f(x)) = x \log x$. Differentiating it w.r.t. x we have

$$\frac{1}{f(x)} f'(x) = 1 + \log x$$

$$\therefore \quad f'(x) = x^x(1 + \log x) \qquad \text{.... (1)}$$

$$f'(x) = 0 \quad \Rightarrow x^x(1 + \log x) = 0$$

$$\Rightarrow 1 + \log x = 0$$

$$\Rightarrow x = 1/e$$

Interval	Sign of $f'(x)$
$(0, 1/e)$	negatvie
$(1/e, \infty)$	positive

Suppose $x < 1/e$ then $\log x < \log(1/e)$ (since the base $e > 1$). i.e., $\log x < -1$

$1 + \log x < 0 \Rightarrow x^x(1 + \log x) < 0$. i.e, $f'(x) < 0$.

Now suppose $x > 1/e$. Then $\log x > \log(1/e)$ i.e., $\log x > -1$

$$\Rightarrow 1 + \log x > 0$$

$$\Rightarrow x^x(1 + \log x) > 0$$

$$\Rightarrow f'(x) > 0.$$

Hence, f is strictly decreasing on $(0, 1/e)$ and it is strictly increasing on $(1/e, \infty)$.

5. Problem: *Determine the intervals in which $f(x) = \frac{2}{(x-1)} + 18x \ \forall x \in \mathbf{R} \setminus \{0\}$ is strictly increasing and decreasing.*

Solution: Given that $f(x) = \frac{2}{(x-1)} + 18x$. Differentiating it w.r.t. x, we get

$$f'(x) = \frac{-1}{(x-1)^2}.2 + 18 \text{ and } f'(x) = 0 \quad \Rightarrow \frac{2}{(x-1)^2} = 18 \quad \Rightarrow (x-1)^2 = 1/9.$$

$\therefore \quad f'(x) = 0$ if $x - 1 = 1/3$ or $x - 1 = -(1/3)$.

i.e., $x = 4/3$ or $x = 2/3$.

The derivative of $f(x)$ can be expressed as

$$f'(x) = \frac{18}{(x-1)^2} \cdot (x - 2/3)(x - 4/3)$$

Intervals	Sign of $f'(x)$
$(-\infty, \frac{2}{3})$	positive
$(\frac{2}{3}, \frac{4}{3})$	negative
$(\frac{4}{3}, \infty)$	positive

$\therefore$ The given function $f(x)$ is strictly increasing on $(-\infty, \frac{2}{3})$ and $(\frac{4}{3}, \infty)$ and it is strictly decreasing on $(\frac{2}{3}, \frac{4}{3})$.

6. Problem: *Let $f(x) = \sin x - \cos x$ be defined on $[0, 2\pi]$. Determine the intervals in which $f(x)$ is strictly decreasing and strictly increasing.*

Solution: Given that $f(x) = \sin x - \cos x$.

$\therefore \quad f'(x) = \cos x + \sin x$

$\therefore \quad f'(x) = \sqrt{2} \, . \sin(x + \pi/4)$

Let $0 < x < 3\pi/4$. Then $\pi/4 < x + \pi/4 < \pi$.

$\therefore \quad \sin(x + \pi/4) > 0$ i.e., $f'(x) > 0$.

Similarly it can be shown that $f'(x) < 0$ in $(3\pi/4, 7\pi/4)$ and

$f'(x) > 0$ in $(7\pi/4, 2\pi)$.

Intervals	Sign of $f'(x)$
$\left(0, \frac{3\pi}{4}\right)$	positive
$\left(\frac{3\pi}{4}, \frac{7\pi}{4}\right)$	negative
$\left(\frac{7\pi}{4}, 2\pi\right)$	positive

Thus the function $f(x)$ strictly increasing in $\left(0, \frac{3\pi}{4}\right)$ and $\left(\frac{7\pi}{4}, 2\pi\right)$ and

it is strictly decreasing in $\left(\frac{3\pi}{4}, \frac{7\pi}{4}\right)$.

7. Problem : *If* $0 \le x \le \frac{\pi}{2}$ *then show that* $x \ge \sin x$.

Solution: Let $f(x) = x - \sin x$. Then $f'(x) = 1 - \cos x \ge 0 \ \forall x$

$\therefore$ f is an increasing function for all x.

Now, $f(0) = 0$. Hence $f(x) \ge f(0)$ for all $x \in [0, \frac{\pi}{2}]$. Therefore, $x \ge \sin x$.

Exercise 10(g)

I. 1. Without using the derivative, show that

(i) the function $f(x) = 3x + 7$ is strictly increasing on **R**

(ii) the function $f(x) = \left(\frac{1}{2}\right)^x$ is strictly decreasing on **R**

(iii) the function $f(x) = e^{3x}$ is strictly increasing on **R**

(iv) the function $f(x) = 5 - 7x$ is strictly decreasing on **R**

2. Show that the function $f(x) = \sin x$ defined on **R** is neither increasing nor decreasing on $(0, \pi)$.

II. 1. Find the intervals in which the following functions are strictly increasing or strictly decreasing.

(i) $x^2 + 2x - 5$ (ii) $6 - 9x - x^2$

(iii) $(x + 1)^3 (x - 1)^3$ (iv) $x^3(x - 2)^2$

(v) xe^x (vi) $\sqrt{(25 - 4x^2)}$

(vii) $\ln(\ln(x)), x > 1$ (viii) $x^3 + 3x^2 - 6x + 12$

2. Show that $f(x) = \cos^2 x$ is strictly increasing on $(0, \pi/2)$.

3. Show that $x + \frac{1}{x}$ is increasing on $[1, \infty)$.

4. Show that $\frac{x}{1+x} < ln(1 + x) < x \ \forall x > 0$.

III. 1. Show that $\frac{x}{1+x^2} < \tan^{-1} x < x$ when $x > 0$.

2. Show that $\tan x > x$ for all $x \in [0, \pi/2)$.

3. If $x \in (0, \pi/2)$ then show that $2x/\pi < \sin x < x$.

4. If $x \in (0, 1)$ then show that $2x < ln\left(\frac{(1+x)}{(x-1)}\right) < 2x\left(1 + \frac{x^2}{2(1-x^2)}\right)$.

5. At what points the slopes of the tangents $y = \frac{x^3}{6} - \frac{3x^2}{2} + \frac{11x}{2} + 12$ increases?

6. Show that the functions $ln\frac{(1+x)}{x}$ and $\frac{x}{(1+x)ln(1+x)}$ are decreasing on $(0, \infty)$.

7. Find the intervals in which the function $f(x) = x^3 - 3x^2 + 4$ is strictly increasing for all $x \in \mathbf{R}$.

8. Find the intervals in which the function $f(x) = \sin^4 x + \cos^4 x \ \forall x \in [0, \pi/2]$ is increasing and decreasing.

10.9 Maxima and Minima

10.9.1 Definition (Global maximum)

Let D be an interval in $\mathbf{R}$ and $f : D \to \mathbf{R}$ be a real function and $c \in D$. Then f is said to have a global maximum on D if $f(c) \geq f(x) \ \forall x \in D$.

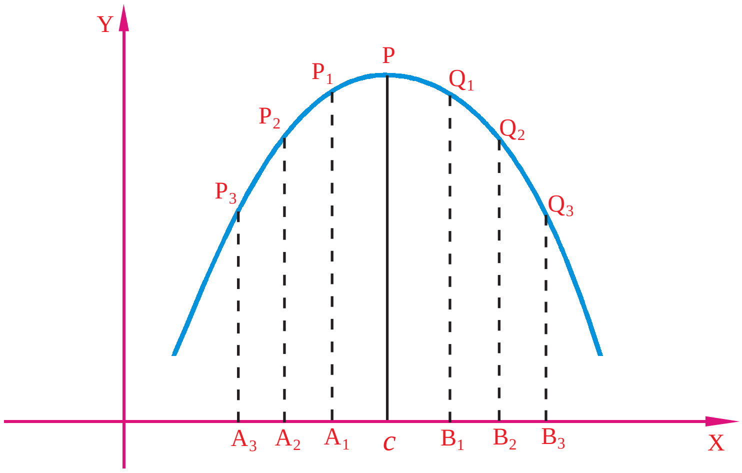


Fig. 10.22

Here, $f(c)$ is called global maximum value of $f(x)$ at $x = c$ and the point $x = c$ is called a point of global maximum.

10.9.2 Note

The global maximum value is also called *absolute maximum value* or *greatest value*.

Example: Let $f : [0, 2] \to \mathbf{R}$ defined by $f(x) = x^2$ be a function. For each $0 \leq x \leq 2$, we have $0 \leq x^2 \leq 4$. Thus,

$$f(x) \leq 4 \qquad \forall \ x \in [0, 2]$$

i.e., $$f(x) \leq f(2) \qquad \forall \ x \in [0, 2].$$

$\therefore f(2) = 4$ is global maximum value of f and $x = 2$ is the point of global maximum.

10.9.3 Definition (Global minimum)

*Let D be an interval in **R** and $f : D \rightarrow R$ be a real function and $c \in D$. Then f is said to have a global minimum on D if $f(x) \geq f(c) \quad \forall x \in D$.*

In this case $f(c)$ is called global minimum value of $f(x)$ at $x = c$ on D and the point $x = c$ is called a point of global minimum (Fig. 10.23).

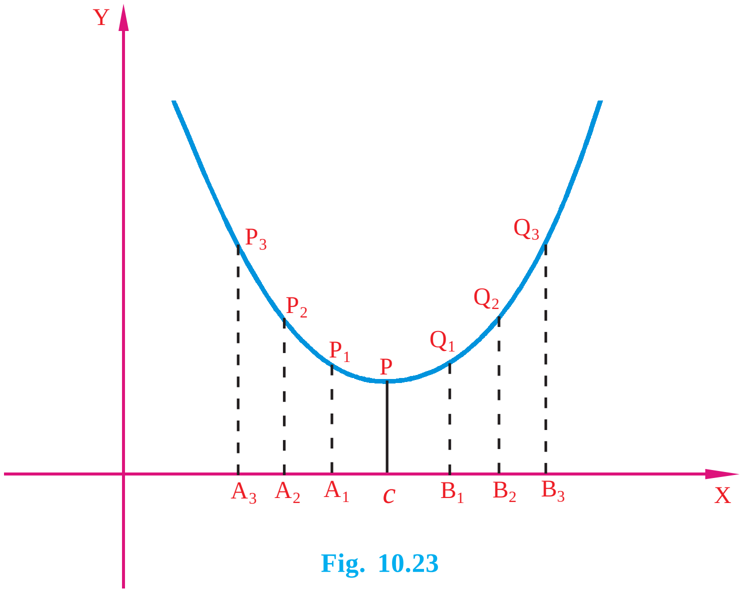


Fig. 10.23

10.9.4 Note : The global minimum value is also called ***absolute minimum*** or ***least value***.

Example: Let $f : [2, 6] \rightarrow \mathbf{R}$ be defined by $f(x) = x^3$. Then

$$f(x) \geq 8 \qquad \forall\ x \in [2, 6].$$

i.e., $$f(x) \geq f(2) \quad \forall\ x \in [2, 6]$$

$\therefore f(2) = 8$ is global minimum value of the function f and $x = 2$ is the point of global minimum.

10.9.5 Definition (Relative maximum)

*Let D be an interval in **R** and $f : D \rightarrow \mathbf{R}$ be a real function, $c \in D$. Then f is said to have a relative maximum at c if $\exists\, \delta > 0$ such that $f(x) \leq f(c) \quad \forall\, x \in (c - \delta, c + \delta)$.*

Here, $f(c)$ is called relative maximum value of $f(x)$ at $x = c$ and the point $x = c$ is called a point of relative maximum.

10.9.6 Note : Relative maximum is also called as *local maximum*.

10.9.7 Note: Relative maximum of a function at $x = c$ does not exist if f is not defined in a neighbourhood of c.

10.9.8 Example: Let $f : [0, 2\pi] \to \mathbf{R}$ be defined by $f(x) = \sin x$.

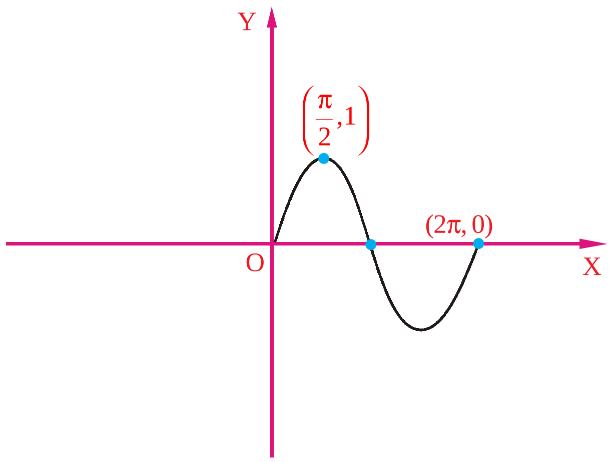


Fig. 10.24

The domain of f is $[0, 2\pi]$. For $x = \frac{\pi}{2}$ (Fig. 10.24), choose $\delta = 0.1$. Then $f(x)$ is defined in $\left(\frac{\pi}{2} - 0.1, \frac{\pi}{2} + 0.1\right)$. Now $f\left(\frac{\pi}{2}\right) \geq f(x) \,\forall\, x \in \left(\frac{\pi}{2} - 0.1, \frac{\pi}{2} + 0.1\right)$.

$\therefore f\left(\frac{\pi}{2}\right)$ is local maximum value of f.

Note that it is also a global maximum on $[0, 2\pi]$.

10.9.9 Definition (Relative minimum)

Let D be an interval in $\mathbf{R}$ and $f : D \to \mathbf{R}$ be a real function, $c \in D$. Then f is said to have a relative minimum at $x = c$ if $\exists\, \delta > 0$ such that $f(x) \geq f(c)$ $\forall\, x \in (c - \delta, c + \delta)$.

Here, $f(c)$ is called relative minimum value of $f(x)$ at $x = c$ and the point $x = c$ is called point of relative minimum.

10.9.10 Note : Relative minimum is also called as ***local minimum***.

10.9.11 Note : Relative minimum of a function f at $x = c$ is defined only when f is defined in some neighbourhood of $x = c$.

10.9.12 Note : If the domain of f is $[a, b]$, then there is no relative maximum and relative minimum at $x = a$ and $x = b$.

Example: Let $f : [0,2\pi] \to \mathbf{R}$, be defined by $f(x) = \cos x$. The function $f(x) = \cos x$ is defined in the neighbourhood of π say $(\pi - 0.2, \pi + 0.2)$ (Fig.10.25).

Now $f(\pi) \leq f(x) \ \forall \ x \in (\pi - 0.2, \pi + 0.2)$

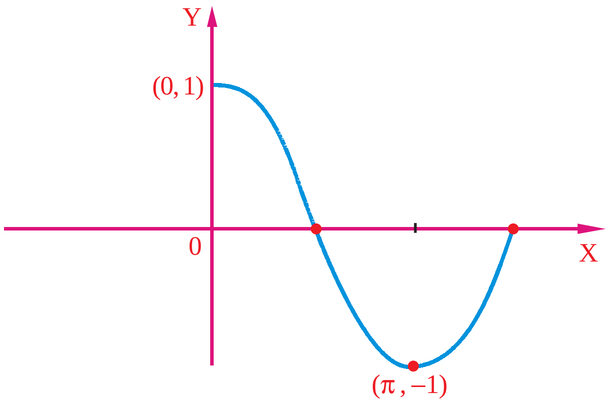


Fig. 10.25

$\therefore \quad f(\pi) = -1$ is relative minimum of f at $x = \pi$.

For the given function it is also the global minimum in $[0, 2\pi]$.

10.9.13 Global extreme value and Local extreme value

Definition

Global minimum or global maximum value at $x = c$ is defined as global extreme value at $x = c$.

Global extreme value is also called global extremum.

Definition (Relative extreme value)

Relative local extreme value of $f(x)$ at $x = c$ is either a relative maximum value at $x = c$ or relative local minimum value at $x = c$.

10.9.14 Solved Problems

1. Problem: *Let $f : \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = 4x^2 - 4x + 11$. Find the global minimum value and a point of global minimum.*

Solution: We have to look for a value $c \in \mathbf{R}$(domain) such that

$$f(x) \geq f(c) \ \forall \ x \in \mathbf{R}$$

so that $f(c)$ is the global minimum value of f. Consider

$$f(x) = 4x^2 - 4x + 11 = (2x - 1)^2 + 10 \geq 0 \ \forall \ x \in \mathbf{R} \quad \text{.... (1)}$$

Now, $f(1/2) = 10$ .... (2)

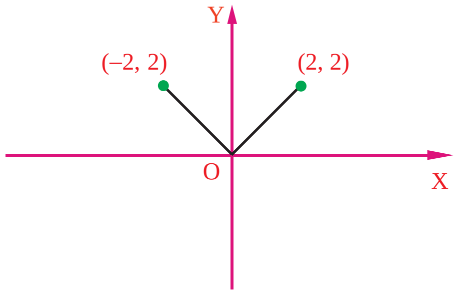
Also $f(x) \geq f(1/2) \ \forall \ x \in \mathbf{R}$.

Hence, $f(1/2) = 10$ is the global minimum value of $f(x)$,

and a point of global minimum is $x = 1/2$.

2. Problem: *Let $f : [-2, 2] \to \mathbf{R}$ be defined by $f(x) = |x|$. Find the global maximum of $f(x)$ and a point of global maximum.*

Solution:



We know that $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$.

Therefore, from the graph of the function

f on $[-2, 2]$ (Fig.10.26)

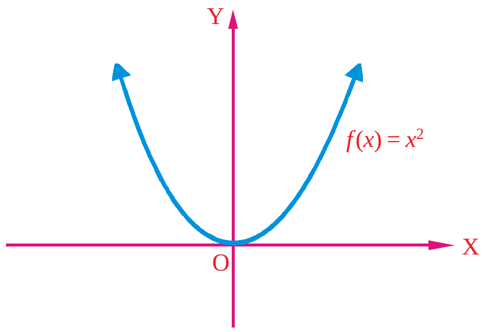
clearly $f(x) \leq f(2)$ and $f(x) \leq f(-2) \ \forall \ x \in [-2, 2]$.

Fig.10.26

$\therefore$ $f(2) = f(-2) = 2$ is the global maximum of $f(x)$, 2 and -2 are the points of global maximum.

3. Problem : *Find the global maximum and global minimum of the function $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = x^2$.*

Solution:



Graph of $f(x) = x^2$

Fig. 10.27

We have $f(x) \geq f(0) \ \forall \ x \in \mathbf{R}$.

Hence the global minimum value of $f(x)$ is 0 and a point of global minimum is $x = 0$.

Suppose f has global maximum at $x_0 \in \mathbf{R}$ $(x_0 > 0)$. Then as per our assumption we have

$$f(x_0) \geq f(x) \quad \forall \ x \in \mathbf{R} \qquad \text{... (3)}$$

Choose $x_1 = x_0 + 1$. Then $x_1 \in \mathbf{R}$ and $x_0 < x_1$

$$\therefore \ x_0^2 < x_1^2 .$$

Hence , $f(x_0) < f(x_1)$.

Thus we got $f(x_1)$ such that $f(x_1) > f(x_0)$ which is a contradiction to (3).

Therefore, $f(x)$ has no global maximum on **R**.

10.9.15 First derivative test

In the present and forthcoming sections we learn to find the relative extremum and global extremum of a function. These can be found by applying the first derivative test and second derivative test. We state the former in the present section and the latter in the next section.

10.9.16 Theorem : If $y = f(x)$ is a differentiable function on (a, b) and $c \in (a, b)$ is a point of local maximum or local minimum then $f'(c) = 0$.

10.9.17 Note : The above theorem states that if $f(x)$ has local maximum or local minimum at $x = c$ then $f'(c) = 0$. If $f'(x) = 0$ at some point of x in (a, b) then f need not have a local maximum or local minimum at x i.e., the converse of the theorem is not true.

10.9.18 Note :

If $f'(c) = 0$ then do not jump into the conclusion that the point $x = c$ is either a point of local maximum or local minimum. For example, for the function $f(x) = x^3$ defined on **R**, we have $f'(x) = 3x^2$ and $f(x) = 0$ at $x = 0$. Since $f(x) < f(0)$ if $x < 0$ and $f(0) < f(x)$ if $x > 0$, f does not have local minimum or local maximum at $x = 0$. (Fig.10.29)

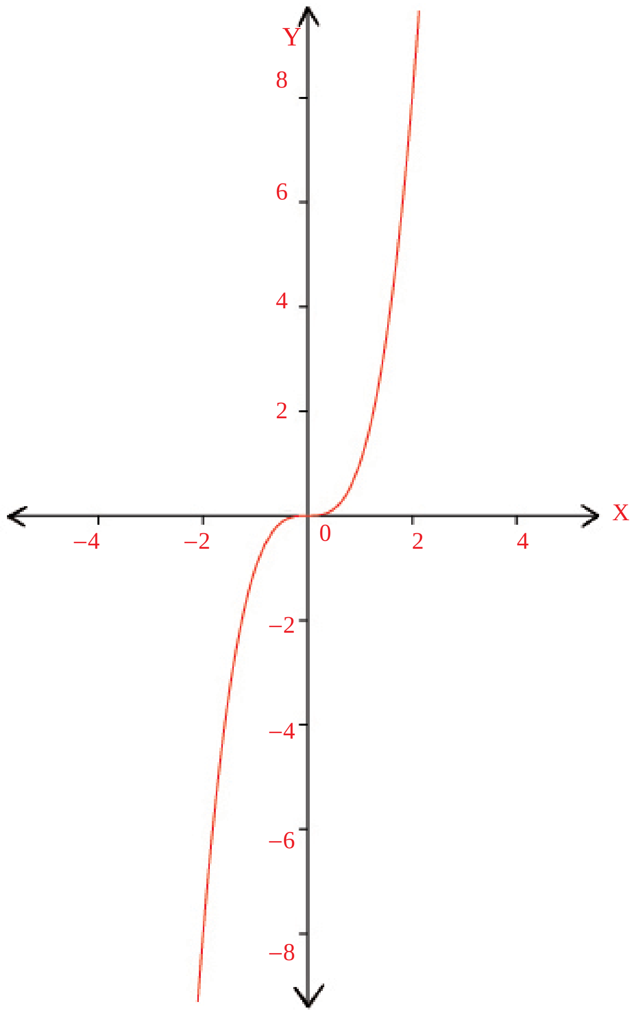


Fig. l0.29

The above theorem helps us to locate the possible points of local maxima or local minima for a function. We can decide whether each of these points is

(i) a local maximum or a local minimum.

(ii) not a local maximum or not a local minimum.

by further investigation. The investigation is done through the first derivative test which is going to be given in this section shortly.

A function f is said to have change of sign at $x = c$

(i) from positive to negative as x goes through the number c if

$\exists\delta > 0$ such that $f(x) > 0$ in $(c - \delta, c)$ and $f(x) < 0$ in $(c, c + \delta)$.

(ii) from negative to positive as x goes through the number c if

$\exists\delta > 0$ such that $f(x) < 0$ in $(c - \delta, c)$ and $f(x) > 0$ in $(c, c + \delta)$.

For example, the function $f(x) = \cos x$ changes sign at $x = \pi/2$ from positive to negative. Because $\cos(\pi/2 - \delta) > 0$ and $\cos(\pi/2 + \delta) < 0$ for some value of $\delta > 0$. In particular if we take $\delta = \left(\frac{1}{10}\right)$ then $\cos\left(\frac{\pi}{2} - \frac{1}{10}\right) > 0$ and $\cos\left(\frac{\pi}{2} + \frac{1}{10}\right) < 0$.

10.9.19 Note : A function f does not have change of sign at $x = c$ means f neither changes sign from positive to negative nor changes sign from negative to positive.

We recall the definitions of stationary and critical points of a function. A point $x = c$ in the domain of the function f is said to be a critical point of f if $f'(c) = 0$ or f is not differentiable at $x = c$. A point $x = c$ of the domain of the function f is said to be a stationary point of f if $f'(c) = 0$.

If c is a stationary point of f then $f(c)$ is called stationary value of f at $x = c$. A stationary point is a critical point but a critical point need not be a stationary point.

10.9.20 Definition (Turning point)

A point $x = c$ of the domain of the function f is said to be a turning point of f if $f'(x)$ changes sign at $x = c$ i.e., positive to negative or negative to positive when the values move from left to the right of c.

10.9.21 Note : A turning point of a function f is a point of either ***relative maximum*** or ***relative minimum***.

1. Example: Let $f(x) = x^2$ on $[-2, 2]$ has a turning point at $x = 0$, since $f'(x)$ changes sign at 0.

2. Example: Let $f(x) = |x|$ on $[-1, 1]$ has a critical point at $x = 0$, since $f'(0)$ does not exist.

l0.9.22 Theorem: (First derivative test)

Let f be a differentiable function on an interval D, $c \in$ D and f is defined in some neighbourhood of c. Suppose c is a stationary point of f such that $(c - \delta, c + \delta)$ does not contain any other stationary point for some $\delta > 0$. Then

(i) c is a point of local maximum if $f'(x)$ changes sign from positive to negative at $x = c$.

(ii) c is a point of local minimum if $f'(x)$ changes sign from negative to positive at $x = c$.

(iii) c is neither a point of local maximum nor a point of local minimum if $f'(x)$ does not change sign at $x = c$.

10.9.23 Solved problems

1. Problem: *Find the stationary points of $f(x) = 3x^4 - 4x^3 + 1, \forall x \in \mathbf{R}$ and state whether the function has local maxima or local minima at those points.*

Solution: Given that $f(x) = 3x^4 - 4x^3 + 1$ and the domain of f is $\mathbf{R}$. Differentiating the function w.r.t. x we have

$$f'(x) = 12x^2(x - 1) \qquad \text{... (1)}$$

The stationary points are the roots of $f'(x) = 0$ i.e., $12x^2(x - 1) = 0$. Hence $x = 0$ and $x = 1$ are the stationary points. Now, we test whether the stationary point $x = 1$ is a local extreme point or not. For,

$$f'(0.9) = 12(0.9)^2 \ (0.9 - 1) \Rightarrow f'(0.9) \text{ is negative,}$$

$$f'(1.1) = 12(1.1)^2 \ (1.1 - 1) \Rightarrow f'(1.1) \text{ is positive,}$$

and $f(x)$ is defined in the neighbourhood i.e.,(0.8, 1.2) of $x = 1$ with $\delta = 0.2$.

Therefore by Theorem 10.9.22 the given function has local(relative) minimum at $x = 1$. Hence $x = 1$ is a local extreme point.

We will now test whether $x = 0$ is a local extreme point or not.

The function $f(x)$ is defined in the neighbourhood of $(-0.2, 0.2)$.

$$f'(-0.1) = 12(-0.1)^2 \ (-0.1 - 1) \Rightarrow f'(-0.1) \text{ is negative}$$

$$f'(0.1) = 12(0.1)^2 \ (0.1 - 1) \Rightarrow f'(0.1) \text{ is negative.}$$

Thus, $f(x)$ has no change in sign at $x = 0$. Therefore, the function f has no local maximum and no local minimum. Hence, $x = 0$ is not a local extreme point.

2. Problem : *Find the points (if any) of local maxima and local minima of the function $f(x) = x^3 - 6x^2 + 12x - 8 \ \forall x \in \mathbf{R}$.*

Solution: Given function is $f(x) = x^3 - 6x^2 + 12x - 8$ and the domain of f is $\mathbf{R}$.

Differentiating the given function w.r.t. x ,we get

$$f'(x) = 3x^2 - 12x + 12 \qquad \text{i.e.,} \quad f'(x) = 3(x - 2)^2.$$

The stationary point of $f(x)$ is $x = 2$, since 2 is a root of $f'(x) = 0$.

Choose $\delta = 0.2$. The 0.2- neighbourhood of 2 is (1.8, 2.2). Now

$f'(1.9) = 3(1.9 - 2)^2 \Rightarrow f'(1.9)$ is positive

$f'(2.1) = 3(2.1 - 2)^2 \Rightarrow f'(2.1)$ is positive.

Thus $f'(x)$ does not change the sign at $x = 2$.

$\therefore$ By Theorem 10.9.22, $x = 2$ is neither a local maximum nor a local minimum.

3. Problem: *Find the points of local minimum and local maximum of the function* $f(x) = \sin 2x \;\forall\; x \in [0, 2\pi]$.

Solution: The given function is $f(x) = \sin 2x$ and domain is $[0, 2\pi]$.

$$f'(x) = 2\cos 2x \qquad \text{.... (1)}$$

The critical points are the roots of $2\cos 2x = 0$ and lying in the domain $[0, 2\pi]$.

They are $\frac{\pi}{4}$ and $\frac{3\pi}{4}$.

Now we apply the first derivative test at $x = \frac{\pi}{4}$.

Clearly $\left(\frac{\pi}{4} - 0.1, \frac{\pi}{4} + 0.1\right)$ is a neighbourhood of $\frac{\pi}{4}$ and the given f is defined on it. Now

$$f'\left(\frac{\pi}{4} - 0.05\right) = 2\cos\left(\frac{\pi}{2} - 0.1\right) > 0$$

$$f'\left(\frac{\pi}{4} + 0.05\right) = 2\cos\left(\frac{\pi}{2} + 0.1\right) < 0.$$

Thus $f'(x)$ changes sign from positive to negative at $x = \frac{\pi}{4}$. Therefore f has a local maximum.

Now we apply the first derivative test at $x = \frac{3\pi}{4}$.

Clearly $\left(\frac{3\pi}{4} - 0.1, \frac{3\pi}{4} + 0.1\right)$ is a neighbourhood of $\frac{3\pi}{4}$ and the given f is defined on it.

Now

$$f'\left(\frac{3\pi}{4} - 0.05\right) = 2\cos\left(\frac{3\pi}{4} - 0.1\right) < 0.$$

$$f'\left(\frac{3\pi}{4} + 0.05\right) = 2\cos\left(\frac{3\pi}{4} + 0.1\right) > 0.$$

Thus $f'(x)$ changes sign from negative to positive at $x = \frac{3\pi}{4}$. Therefore f has a local minimum at $x = \frac{3\pi}{4}$.

10.9.24 Second derivative test

Now we learn another test known as second derivative test to find the local extrema and the points of local extrema of a function.

10.9.25 Theorem

Let f be a function defined on an open interval I and $c \in$ I. Let f be twice differentiable at c. Then

(i) $x = c$ is a point of local maximum of f if $f'(c) = 0$ and $f''(c) < 0$, and local maximum value of f is $f(c)$.

(ii) $x = c$ is a point of local minimum of f if $f'(c) = 0$ and $f''(c) > 0$ and $f(c)$ is local minimum value at $x = c$.

(iii) the test fails if $f'(c) = 0$ and $f''(c) = 0$ (further investigation is needed).

10.9.26 Note

When $f''(x) = 0$ and $f'(x) = 0$ at $x = c$ then apply first derivative test to verify whether $x = c$ is point of local extremum.

10.9.27 Solved problems

1. Problem: *Find the points of local extrema of the function $f(x) = x^3 - 9x^2 - 48x + 6$ $\forall x \in \boldsymbol{R}$. Also find its local extrema.*

Solution : Given function is

$$f(x) = x^3 - 9x^2 - 48x + 6 \qquad \text{... (1)}$$

and the domain of the function is **R**.

Differentiating (1) w.r.t. x we get

$$f'(x) = 3x^2 - 18x - 48 = 3(x - 8)(x + 2). \qquad \text{... (2)}$$

Thus the stationary points are -2 and 8.

Differentiating (2) w.r.t. x we get,

$$f''(x) = 6(x - 3) \qquad \text{... (3)}$$

Let $x_1 = -2$ and $x_2 = 8$. Now we have to find f'' at each of these points to know the sign of second derivative.

At $x_1 = -2$, $f''(-2) = -30$. The sign of it is negative.

$\therefore$ $x_1 = -2$ is a point of local maximum of f and its local maximum value is $f(-2) = 58$.

Now, at $x_2 = 8$, $f''(8) = 30$. Thus the sign of $f''(x_2)$ is positive. Therefore,

$x_2 = 8$ is a point of local minimum of f and its local minimum value is

$$f(8) = -442.$$

2. Problem: *Find the points of local extrema of* $f(x) = x^6$ $\forall$ $x \in \boldsymbol{R}$. *Also find its local extrema.*

Solution:
$$f(x) = x^6 \quad \text{.... (1)}$$

Differentiating (1) w.r.t. x we get

$$f'(x) = 6x^5 \quad \text{.... (2)}$$

and again differentiating (2) w.r.t. x we get,

$$f''(x) = 30x^4 \quad \text{.... (3)}$$

The stationary point of $f(x)$ is $x = 0$ only (since $f'(x) = 0$ only at $x = 0$).

Now $f''(0) = 0$. At $x = 0$, we can not conclude anything about the local extrema by the second derivative test. Therefore, we apply the first derivative test. As the domain of f is $\mathbf{R}$, the function f is defined on $(-0.2, 0.2)$ which is a neighbourhood of $x = 0$. Now

$f'(-0.1) = 6(-0.1)5 < 0$, $f'(0.1) = 6(0.1)5 > 0$.

Thus $f'(x)$ changes sign from negative to positive at $x = 0$.

$\therefore$ $x = 0$ is a point of local minimum and its local minimum value is $f(0) = 0$.

3. Problem: *Find the points of local extrema and local extrema for the function*

$$f(x) = \cos 4x \textit{ defined on } \left(0, \frac{\pi}{2}\right).$$

Solution: Here

$$f(x) = \cos 4x \quad \text{.... (1)}$$

and its domain is $\left(0, \frac{\pi}{2}\right)$.

$$\therefore \quad f'(x) = -4 \sin 4x \quad \text{.... (2)}$$

and

$$f''(x) = -16 \cos 4x \quad \text{.... (3)}$$

The stationary points are the roots of $f'(x) = 0$ and lying in the domain $\left(0, \frac{\pi}{2}\right)$.

$$f'(x) = 0 \Rightarrow -4 \sin 4x = 0$$
$$\Rightarrow 4x = 0, \pi, 2\pi, 3\pi, 4\pi \;\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\Rightarrow x = 0, \pi/4, \pi/2, 3\pi/4, \pi \;\ldots\ldots\ldots\ldots\ldots\ldots$$

The point lying in the domain is $x = \frac{\pi}{4}$ only. Thus $x = \frac{\pi}{4}$ is the stationary point of the given function. Now

$$f''(\pi/4) = -16\cos(\pi) = 16 > 0.$$

$\therefore$ The function f has local minimum at $x = \frac{\pi}{4}$ and its local minimum value is

$$f\left(\frac{\pi}{4}\right) = -1.$$

10.9.28 Applications of maxima and minima

Applications of maxima and minima can be found in Engineering, Economics, Commerce, Business management and in Pure Mathematics. Constructing the rectangle with largest area and having its perimeter constant, finding the maximum profit for a given profit function, the shape of steel beam to have the maximum strength and design of pipe to reduce the cost to supply gas.

10.9.29 Solved problems

1. Problem: *Find two positive numbers whose sum is 15 so that the sum of their squares is minimum.*

Solution: Suppose one numbers is x and the other number $15 - x$. Let S be the sum of squares of these numbers. Then

$$S = x^2 + (15 - x)^2 \qquad \ldots\ldots (1)$$

Note that the quantity S, to be minimized, is a function of x.

Differentiating (1) w.r.t. x, we get

$$\frac{dS}{dx} = 2x + 2(15 - x)(-1)$$
$$= 4x - 30 \qquad \ldots\ldots (2)$$

and again differentiating (2) w.r.t. x, we get

$$\frac{d^2S}{dx^2} = 4 \qquad \ldots\ldots (3)$$

The stationary point can be obtained by solving $\frac{dS}{dx} = 0$ i.e., $4x - 30 = 0$.

$\therefore\ x = 15/2$ is the stationary point of (l).

Since $\frac{d^2S}{dx^2} = 4 > 0$, S is minimum at $x = \frac{15}{2}$.

$\therefore$ The two numbers are $\frac{15}{2}$, $15 - \frac{15}{2}$ i.e., $\frac{15}{2}$ and $\frac{15}{2}$.

2. Problem: *Find the maximum area of the rectangle that can be formed with fixed perimeter 20.*

Solution: Let x and y denote the length and the breadth of a rectangle respectively. Given that the perimeter of the rectangle is 20.

i.e., $2(x + y) = 20$

i.e., $x + y = 10$ ..... (1)

Let A denote the area of rectangle. Then

$$A = x\,y \quad \text{.... (2)}$$

which is to be minimized. Equation (1) can be expressed as

$$y = 10 - x \quad \text{.... (3)}$$

From (3) and (2), we have

$$A = x\,(10 - x)$$

$$A = 10x - x^2 \quad \text{.... (4)}$$

Differentiating (4) w.r.t. x we get

$$\frac{dA}{dx} = 10 - 2x \quad \text{.... (5)}$$

The stationary point is a root of $10 - 2x = 0$.

$\therefore\ x = 5$ is the stationary point.

Differentiating (5) w.r.t. x, we get

$$\frac{d^2A}{dx^2} = -2$$

which is negative. Therefore by second derivative test the area A is maximized at $x = 5$ and hence $y = 10 - 5 = 5$, and the maximum area is $A = 5(5) = 25$.

3. Problem: *Find the point on the graph $y^2 = x$ which is the nearest to the point (4, 0).*

Solution:

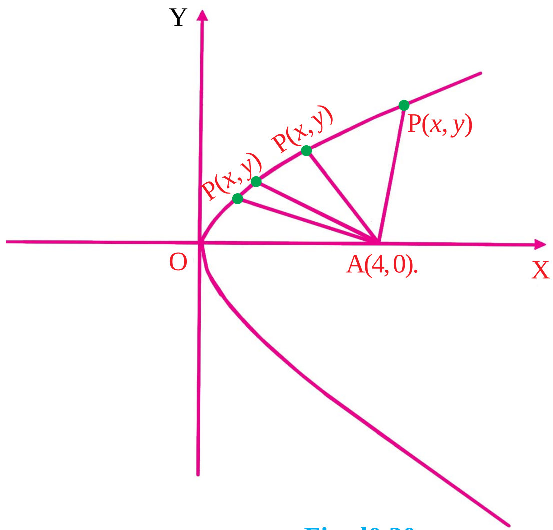


Fig. 10.30

Let P(x, y) be any point on $y^2 = x$ and A(4, 0). We have to find P such that PA is minimum (Fig. 10.30).

Suppose PA =D. The quantity to be minimized is D.

$$D = \left(\sqrt{(x-4)^2 + (y-0)^2}\right) \quad .... (1)$$

P(x, y) lies on the curve, therefore

$$y^2 = x \quad .... (2)$$

From (1) and (2) , we have

$$D = \sqrt{\left((x-4)^2 + x\right)}$$

$$D = \sqrt{\left(x^2 - 7x + 16\right)} \quad .... (3)$$

Differentiating (3) w.r.t. x, we get

$$\frac{dD}{dx} = \frac{2x-7}{2} \cdot \frac{1}{\sqrt{x^2 - 7x + 16}}$$

Now $\frac{dD}{dx} = 0$

gives $x = \frac{7}{2}$. Thus $\frac{7}{2}$ is a stationary point of the function D. We apply the first derivative test to verify whether D is minimum at $x = \frac{7}{2}$.

$$\left(\frac{dD}{dx}\right)_{x=3} = -\frac{1}{2}\frac{1}{\sqrt{9-12+16}}$$

and it is negative

$$\left(\frac{dD}{dx}\right)_{x=4} = \frac{1}{2}\frac{1}{\sqrt{16-28+16}}$$

and it is positive

$\frac{dD}{dx}$ changes sign from negative to positive.

Therefore, D is minimum at $x = 7/2$. Substituting $x = 7/2$ in (2) we have $y^2 = 7/2$.

$$\therefore y = \pm\sqrt{\left(\frac{7}{2}\right)}$$

Thus the points $\left(\left(\frac{7}{2}\right), \sqrt{\left(\frac{7}{2}\right)}\right)$ and $\left(\left(\frac{7}{2}\right), -\sqrt{\left(\frac{7}{2}\right)}\right)$ are nearest to A(4,0).

4. Problem: *Prove that the radius of the right circular cylinder of greatest curved surface area which can be inscribed in a given cone is half of that of the cone.*

Solution: Let O be the centre of the circular base of the cone and its height be h (Fig. 10.31). Let r be the radius of the circular base of the cone. Then AO = h, OC = r.

Let a cylinder with radius x(OE) be inscribed in the given cone. Let its height be u.

i.e., RO = QE = PD = u

Now the triangles AOC and QEC are similar.

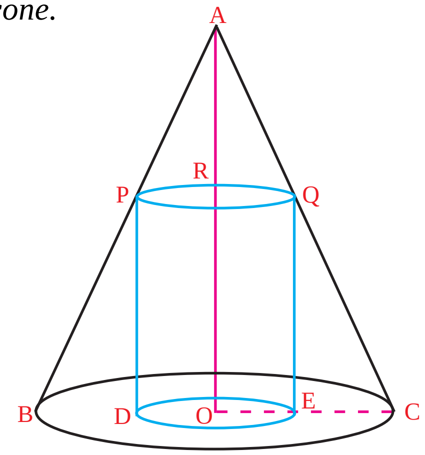


Fig. 10.31

Therefore,

$$\frac{QE}{OA} = \frac{EC}{OC}$$

i.e., $$\frac{u}{h} = \frac{r-x}{r}$$

$$\therefore u = \frac{h(r-x)}{r} \qquad \ldots (1)$$

Let S denote the curved surface area of the chosen cylinder. Then

$$S = 2\pi xu.$$

Inview of (1) , we have

$$S = 2\pi h\,(r x - x^2)/r$$

As the cone is fixed one, the values of r and h are constants. Thus S is function of x only. Now,

$$\frac{dS}{dx} = 2\pi h(r-2x)/r \quad \text{and} \quad \frac{d^2S}{dx^2} = -4\pi h/r.$$

The stationary point of S is a root of

$$\frac{dS}{dx} = 0$$

i.e., $\pi(r - 2x) / r = 0$

i.e., $x = r/2$

$$\frac{d^2S}{dx^2} < 0 \quad \text{for all } x, \text{ Therefore } \left(\frac{d^2S}{dx^2}\right)_{x=r/2} < 0.$$

Hence, the radius of the cylinder of greatest curved surface area which can be inscribed in a given cone is $r/2$.

5. Problem: *The profit function P(x) of a company, selling x items per day is given by P(x) = (150 − x)x −1600. Find the number of items that the company should sell to get maximum profit. Also find the maximum profit.*

Solution: Given that the profit function is

$$P(x) = (150 - x)x - 1600. \qquad \text{.... (1)}$$

For maxima or minima $\frac{dP(x)}{dx} = 0$.

$$\therefore\ (150 - x)(1) + x(-1) = 0.$$

i.e., $x = 75$

Now $\frac{d^2P(x)}{dx^2} = -2$ and $\left[\frac{d^2P(x)}{dx^2}\right]_{x=75} < 0.$

$\therefore$ The profit P(x) is maximum for $x = 75$.

$\therefore$ The company should sell 75 items a day to make maximum profit.

The maximum profit will be P(75) = 4025.

6. Problem: *A manufacturer can sell x items at a price of rupees (5 – x /100) each. The cost price of x items is Rs. (x/5 + 500). Find the number of items that the manufacturer should sell to earn maximum profits.*

Solution: Let S(x) be the selling price of x items and C(x) be the cost price of x items. Then, we have

$$S(x) = \{\text{cost of each item}\} \, . \, x$$

$$\therefore \quad S(x) = (5 - x/100)\, x = 5x - x^2/100$$

and

$$C(x) = x/5 + 500$$

Let P(x) denote the profit function. Then,

$$P(x) = S(x) - C(x)$$

i.e.,

$$P(x) = (5x - x^2/100) - (x/5 + 500)$$

$$= (24x/5) - (x^2/100) - 500 \qquad \text{.... (1)}$$

For maxima or minima

$$\frac{dP(x)}{dx} = 0$$

i.e., $24/5 - x/50 = 0$

The stationary point of P(x) is $x = 240$ and

$$\left[\frac{d^2P(x)}{dx^2}\right] = -\frac{1}{50} \quad \text{for all } x.$$

Hence the manufacturer can earn maximum profit if he sells 240 items.

10.9.30 Maximum and minimum values of a continuous function on a closed interval [a, b]

The function $f : (0, 1) \to \mathbf{R}$ defined by $f(x) = x$ has no global maximum and no global minimum. However, if we extend the domain of f to $[0, 1]$ then $f(x) \le f(1)\ \forall x \in [0, 1]$ and the function f has global maximum for f. It also has a global minimum value 0. Thus the function f has global minimum and global maximum when domain is [0, 1]. Note that $f(x) = x$ is continuous.

We state the following which can be used to find global maximum (minimum) when a function is continuous and its domain is a closed interval.

Let a function f be continuous on $[a, b]$ and differentiable on (a, b). Suppose $x_1, x_2, x_3, ..., x_k$ are the points of local extrema of f in (a, b). Then

(i) the greatest value of $f(x_1), f(x_2), ..., f(x_k), f(a)$ and $f(b)$ will be the absolute maximum of f on $[a, b]$.

(ii) the least value of $f(x_1), f(x_2), \ldots\ldots, f(x_k), f(a)$ and $f(b)$ will be the absolute minimum of f on $[a, b]$.

10.9.31 Solved problems

1. Problem: *Find the absolute extremum of $f(x) = x^2$ defined on $[-2, 2]$.*

Solution: The given function $f(x) = x^2$ is continuous on $[-2, 2]$. It can be shown that it has only local minimum and the point of local minimum is 0. The absolute(global) maximum of f is the largest value of $f(-2)$, $f(0)$ and $f(2)$ i.e., 4, 0 , 4.

Hence, the absolute maximum value is 4. Similarly the absolute minimum is the least value of 4,0,4. Hence 0 is the absolute minimum value.

2. Problem : *Find the absolute maximum of $x^{40} - x^{20}$ on the interval [0, 1]. Find also its absolute maximum value.*

Solution: Let $f(x) = x^{40} - x^{20} \quad \forall x \in [0, 1]$ .... (1)

The function f is continuous on [0, 1] and the interval [0, 1] is closed.

From (1) we have

$f'(x) = 40\,x^{39} - 20x^{19} = 20x^{19}(2x^{20} - 1)$. Thus $f'(x) = 0$ at $x = 0$ or

$$x = \left(\frac{1}{2}\right)^{\frac{1}{20}}.$$

Therefore, the critical points of f are 0 and $\left(\frac{1}{2}\right)^{\frac{1}{20}}$,

and 0 is one of the end points of the domain. Therefore no local maximum exists at $x = 0$. Now

$$f''(x) = 40(39)\,x^{38} - 20(19)x^{18}$$

$$= 20x^{18}(78x^{20} - 19)$$

$$[f''(x)]_{x=\left(\frac{1}{2}\right)^{\frac{1}{20}}} = 20(1/2)^{(18/20)}[39 - 19] > 0.$$

Therefore f has local minimum at $x = (1/2)^{(1/20)}$

and its value is $f\left(\left(\frac{1}{2}\right)^{\frac{1}{20}}\right) = -\frac{1}{4}$.

Therefore the absolute maximum value of the function f is the largest value of $f(0)$, $f(1)$ and $f\left(\left(\frac{1}{2}\right)^{\frac{1}{20}}\right)$ i.e., the largest value of $\left\{0, 0, -\frac{1}{4}\right\}$.

Hence, the absolute maximum of f is 0 and the points of absolute maximum are 0 and 1. Further the absolute minimum is the least of 0, 0, $-\frac{1}{4}$.

Hence the absolute minimum is $-\frac{1}{4}$ and the point of absolute minimum is $x = \left(\frac{1}{2}\right)^{\left(\frac{1}{20}\right)}$.

Exercise 10(h)

I. 1. Find the points of local extrema (if any) and local extrema of the following functions each of whose domain is shown against the function.

(i) $f(x) = x^2, \ \forall x \in \mathbf{R}$

(ii) $f(x) = \sin x, \ [0, 4\pi]$

(iii) $f(x) = x^3 - 6x^2 + 9x + 15 \ \forall x \in \mathbf{R}$

(iv) $f(x) = x\sqrt{(1-x)} \ \forall x \in (0, 1)$

(v) $f(x) = 1/(x^2 + 2) \ \forall x \in \mathbf{R}$

(vi) $f(x) = x^3 - 3x \ \forall x \in \mathbf{R}$

(vii) $f(x) = (x - 1)(x + 2)^2 \ \forall x \in \mathbf{R}$

(viii) $f(x) = \frac{x}{2} + \frac{2}{x} \ \forall x \in (0, \infty)$

(ix) $f(x) = -(x - 1)^3 (x + 1)^2 \ \forall x \in \mathbf{R}$

(x) $f(x) = x^2 e^{3x} \ \forall x \in \mathbf{R}$

2. Prove that the following functions do not have absolute maximum and absolute minimum.

(i) e^x in $\mathbf{R}$ (ii) $\log x$ in $(0, \infty)$ (iii) $x^3 + x^2 + x + 1$ in $\mathbf{R}$

II.1. Find the absolute maximum value and absolute minimum value of the following functions on the domain specified against the function.

(i) $f(x) = x^3$ on $[-2, 2]$

(ii) $f(x) = (x - 1)^2 + 3$ on $[-3, 1]$

(iii) $f(x) = 2\,|x|$ on $[-1, 6]$

(iv) $f(x) = \sin x + \cos x$ on $[0, \pi]$

(v) $f(x) = x + \sin 2x$ on $[0, \pi]$.

2. Use the first derivative test to find local extrema of $f(x) = x^3 - 12x$ on $\mathbf{R}$.

3. Use the first derivative test to find local extrema of $f(x) = x^2 - 6x + 8$ on **R**.

4. Use the second derivative test to find local extrema of the function
$f(x) = x^3 - 9x^2 - 48x + 72$ on **R**.

5. Use the second derivative test to find local extrema of the function
$f(x) = -x^3 + 12x^2 - 5$ on **R**.

6. Find local maximum or local minimum of $f(x) = -\sin 2x - x$ defined on $[-\pi/2, \pi/2]$.

7. Find the absolute maximum and absolute minimum of $f(x) = 2x^3 - 3x^2 - 36x + 2$ on the interval $[0, 5]$.

8. Find the absolute extremum of $f(x) = 4x - \frac{x^2}{2}$ on $\left[-2, \frac{9}{2}\right]$.

9. Find the maximum profit that a company can make, if the profit function is given by $P(x) = -41 + 72x - 18x^2$.

10. The profit function $P(x)$ of a company selling x items is given by
$P(x) = -x^3 + 9x^2 - 15x - 13$ where x represents thousands of units. Find the absolute maximum profit if the company can manufacture a maximum of 6000 units.

III.1. The profit function $P(x)$ of a company selling x items per day is given by
$P(x) = (150 - x)x - 1000$. Find the number of items that the company should manufacture to get maximum profit. Also find the maximum profit.

2. Find the absolute maximum and absolute minimum of $f(x) = 8x^3 + 81x^2 - 42x - 8$ on $[-8, 2]$.

3. Find two positive integers whose sum is 16 and the sum of whose squares is minimum.

4. Find two positive integers x and y such that $x + y = 60$ and xy^3 is maximum.

5. From a rectangular sheet of dimensions 30 cm $\times$ 80 cm., four equal squares of side x cm. are removed at the corners, and the sides are then turned up so as to form an open rectangular box. Find the value of x, so that the volume of the box is the greatest.

6. A window is in the shape of a rectangle surmounted by a semicircle. If the perimeter of the window is 20 ft. , find the maximum area.

7. If the curved surface of right circular cylinder inscribed in a sphere of radius r is maximum, show that the height of the cylinder is $\sqrt{2}\,r$.

8. A wire of length l is cut into two parts which are bent respectively in the form of a square and a circle. What are the lengths of the pieces of the wire respectively so that the sum of the areas is the least.

Key Concepts

❖ Δx = Small change in x

Δy = Small change in y corresponding to Δx in x when $y = f(x)$.

❖ Differential of y is denoted by dy.

$dy = f'(x)dx$.

Ex.: $d(\sin x) = \cos x\, dx; d(p^2) = 2p\,dp;\ d\left(\frac{1}{y}\right) = \frac{-1}{y^2}dy$.

❖ (i) Δy is called error or absolute error in y.

(ii) $\frac{\Delta y}{y}$ is called relative error in y.

(iii) $\frac{\Delta y}{y} \times 100$ is called the percentage error in y.

❖ (i) $f'(x)\,\Delta x$ is called an approximate error in $y = f(x)$.

(ii) $\frac{f'(x)}{f(x)}\,\Delta x$ is called an approximate relative error in $y = f(x)$.

(iii) $\frac{f'(x)}{f(x)} \times 100 \times \Delta x$ is called an approximate percentage of error in $y = f(x)$.

❖ $f(x+\Delta x) \cong f(x) + f'(x)\Delta x$.

❖ Let $y = f(x)$ be a curve and P(a, b) be a point on it. Then

(i) slope of tangent at P = m = $f'(a)$ or $\left.\frac{dy}{dx}\right|_{(a,b)}$.

(ii) equation of tangent at P is $y - b = m(x-a)$.

(iii) equation of normal at P is : $y - b = \frac{-1}{m}(x-a)$.

(iv) length of tangent = $\left|\frac{f(a)\sqrt{1+m^2}}{m}\right|$.

(v) length of normal = $\left|f(a)\sqrt{1+m^2}\right|$.

(vi) length of subtangent $= |f(a)/m|$.

(vii) length of subnormal $= |f(a).m|$.

❖ $y = f(x)$ and $y = g(x)$ are two curves intersecting at P. Let $m_1 = f'(x)|_P$ and $m_2 = g'(x)|_P$. Then

(i) curves touch each other at P if $m_1 = m_2$.

(ii) curves intersect each other orthogonally if $m_1 m_2 = -1$.

(iii) If $m_1 \neq m_2$, $m_1 m_2 \neq -1$, then the angle between the curves θ, at P is given by

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|.$$

(iv) If one of m_1 or m_2 say $m_2 = 0$, then the angle between the curves is $\left|\text{Tan}^{-1}(m_1)\right|$.

❖ $\dfrac{\Delta y}{\Delta x} = \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$ is the average rate of change in y between $x = x_1$ and $x = x_2$.

❖ (i) $\dfrac{dy}{dx}$ can be viewed as the rate of change of y with respect to x.

(ii) velocity $v = \dfrac{ds}{dt}$.

(iii) acceleration $a = \dfrac{dv}{dt} = \dfrac{d^2 s}{dt^2}$.

❖ Suppose $a, b, (a < b)$ are two real numbers.

Let $f : [a, b] \rightarrow \mathbf{R}$ be a function satisfying the conditions

(i) f is continuous on $[a, b]$

(ii) f is differentiable on (a, b)

and

(iii) $f(a) = f(b)$.

Then $\exists\ c \in (a, b)$ such that $f'(c) = 0$.

❖ Suppose a, b $(a < b)$ are two real numbers.

Let $f : [a, b] \to \mathbf{R}$ be a function satisfying the following conditions

(i) f is continuous on $[a, b]$

(ii) f is differentiable on (a, b)

Then $\exists$ at least one $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

❖ Let f be a real function on the interval I.

Then f is said to be

(i) a strictly increasing function on I if $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$ $\forall x_1, x_2 \in$ I

(ii) a strictly decreasing function on I if $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$ $\forall x_1, x_2 \in$ I

(iii) increasing on I if $x_1 < x_2 \Rightarrow f(x_1) \le f(x_2)$ for all $x_1, x_2 \in$ I.

(iv) decreasing on I if $x_1 < x_2 \Rightarrow f(x_1) \ge f(x_2)$ for all $x_1, x_2 \in$ I.

❖ A function is called monotonic, if it is either increasing or decreasing.

❖ (i) f is increasing in $[a, b]$ if $f'(x) > 0$ for each $x \in (a, b)$.

(ii) f is decreasing in $[a, b]$ if $f'(x) < 0$ for each $x \in (a, b)$.

(iii) f is constant in $[a, b]$ if $f'(x) = 0$ for each $x \in (a, b)$.

❖ A function f is said to have a maximum value (or minimum value) in I if there exists a point $c \in$ I such that $f(c) \ge f(x)$ (or $f(c) \le f(x)$) for all $x \in$ I.

The number $f(c)$ is called the maximum value (or the minimum value) of f in I and the point 'c' is called a point of maximum value (or a point of minimum value) of f in I. f is said to have an extreme value if it has a maximum value or minimum value in I.

❖ First Derivative Test

(a) x_0 is a point of local maximum of $f(x)$ if

(i) $f'(x_0) = 0$

(ii) $f'(x) > 0$ at every point x sufficiently close to x_0 and to the left of x_0; and $f'(x) < 0$ at every point x sufficiently close to x_0 and to the right of x_0.

(b) x_0 is a point of local minimum of $f(x)$ if

(i) $f'(x_0) = 0$

(ii) $f'(x) < 0$ at every point x close to x_0 and to the left of x_0; and $f'(x) > 0$ at every point x close to x_0 and to the right of x_0.

(c) If $f'(x_0) = 0$ and if $f'(x)$ does not change sign as x increases through x_0, then x_0 is neither a point of local maximum nor a point of local minimum.

❖ Second Derivative Test.

(a) x_0 is a local maximum if $f'(x_0) = 0$ and $f''(x_0) < 0$

(b) x_0 is a local minimum if $f'(x_0) = 0$ and $f''(x_0) > 0$.

(c) Test fails if $f''(x_0) = 0$.

Historical Note

One of the most distinguished families in the history of mathematics is the *Bernoulli* family of Switzerland. The family record started with two brothers; *Jakob Bernoulli* (1654 - 1705) and *Johann Bernoulli* (1667 - 1748). They are among the first mathematicians to realise the surprising power of calculus and apply the tool to a great diversity of problems. *Johann Bernoulli*, who taught *Euler*, was very influential in making the power of the new subject appreciated in Europe.

There are many applications of differentiation in science and engineering. Differentiation is also used in analysis of finance and economics besides physical and natural sciences. One important application of differentiation is in the area of optimization, which means finding the condition for maximum or minimum to occur.

During 15th century, an early version of the mean value theorem was first described by *Parameshwara* (1370 - 1460) of Kerala School of Mathematics. The first proof of Rolle's theorem was given by *Michel Rolle* in 1691. The mean value theorem in its modern form was stated by *Augustin Louis Cauchy* (1789 - 1857).

Answers

Exercise 10(a)

I. 1. (i) $\Delta y = 0.2301, \quad dy = 0.23$ (ii) $\Delta y = e^5(e^{0.02} - 1) + 0.02, \quad dy = (e^5 + 1)(0.02)$

(iii) $\Delta y = 0.026005, \quad dy = 0.026$ (iv) $\Delta y = -0.0001996, \quad dy = -0.0002$

(v) $\Delta y = -0.0152, \quad dy = -0.01516;$

II. 1. (i) 9.056 (ii) 4.0208 (iii) 5.0001 (iv) 1.9834

(v) 0.8834 (vi) 0.4987 (vii) 2.03125

2. 8 **3.** 7.04 sq. cm **4.** $16\pi, \ 1.6\pi$ **5.** $\frac{1}{2}$

Exercise 10(b)

I. **1.** 764 **2.** $\frac{-1}{64}$ **3.** 11 **4.** 24

5. 1 **6.** $\frac{-a}{2b}$ **7.** (3, –20) and (–1, 12) **8.** (3, 1)

9. $(2, \ -9)$ **10.** $y = \frac{1}{2}$.

II. 1. (i) Tangent $10x + y = 5$; Normal $x - 10y + 50 = 0$

(ii) Tangent $y = 3x - 2$; Normal $x + 3y - 4 = 0$

(iii) Tangent $y = 0$; Normal $x = 0$.

(iv) Tangent $x + y - \sqrt{2} = 0$; Normal $x = y$

(v) Tangent $4x - y - 14 = 0$; Normal $x + 4y - 12 = 0$

(vi) Tangent $y - 1 = 0$; $x = 0$

2. Tangent $5x + 2y - 20 = 0$; Normal $2x - 5y + 21 = 0$

3. Tangent $5x + y + 2 = 0$; Normal $x - 5y + 16 = 0$

4. at $\left(1, \frac{-1}{2}\right)$ tangent : $3x + 2y - 2 = 0$, normal : $4x - 6y - 7 = 0$

at $\left(3, \frac{9}{2}\right)$ tangent : $3x + 2y - 18 = 0$, normal : $4x - 6y + 15 = 0$

5. $3x - 2y - 2\sqrt{e} = 0$; $4x + 6y - 7\sqrt{e} = 0$.

6. $2x + 3y = 6$; $3x - 2y + 4 = 0$

III. **2.** $\left(2\sqrt{\frac{2}{3}}, \sqrt{\frac{2}{3}}\right)$, $\left(-2\sqrt{\frac{2}{3}}, -\sqrt{\frac{2}{3}}\right)$

Exercise 10(c)

I. **1.** $\left|a \tan \frac{x}{a}\right|$; $\left|\frac{b^2}{2a} \sin \frac{2x}{a}\right|$

II. **1.** $k = -2$

2. $2a \sin \frac{t}{2}$; $2a \sin \frac{t}{2} \tan \frac{t}{2}$; $a \sin t$; $2a \sin^2 \frac{t}{2} \tan \frac{t}{2}$.

3. $a \cosh^2 \frac{x}{a}$; $\frac{a}{2} \sinh \frac{2x}{a}$

4. $a(\sin t - t\cos t)\cot t$; $a(\sin t - t\cos t)\tan t$.

Exercise 10(d)

I. **1.** $\mathrm{Tan}^{-1}\left(\frac{1}{7}\right)$ **2.** $\mathrm{Tan}^{-1}(3)$ **3.** $\mathrm{Tan}^{-1}\left(\frac{22\sqrt{6}}{69}\right)$ **4.** $\frac{\pi}{2}$

5. $\mathrm{Tan}^{-1}\left(\frac{9}{13}\right)$ **6.** $\mathrm{Tan}^{-1}(3)$ **7.** $\mathrm{Tan}^{-1}\left(\frac{1}{3}\right)$

Exercise 10(e)

I. **1.** –38 unit/sec. **2.** 144 **3.** 50 units/sec, 24 units/sec^2.

4. $t = 2$ sec, $s = 2$ units, $t = 4$ sec, $s = -2$ units. **5.** 9

II. **1.** $\frac{8}{3}$ cm^2/sec **2.** 80π cm^2/sec **3.** 1.4π cm/sec **4.** $\frac{1}{\pi}$ cm/sec

5. 2π cm^3/sec **6.** 49000 units **7.** 180 **8.** $\frac{1}{80}$

9. $\frac{2}{9\pi}$ **10.** 20.967 **11.** 208 **12.** 32 units/sec

Exercise 10(f)

I. **1.** (i) $c=0$ (ii) $c=\mathrm{Cos}^{-1}\left(\frac{1\pm\sqrt{33}}{8}\right)$ (iii) $c=0$

2. $a=11, b=-6$ **4.** $\left(\frac{7}{2},\frac{1}{4}\right)$ **5.** $\left(\frac{\sqrt{39}}{3},\frac{13\sqrt{39}}{9}\right)$

6. (i) $\frac{1}{7}$ (ii) $\log(e-1)$ **7.** $\frac{2\pm\sqrt{7}}{3}$

8. (i) $c=\frac{5}{2}$ (ii) $c=\mathrm{Cos}^{-1}\left(\frac{1\pm\sqrt{33}}{8}\right)$ (iii) $c=\log_2 e$.

Exercise 10(g)

II. **1.** (i) $(-\infty,-1)$ strictly decreasing, $(-1,\infty)$ strictly increasing.

(ii) $\left(-\infty,-\frac{9}{2}\right)$ strictly increasing, $\left(-\frac{9}{2},\infty\right)$ strictly decreasing.

(iii) $(-\infty,-1)\cup(-1,0)$ strictly decreasing, $(0,1)\cup(1,\infty)$ strictly increasing.

(iv) $\left(-\infty,\frac{6}{5}\right)\cup(2,\infty)$ increasing, $\left(\frac{6}{5},2\right)$ decreasing.

(v) $(-1,\infty)$ increasing, $(-\infty,-1)$ decreasing.

(vi) $\left(-\frac{5}{2},0\right)$ increasing, $\left(0,\frac{5}{2}\right)$ decreasing.

(vii) $(1,\infty)$ increasing.

(viii) $(1-\sqrt{3},1+\sqrt{3})$ decreasing, $(-\infty,1-\sqrt{3})\cup(1+\sqrt{3},\infty)$ increasing.

5. $(3,\infty)$

7. $(-\infty,0)\cup(2,\infty)$ increasing, $(0,2)$ decreasing.

8. $\left(0,\frac{\pi}{4}\right)$ decreasing, $\left(\frac{\pi}{4},\frac{\pi}{2}\right)$ increasing.

Exercise 10(h)

I. 1. (i) Point of local minimum $x = 0$, local minimum = 0

(ii) Point of local minimum $x = \frac{3\pi}{2}$, local minimum = – 1

Point of local minimum $x = \frac{7\pi}{2}$, local minimum = – 1

Point of local maximum $x = \frac{\pi}{2}$, local maximum = 1

Point of local maximum $x = \frac{5\pi}{2}$, local maximum = 1

(iii) Point of local maximum $x = 1$, local maximum = 19

Point of local minimum $x = 3$, local minimum = 15

(iv) Point of local maximum $x = \frac{2}{3}$, local maximum = $\frac{2\sqrt{3}}{9}$

(v) Point of local maximum $x = 0$, local maximum = $\frac{1}{2}$

(vi) Point of local minimum $x = 1$, local minimum = – 2

Point of local maximum $x = -1$, local maximum = 2

(vii) Point of local minimum $x = 0$, local minimum = – 4

Point of local maximum $x = -2$, local maximum = 0

(viii) Point of local minimum $x = 2$, local minimum = 2

(ix) Point of local minimum $x = -1$, local minimum = 0

Point of local maximum $x = -\frac{1}{5}$, local maximum = $\frac{3456}{3125}$

(x) Point of local minimum $x = 0$, local minimum = 0

Point of local maximum $x = -\frac{2}{3}$, local maximum = $\frac{4}{9e^2}$

II. 1. (i) absolute minimum = – 8, absolute maximum = 8

(ii) absolute minimum = 3, absolute maximum = 19

(iii) absolute minimum = 0, absolute maximum = 12

(iv) absolute minimum = – 1, absolute maximum = $\sqrt{2}$

$x \neq$

(v) abs. min $= \frac{2\pi}{3} - \frac{\sqrt{3}}{2}$, abs max $= \frac{\pi}{3} + \frac{\sqrt{3}}{2}$

2. Point of local minimum $x = 2$, local minimum = – 16

Point of local maximum $x = -2$, local maximum = 16

3. Point of local minimum $x = 3$, local minimum = – 1

4. Local minimum $x = -376$, Local maximum = 124

5. Local minimum $x = -5$, Local maximum = 251

6. Local minimum $= -\frac{\sqrt{3}}{2} - \frac{\pi}{3}$, Local maximum $= \frac{\sqrt{3}}{2} + \frac{\pi}{3}$.

7. Absolute minimum = –79, Absolute maximum = 2

8. Absolute minimum = –10, Absolute maximum = 8.

9. Maximum profit = 31

10. 12

III. 1. No. of items = 75; Maximum profit = 4625.

2. Absolute maximum = 1416, Absolute minimum $= -\frac{213}{16}$

3. 8, 8 **4.** 15, 45 **5.** $x = \frac{20}{3}$ cm

6. $\frac{200}{\pi + 4}$ **8.** $\frac{\pi l}{\pi + 4}, \frac{4l}{\pi + 4}$

Reference Books

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- Geometry - Schaum's Outline series; Rich; McGraw - Hill Education (India) Ltd.; 2007.
- The Elements of Coordinate Geometry; S.L. Loney; Macmillan India Ltd., London; 1975.
- An Elementary Treatise on Coordinate Geometry of Three Dimensions; R.J.T. Bell; Macmillan & Co., London; 1959.
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Syllabus in MATHEMATICS PAPER - IB

To be effective from the academic year 2012-2013

Name of Topic and Sub Topics	No. of Periods
COORDINATE GEOMETRY	
1. Locus	**08**
1.1 Definition of locus - Illustrations	
1.2 To find equations of locus - Problems connected to it	
2. Transformation of Axes	**08**
2.1 Transformation of axes - Rules, Derivations and Illustrations	
2.2 Rotation of axes - Derivations - Illustrations.	
3. The Straight Line	**25**
3.1 Revision of fundamental results	
3.2 Straight line - Normal form - Illustrations	
3.3 Straight line - Symmetric form	
3.4 Straight line - Reduction into various forms	
3.5 Intersection of two Straight lines	
3.6 Family of straight lines - Concurrent lines	
3.7 Condition for Concurrent lines	
3.8 Angle between two lines	
3.9 Length of perpendicular from a point to a line	
3.10 Distance between two parallel lines	
3.11 Concurrent lines - properties related to a triangle	
4. Pair of Straight lines	**24**
4.1 Equations of pair of lines passing through origin, angle between a pair of lines	
4.2 Condition for perpendicular and coincident lines, bisectors of angles	
4.3 Pair of bisectors of angles	
4.4 Pair of lines - second degree general equation.	
4.5 Conditions for parallel lines - distance between them, Point of intersection of pair of lines	
4.6 Homogenising a second degree equation with a first degree equation in x and y	

5. Three Dimensional Coordinates	**04**
5.1 Coordinates	
5.2 Section formulas - Centroid of a triangle and tetrahedron	
6. Direction Cosines and Direction Ratios	**10**
6.1 Direction Cosines	
6.2 Direction Ratios	
7. Plane	**04**
7.1 C artesian equation of Plane - Simple Illustrations	
CALCULUS	
8. Limits and Continuity	**15**
8.1 Intervals and neighbourhoods	
8.2 Limits	
8.3 Standard Limits	
8.4 Continuity	
9. Differentiation	**24**
9.1 Derivative of a function	
9.2 Elementary Properties	
9.3 Trigonometric, Inverse Trigonometric, Hyperbolic, Inverse Hyperbolic Function - Derivatives.	
9.4 Methods of Differentiation	
9.5 Second Order Derivatives	
10. Applications of Derivatives	**28**
10.1 Errors and Approximations	
10.2 Geometrical interpretation of a derivative	
10.3 Equations of tangents and normals	
10.4 Lengths of tangent, normal, sub tangent and subnormal.	
10.5 Angle between two curves and condition for orthogonality of curves	
10.6 Derivative as Rate of change	
10.7 Rolle's Theorem and Lagrange's Mean value theorem without proofs and their geometrical interpretation	
10.8 Increasing and decreasing functions	
10.9 Maxima and Minima	
Total	**150**

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MODEL QUESTION PAPER w.e.f. 2012-13
MATHEMATICS - IB

(English Version)

Time : 3 Hours **Max. Marks : 75**

Note : The Question Paper consists of three sections A, B and C

Section - A

$10 \times 2 = 20$ Marks

I. Very Short Answer Questions :

(i) Answer All questions

(ii) Each Question carries Two marks

1. Find the value of x, if the slope of the line passing through (2, 5) and (x, 3) is 2.

2. Transform the equation $x + y + 1 = 0$ into the normal form.

3. Show that the points (1, 2, 3), (2, 3, 1) and (3, 1, 2) form an equilateral Triangle.

4. Find the angle between the planes $2x - y + z = 6$ and $x + y + 2z = 7$.

5. Show that $\underset{x \to 0+}{\text{Lt}} \left\{ \frac{2|x|}{x} + x + 1 \right\} = 3$.

6. Find $\lim_{x \to 0} \frac{e^{x+3} - e^3}{x}$.

7. If $f(x) = a^x e^{x^2}$ find $f'(x)$ (where $a > 0, a \neq 1$).

8. If $y = \log[\sin(\log x)]$, find $\frac{dy}{dx}$.

9. Find the approximate value of $\sqrt[3]{65}$.

10. Find the value of 'C' in Rolle's theorem for the function $f(x) = x^2 + 4$ on $[-3, 3]$.

Section - B $\quad 5 \times 4 = 20$ Marks

II. Short Answer Questions

(i) Answer any Five questions.

(ii) Each Question carries Four marks.

11. A(2, 3) and B(–3, 4) are two given points. Find the equation of the Locus of P, so that the area of the triangle PAB is 8.5 sq. units.

12. When the axes are rotated through an angle $\frac{\pi}{6}$ find the transformed equation of $x^2 + 2\sqrt{3}xy - y^2 = 2a^2$.

13. Find the points on the line $3x - 4y - 1 = 0$ which are at a distance of 5 units from the point (3, 2).

14. Show that $f(x) = \begin{cases} \dfrac{\cos ax - \cos bx}{x^2} & \text{if } x \neq 0 \\ \dfrac{1}{2}(b^2 - a^2) & \text{if } x = 0 \end{cases}$

where a and b are real constants is continuous at '0'.

15. Find the derivative of $\sin 2x$ from the first principle.

16. A particle is moving in a straight line so that after t seconds its distance s (in cms) from a fixed point on the line is given by $s = f(t) = 8t + t^3$. Find (i) the velocity at time $t = 2$ sec (ii) the initial velocity (iii) acceleration at $t = 2$ sec.

17. Show that the tangent at any point θ on the curve $x = c\sec\theta$, $y = c\tan\theta$ is $y\sin\theta = x - c\cos\theta$.

Section - C $\quad 5 \times 7 = 35$ Marks

III. Long Answer Questions

(i) Answer any Five questions.

(ii) Each Question carries Seven marks.

18. Find the equation of straight lines passing through (1, 2) and making an angle of 60^0 with the line $\sqrt{3}x + y + 2 = 0$.

19. Show that the area of the triangle formed by the lines $ax^2 + 2hxy + by^2 = 0$ and $lx + my + n = 0$ is $\left| \dfrac{n^2\sqrt{h^2 - ab}}{am^2 - 2hlm + bl^2} \right|$.

20. Find the value of k, if the lines joining the origin to the points of intersection of the curve $2x^2 - 2xy + 3y^2 + 2x - y - 1 = 0$ and the line $x + 2y = k$ are mutually perpendicular.

21. If a ray with d.c's l, m, n makes angles α, β, γ and δ with four diagonals of a cube, then show that

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = \frac{4}{3}.$$

22. If $x = \dfrac{3at}{1+t^3}$, $y = \dfrac{3at^2}{1+t^3}$ then find $\dfrac{dy}{dx}$.

23. At any point t on the curve $x = a(t + \sin t)$; $y = a(1 - \cos t)$ find lengths of tangent and normal.

24. A wire of length l is cut into two parts which are bent respectively in the form of a square and a circle. Find the lengths of the pieces of the wire, so that the sum of the areas is the least.

Chapter - I
Planes

1.1 Definitions and Formulae

1.1.1 General equation of a plane is $ax + by + cz + d = 0$

1.1.2 one point form of a plane (cartesion form)

The equation of plane through the point (x_1, y_1, z_1) is

$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$

1.1.3 Normal form of a plane. It l, m, n are the direction cosines it the perpendicular from the origin to a plane and p is the length of teh perpendicular, then the equation of the plane is

$lx + my + xz \;=\; p$ or $x\cos\alpha + y\cos\beta + z\cos\gamma = p$

1.1.4 Equation of a plane possing through the points (x_1, y_1, z_1) (x_2, y_2, z_2) and (x_3, y_3, z_3) is

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & x_2 & x_2 & 1 \\ x_3 & x_3 & x_3 & 1 \end{vmatrix} = 0 \text{ ಲೇದಾ } \begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - z_1 & y_3 - z_1 & z_3 - z_1 \end{vmatrix} = 0$$

1.1.5 Intercept form at a plane

The equation of a plane which makes intercepts a, b, c on the coordinate axes respectively is given by $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.

1.1.6 The length of the perpendicular from from the point (x_1, y_1, z_1) to the plane $ax + by + cz + d = 0$ is

$$\left|\frac{ax_1 + by_1 + cz_1 + d}{\sqrt{a^2 + b^2 + c^2}}\right|$$

1.1.7 The equation of the plane passing through (x_1, y_1, z_1) and perpendicular to the line whose direction ratios are a, b, c is

$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$

1.1.8 Definition: The angle between the two planes is equal to the angle between their normals.

1.1.9 Angle between the planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$. Let θ be one of the angles between the planes, a_1, b_1, c_1 and $a_2, b_2, c_2 = 0$ are direction ratios of the normals to the given planes

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

The other angle between the planes = $\pi - \theta$

(i) Condition of perpendicularity of two planes is $a_1a_2 + b_1b_2 + c_1c_2 = 0$

(ii) Condition of parallelism is $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

1.1.10 If $\pi_1 = a_1x + b_1y + c_1 z + d_1 = 0$ and $a_2x + b_2y + c_2 z + d_2 = 0$ are intersecting planes then the plane passing through their line of intersection is

$\pi_1 + x\,\pi_2 = 0$

1.1.11 If (x_1, y_1, z_1), (x_2, y_2, z_2) (x_3, y_3, z_3) and (x_4, y_4, z_4) are coplanar then

$$\begin{vmatrix} x_4 - x_1 & y_4 - y_1 & z_4 - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_3 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_2 - z_1 \end{vmatrix} = 0$$

1.1.12 Planes bisecting the angle between two planes.

Let the equations of two given planes be $a_1x + b_1y + c_1z + d_1 = 0$, $a_2x + b_2y + c_2z + d_2 = 0$ these being written in such a way that their constant terms are either positive or both negative

$$\frac{(a_1x + b_1y + c_1z + d_1)}{\sqrt{a_1^2 + b_1^2 + c_1^2}} = \frac{(a_2x + b_by + c_2z + d_2)}{\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

1.1.13 The homogeneous equation of second degree

$ax^2 + by^2 + cz^2 + 2fyx + 2\,gzx + 2hxy = 0$ in x, y, z represents a pair of planes if $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$; $h^2 \geq ab$, $g^2 \geq ac$, $f^2 \geq bc$.

1.1.14 Angle between pair of planes is

$$\theta = \tan^{-1}\left\{\frac{2\sqrt{f^2 + g^2 + h^2 - ab - bc - ca}}{(a + b + c)}\right\}$$

(i) Both planes are perpendicular it $a + b + c = 0$

Examples

Ex 1: The equation of the plane passing the point (4, 0, 1) and parallel to the plane $4x + 3y - 12z + 6 = 0$ is $4x + 3y - 12z + 6 = 0$.

Sol: Let the equation of the plane parallel to the plane $4x + 3y - 12z + 6 = 0$ be

$4x + 3y - 12z + k = 0$ --- (1)

The plane (1) passing through (4, 0, 1)

$\therefore\ 4(4) + 3(0) - 12(1) + k = 0$

$\Rightarrow k = -4$

$\therefore$ The requires plane is $4x + 3y - 12z - 4 = 0$

Ex 2: If a plane meets the coordinate axes in A, B, C such that the controid of ΔABC is the point (p, q, r) then the equation of the plane is $\frac{x}{È}+\frac{y}{q}+\frac{z}{r}=3$.

Sol: Let the intercepts made by the plane with the coordinate axes be a, b, c respectively, then the equation of the plane is $\frac{x}{È}+\frac{y}{q}+\frac{z}{r}=1$

The coordinates of A, B, C are A(a,0,0) B(0, b, 0), C(0, 0, c)

The centroid of Δ ABC is $\left(\frac{a}{3},\frac{b}{3},\frac{c}{3}\right)$

$$\therefore \quad \left(\frac{a}{3},\frac{b}{3},\frac{c}{3}\right) = (p, q, r)$$

$$\Rightarrow \quad \frac{a}{3}=p,\ \frac{b}{3}=q,\ \frac{c}{3}=r$$

$$\Rightarrow \quad a = 3p, b = 3q, c = 3r$$

putting these values in (1), $\frac{x}{3p}+\frac{y}{3q}+\frac{z}{3r}=1 \Rightarrow \frac{x}{p}+\frac{y}{q}+\frac{z}{r}=3$

Ex 3: The equation of the plane through the point (2, 3, –1) and is perpendicular to the line through the points (3, 4, –1) and (2, –1, 5) is $x+5y-6z-23=0$ neÚÔáT+~.

Sol: The equation of the required plane be

$$(3-2)(x-2)+(4+1)(y-3)+(-1-5)(x+1)=0$$

$$\Rightarrow \quad x+5y-6z-23=0$$

Ex 4: Find the angle between the pair of planes represented by $2x^2-6y^2-12z^2+18yz+2zx+xy=0$

Ans: $\theta=\cos^{-1}\left(\frac{10}{21}\right)$

Sol: Given equation of the pair of planes

$$2x^2-6y^2-12z^2+18yz+2zx+xy=0$$

$$a = 2,\ b = -b,\ c = -12,\ f = 9,\ g = 1,\ h = \frac{1}{2}$$

Angle between the pair planes

$$\cos\theta = \frac{\|a+b+c\|}{\sqrt{(a+b+c)^2+4(f^2+g^2+h^2-ab-bc-(a)}}$$

$$= \frac{\|2-6-12\|}{\sqrt{(-16)^2+4(81+1+\frac{1}{4}+12-74+24)}}$$

$$\theta = \cos^{-1}\left(\frac{16}{21}\right)$$

Ex 5: The angle between the pair of planes $6x^2+4y^2-10z^2-11xy+4zx=0$

Ans : $\frac{\pi}{2}$

Sol: Given pair of planes $6x^2+4y^2-10z^2-11xy+3yz+4zx+4zx=0$

$a+b+c=6+4-10=10$

Angle between the pair of planes is a + b + c = 6 + 4 – 10 = 0

$\Rightarrow \quad \cos\theta = \frac{\pi}{2}$

Exercise

1. The equation of the plane passing through the points (1, 1, 1), (1, –1, 1) and (–7, –3, –5) is

   Ans: $3x-48+1=0$

2. The plane passing through the point (1, 2, 3) and whose normal makes equal angles with the coordinate axes is Ans: $x+y+z-6=0$

3. The distance between the planes $2x-y+3z=6$ abd $-6x+3x-9z=5$ is

   Ans: $\frac{23}{3\sqrt{14}}$

4. The equation of plane making intercepts 4, 5, 2 on the axes is

   Ans: $5x+4y+10z=0$

5. The direction cosines of the normal to the plane $2x-3y+6z+14=0$

   Ans: $\left(\frac{-2}{7},\frac{3}{7},\frac{-6}{7}\right)$

6. The normal form of the plane $2x-2y+z=5$ Ans: $\frac{2}{3}x-\frac{2}{3}y+\frac{1}{3}z=\frac{5}{3}$

7. The equation of the plane bisecting the line segment joining the points (2, 0, 6) and (–6, 2, 4) is

   Ans: $4x-y+z+4=0$

8. If a, b, c are X, Y, Z intercepts of x + 2y + 3z = 6 on the coordinate axes Ans: 6

9. A plane $\frac{x}{2}+\frac{y}{4}+\frac{z}{5}=1$ meets the coordinate axes at A, B, C this the coordinates of the cenroid of ΔABC are Ans: $\left(\frac{2}{3},\frac{4}{3},\frac{5}{3}\right)$

10. The equation of the plane through the point (–1, 3, 2) and perpendicular to the planes $x+2y+2z-5=0$ and $3x+3y+2z=8$ is Ans: $2x-y-2z=21$

11. The length of the perpendicular from the origin to the plane $2x-3y+6z-7=0$

    Ans: 1

12. The angle between teh planes $2x + 4y - 6z = 11$ and $3x + 6y + 5z + 4 = 0$

Ans: $\frac{\pi}{2}$

13. The equation of the plane which passes through the point (2, –4, 5) and is parallel to the plane $4x + 2y - 7z + 6 = 0$ is Ans: $4x + 2y - 7z + 35 = 1$

14. The planes bisecting the angles between the planes $2x - y + 2z + 3 = 0$ and $3x - 2y + 6z + 8 = 0$ are

Ans: $2x - y + 2z + 3 = 0$

$5x - y - 4z - 3 = 0$

15. The angle between the pair of planes represented by $x^2 + y^2 - z^2 + 4xy = 0$ is

Ans: $\cos^{-1}\frac{2}{3}$

Chapter – II

The Straight line

2.1.1 Definitions and Formulae

2.1.1 The general form of the line is $a_1 x + b_1 y + c_1 z + d_1 = 0$

$$= a_2 x + b_2 y + c_2 z + d_2$$

2.1.2 (i) The symmetrical form of the line The equation of the line the point (x, y, z) and with drs l, m, n is $\dfrac{x-x_1}{l} = \dfrac{y-y_1}{m} = \dfrac{z-z_1}{n}$

ii) The parameteric form of the line is : $x = x_1 + lr,\ y = y_1 + mr,\ z = z_1 + nr$

2.1.3 Equations of the line through the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ is

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1} \text{ or } \frac{x-x_2}{x_1-x_2} = \frac{y-y_2}{y_1-y_2} = \frac{z-z_2}{z_1-z_2}$$

2.1.4 Transformation of the equation of a line from general form to the symmetrical form $a_1x + b_1y + c_1z + d_1 = 0 = a_2x + b_2y + c_2z + d_2$

$$\frac{x-\left(\dfrac{b_1d_2-b_2d_1}{a_1b_2-a_2b_1}\right)}{b_1c_2-b_2c_1} \quad R \quad \frac{y-\left(\dfrac{d_1a_2-d_2a_1}{a_1b_2-a_2b_1}\right)}{b_1c_2-b_2c_1} \quad R \quad \frac{z}{a_1b_2-a_2b_1}$$

2.1.5 The coordinates of the points of intersectionof a line and a plane

Let $\dfrac{x-x_1}{l} = \dfrac{y-y_1}{m} = \dfrac{z-z_1}{n}$ be the equation of a line and

$ax + by + cz + d = 0$ be the equation of a plane are

$$\left(x - \frac{l(ax_1+by_1+cz_1)}{al+bm+cn}, y_1 - \frac{m(ax_1+bz_1+cz_1)}{al+bm+cn}, z = \frac{n(ax_1+by_1+cz_1)}{al+bm+cn}\right)$$

2.1.6 The image of the point $p(\alpha,\beta,\gamma)$ in the plane $ax + by + cz + d = 0$ is given by

$$(\alpha + ar, \beta + br, \gamma + cr),\ ,¿£Ø\&f\ r = \frac{-2(a\alpha + b\beta + c\gamma)}{a^2+b^2+c^2}$$

2.1.7 Angle between two lines

Let (l_1, m_1, n_1) and (l_2, m_2, n_2) be the dcs of two lines and θ be the angle between the lines, then

$\cos q = l_1l_2 + m_1m_2 + n_1n_2$.

(i) The lines are perpendicular it $l_1l_2 + m_1m_2 + n_1n_2 = 0$

(ii) The lines are parallel if $\dfrac{l_1}{l_2} = \dfrac{m_1}{m_2} = \dfrac{n_1}{n_2}$

2.1.8 Angle between a line and a plane

If θ is the acute angle between the line $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$ and the plane $ax+by+cz+d=0$ then

$$\sin\theta = \pm\frac{(al+bm+cn)}{\sqrt{a^2+b^2+c^2}\sqrt{l^2+m^2+n^2}}.$$

(i) Condition for the line to be parallel to the plane is al + bm + cn = 0

(ii) Condition for the line to be perpendicular to the palne is $\frac{a}{l}=\frac{b}{m}=\frac{c}{n}$

(iii) Condition for the line to lies in the plane is al + bm + cn = 0 and

$ax_1 + by_1 + cz_1 + d_1 = 0$

2.1.9 Coplanar lines. The lines L_1 and L_2 whose equations are given by

$$\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1},\ \frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_3}{n_3}$$

are coplanar if only it $\begin{vmatrix} x_1-x_2 & y-y_2 & z-z_2 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0$

2.1.10 Equation of the plane containing the line $\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1}$ and parallel to the line whose dcs (l_2, m_2, n_2) is

$$\begin{vmatrix} x_1-x_2 & y-y_2 & z-z_2 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0$$

2.1.11 The equation of the plane containing the lines $\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1}$ and $\frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_3}{n_3}$ is

$$\begin{vmatrix} x_1-x_2 & y-y_2 & z-z_2 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0$$

2.1.12 $\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1}$ and $a_1x+b_1y+c_1z+d_1=0$,

$a_2x+b_2y+c_2z+d_2 = 0$ are coplanar then

$$\frac{a_1x+b_1y+c_1z+d_1}{a_1l_1+b_1m_1+c_1n_1} = \frac{a_2x+b_2y+c_2z+d_2}{a_2l_1+b_2m_1+c_2n_1}$$

2.1.13 Skew lines: Two non parallel straight lines which do not intersect are called skew lines.

2.1.14 Definition: The line intercepted by two skew lines L_1 and L_2 on the common perpendicular to both the lines is called the shortest distance between the lines.

2.1.15 The shortest distance between the lines $\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1}$ and $\frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_3}{n_3}$

is

$$\begin{vmatrix} x_1-x_2 & y-y_2 & z-z_2 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} \div \sqrt{(m_1n_2-m_2n_1)^2}$$

2.1.16 Volume of the tetrahedran

If the vertices of the tetrahedran are (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) and (x_4, y_4, z_4) then its volume is

$$V = \frac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}$$

Examples

Ex 1: The symmetrical equations of the line $x = ay+b$ and $z = cy+d$

Ans: $\frac{x-b}{a}=\frac{y}{1}=\frac{z-d}{a}$

Sol: $x = ay+b \Rightarrow \frac{x-b}{a} = y$

$z = cy+d \Rightarrow \frac{z-d}{c} = y \qquad \therefore \frac{x-b}{a}=\frac{y}{1}=\frac{z-d}{c}$

Ex 2: The equation of the plane through the given line is $3x-4y+5z=10$, $2x+2y-3z=4$ and parallel to $x=2y=3z$ is

Ans: $x-20y+27z-14=0$

Sol: The equation of the plane through the given line is $\pi + \lambda\ \pi_2 = 0$ $(3x-4y+5z-10=0)+$ K $(2x+2y-3z=4)=0$

$(3+2K)x+(-4+2K)y+(5-3K)z-(10+4K)=0$ ... (1)

It is parallel to $x=2y=3z$ or $\frac{x}{1}=\frac{y}{1/2}=\frac{z}{1/3}$ ... (2)

Since plane (1) is parallel to line (2)

$$(3+2k)+(-4+2k)\frac{1}{2}+(5-3k)\frac{1}{2}=0$$

$$\Rightarrow \quad k=\frac{-4}{3}$$

Putting the value of K is (1), the required plane is $x-20y+27z-14=0$

Ex 3: If the lines $\frac{x-1}{-3}=\frac{y-2}{2k}=\frac{z-3}{2}$ and $\frac{x-1}{3k}=\frac{y-5}{1}=\frac{z-6}{-5}$ are pendicular then the value of K is

Ans: $\frac{-10}{7}$

Sol: Since two lines are perpendicular we have $l_1l_2+m_1m_2+n_1n_2=0$

$$\therefore \quad (-3)(3k)+(2k)(1)+2(-5)=0$$

$$\Rightarrow \quad -7k-10=0$$

$$\Rightarrow \quad k=\frac{-10}{7}$$

Ex 4: The equation of the plane containing the line $\frac{x-1}{3}=\frac{y+6}{4}=\frac{z+1}{2}$ and parallel to the line $\frac{x-2}{-2}=\frac{y-1}{-3}=\frac{z+4}{5}$ is

Ans: $26x-11y-27z-109=0$

Sol: From the given lines (x_1,y_1,z_1) = (1, –6, –1)

(l_1,m_1,n_1)= (3, 4, 2) and (l_2,m_2,n_2)= (2, –3, 5)

The required plane is $\begin{vmatrix} x-1 & y+6 & z+1 \\ 3 & 4 & 2 \\ 2 & -3 & 5 \end{vmatrix}=0$

$$\Rightarrow \quad (x-1)(20+6)-(y+6)(15-4)+(z+1)(-9-8)=0$$

$$\Rightarrow \quad 26x-11y-17z-109=0$$

Ex 5: The shortest distance between the lines $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ and $\frac{x-2}{3}=\frac{y-4}{4}=\frac{z-5}{5}$ is

Ans: $\frac{1}{\sqrt{6}}$

Sol: From the given liens

$(x_1,y_1,z_1)=(1,2,3)$

$(l_1,m_1,n_1)=(2,3,4)$ and

$(x_2,y_2,z_2)=(2,4,5)$

$(l_2,m_2,n_2)=(3,4,5)$

$$\begin{vmatrix} x_1 - x_2 & y_1 - y_2 & z_1 - z_2 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} \quad R \quad \begin{vmatrix} -1 & -2 & -2 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix}$$

$$= -1\,(15 - 16) + 2(10-12) - 2(8-1) = -1$$

Also $\sum(m_1n_2 - m_2n_1)^2 \quad = \quad (8-9)^2 + (15-16)^2 + (10-12)^2 = 6$

The shortest distance between the lines $= \left|\dfrac{-1}{\sqrt{6}}\right| = \dfrac{1}{\sqrt{6}}$

Exercise

1. The equations of the line through the points (1, –1, 2) and (3, 2, 5)

Ans: $\dfrac{x-1}{2} = \dfrac{y+1}{3} = \dfrac{z-2}{3}$

2. The symmetrical form of the equations of the line $x + y + z + 1 = 0$ and $4x + y - 2z + 2 = 0$ is

Ans: $\dfrac{x - \frac{1}{3}}{-3} = \dfrac{y + \frac{2}{3}}{6} = \dfrac{z}{-3}$

3. The coordinates of the point of intersection of the line $\dfrac{x-1}{2} = \dfrac{y-2}{3} = \dfrac{z-3}{4}$ with the plane 3x – 2y + z = 6

Ans: (3, 5 7)

4. The foot of the perpendicular from the point $p(2,3,4)$ to the plane $x + y - z + 4 = 0$ is

Ans: $\left(\dfrac{1}{3}, \dfrac{4}{3}, \dfrac{17}{3}\right)$

5. The equation of the line through (3, 1, 2) and equally inclined to the axes

Ans: $\dfrac{x-3}{1} = \dfrac{y-1}{1} = \dfrac{z}{1}$

6. The image of the point (1, 3, 4) in the plane $2x - y + z + 3 = 0$ is

Ans: (-3, 5, 2)

7. The angle between the lines $\dfrac{(x-1)}{2} = \dfrac{y-2}{-1} = \dfrac{z+1}{1}$ and $\dfrac{x+2}{1} = \dfrac{y-1}{1} = \dfrac{z+3}{2}$ is

Ans: $\theta = \dfrac{\pi}{3}$

8. The length of the perpendicular from (2, –3, 1) to the line $\frac{x+1}{2}=\frac{y-1}{1}=\frac{z+3}{2}$ is

Ans: $\sqrt{\frac{531}{14}}$

9. The condition for the lines $x = ay + b,\ z = cy + d$ and $x = a'y + b',\ z = c'y + d'$ to be perpendicular is

Ans: $aa'+bb'+cc'$

10. The angle between the line $\frac{x+1}{2}=\frac{y-1}{1}=\frac{z-3}{-2}$ and the plane Ans: $\frac{\pi}{6}$

11. The equation to the plane through the line $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ and parallel to the X – axis is

Ans: $4y-3z+1=0$

12. The equation to the plane through the line $x-y+3z+5=0=2x+y-2z+6$ and passing through the point (3, 1, 1) Ans: $9x+3y-5z+29=0$

13. The length of the perpendicular from the point (3, 4, 5) on the line $\frac{x-2}{2}=\frac{y-3}{5}=\frac{z-1}{3}$ is

Ans: $\sqrt{\frac{17}{2}}$

14. The shortest distance between the lines $\frac{x-3}{1}=\frac{y-5}{-2}=\frac{z-7}{1}$ and $\frac{x+1}{7}=\frac{y+1}{-6}=\frac{z+1}{1}$ is

Ans: $\frac{2}{\sqrt{29}}$

15. The volume of the tetrahedron whose vertices are (0, 0, 0) (1, 1, 1) (1, 1, –1) and (1, –1, 1) is

Ans: $\frac{2}{3}$ cubic units

Chapter – 3 : L' Hopital Rule

3.1. Introduction

When evaluating limits of functions we come across some limits of the form

$\lim\limits_{x\to a}\dfrac{f(x)}{g(x)}$ where $f(a) = g(a) = 0$. This limit can not be evaluated by substituting x = a, since

this produces $\dfrac{0}{0}$, a meaning less expression known as an "indeterminate form". One way to attack the problems on indeterminate forms is to use standard limits or to obtain polynomial approximations to *f(x)* and *g(x)*. Sometimes the work can be shortened by the use of a "differentiation technique" known as L'Hopitals rule which is named after Guillaume Francois Antonio de L'Hopital (1661 – 1704) a French mathematician who wrote the first text book on Differential calculus. The rule is this

3.2 L" Hopital Rule : If $f(x)$ and $g(x)$ are two functions such that $f(a) = g(a) = 0$ and that $f'(a)$ and $g'(a)$ exist and that $g'(a) \neq 0$. Then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{f'(x)}{g'(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

Note: The proof is not needed at this level

Example: We shall use this rule to evaluate the familiar limit $\lim\limits_{x\to 0}\dfrac{\sin x}{x} = 1$ which is of the form $\dfrac{0}{0}$. Here

$f(x) = \sin x$ and $g(x) = x$. We have $f'(x) = \cos x$ and $g'(x) = 1$. So by the rule we have

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to 0}\frac{\sin x}{x} = \lim_{x\to a}\frac{f'(x)}{g'(x)} = \lim_{x\to 0}\frac{\cos x}{1} = 1$$

Note:

1. L' HOPITAL RULE is applicable only for forms $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$

2. For other indeterminate forms like $\infty - \infty, 0\times\infty, 1^{\infty}, 0^{0}, \infty^{0}$ we have to convert them into $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$.

3. To find $\lim\limits_{x\to a}\dfrac{f(x)}{g(x)}$ by L'Hopital's rule we proceed to differentiate '*f* and 'g' so long as we still get the form $\dfrac{0}{0}$ at $x = a$. But as soon as one or the other of these derivatives is different from zero at $x = a$ we stop differentiating. L' Hopital's rule does not apply when either of the numerator or the denominator has a finite non – zero limit.

3.3 Examples

3.3.1 Evaluate $\lim_{x\to 0}\frac{\sqrt{1+x}-1-\frac{x}{2}}{x^2}$

Solution: $\lim_{x\to 0}\frac{\sqrt{1+x}-1-\frac{x}{2}}{x^2}$ $\left\{\frac{0}{0}\right\}$

$$= \lim_{x\to 0}\frac{\frac{1}{2\sqrt{1+x}}-\frac{1}{2}}{2x} = \left(\lim_{x\to 0}\frac{f'(x)}{g'(x)}\right), \text{ still } \left\{\frac{0}{0}\right\}$$

$$= \lim_{x\to 0}\frac{\frac{1}{4}(1+x)^{\frac{-3}{2}}}{2} = \left(\lim_{x\to 0}\frac{f''(x)}{g''(x)}\right)$$

$$= \frac{-1}{8}$$

3.3.2 Evaluate $\lim_{x\to 0}\frac{1-\cos x}{x+x^2}$

Solution: $\lim_{x\to 0}\frac{1-\cos x}{x+x^2}$ $\left\{\frac{0}{0}\right\}$

$$= \lim_{x\to 0}\frac{\sin x}{1+2x} = 0$$

Here we stop the process. If we continue to differentiate once more in an attempt to apply L'Hopital\s rule again we get $\lim_{x\to 0}\frac{\cos x}{2} = \frac{1}{2}$ which is wrong

3.3.3 Evaluate $\lim_{x\to\infty}\frac{x-2x^2}{3x^2+5x}$

Solution: $\lim_{x\to 0}\frac{x-2x^2}{3x^2+5x}$ $\left\{\frac{\infty}{\infty}\right\}$

$$= \lim_{x\to 0}\frac{1-4x}{6x+5} \quad \left\{\frac{-\infty}{\infty}\right\}$$

$$= \lim_{x\to 0}\frac{-4}{6} = \frac{-2}{3}$$

3.3.4 Evaluate $\lim_{x\to\infty} x\sin\frac{1}{x}$ $\{\infty, 0\}$

Solution: Let $x=\frac{1}{t}$ then $x\to\infty \Rightarrow t\to 0$, then $\lim_{x\to\infty} x\sin\frac{1}{x} = \lim_{t\to 0}\frac{1}{t}\sin t$

$$= \lim_{t\to 0}\frac{\sin t}{t} \quad \left\{\frac{0}{0}\right\}$$

$$= 1$$

3.3.5 Evaluate $\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{x}\right)$ $(\infty-\infty)$

Solution: $\lim_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{x}\right)$ $= \lim_{x\to 0}\left(\frac{x-\sin x}{x\sin x}\right)$ $\left\{\frac{0}{0}\right\}$

$$= \lim_{x\to 0}\frac{1-\cos x}{x\cos x+\sin x}\left(\lim_{x\to 0}\frac{f'(x)}{g'(x)}\right) still\left\{\frac{0}{0}\right\}$$

$$= \lim_{x\to 0}\frac{\sin x}{2\cos x-\sin x}\left(\lim_{x\to 0}\frac{f''(x)}{g''(x)}\right)$$

$$= \frac{\sin 0}{2\cos 0-0.\sin 0}=\frac{0}{2}=0$$

3.3.6 Evaluate $\lim_{x\to 0}\left\{\frac{xe^x-\log(1+x)}{x^2}\right\}$

Solution: $\lim_{x\to 0}\left\{\frac{xe^x-\log(1+x)}{x^2}\right\}$ .... of the form $\left(\frac{0}{0}\right)$

$$= \lim_{x\to 0}\left\{\frac{xe^x-e^x\frac{1}{1+x}}{2x}\right\}------\left\{\lim_{x\to 0}\frac{f'(x)}{g'(x)}\right\} \text{ still } \frac{0}{0}$$

$$= \lim_{x\to 0}\left\{\frac{xe^x+2e^x+(1+x)^{-2}}{2}\right\}----\left\{\lim_{x\to 0}\frac{f''(x)}{g''(x)}\right\}$$

$$= \frac{0+1+1+1}{2}=\frac{3}{2}$$

3.3.7 Evaluation $\lim_{x\to a}\frac{\ln(x-a)}{\ln(e^x-e^a)}$

Solution: $\lim_{x\to a}\frac{\ln(x-a)}{\ln(e^x-e^a)}$ ---- of the form $\frac{0}{0}$

$$= \lim_{x\to a}\frac{\frac{1}{x-a}}{\frac{e^x}{e^x-e^a}} -----\left\{\lim_{x\to a}\frac{f'(x)}{g'(x)}\right\}$$

$$= \lim_{x\to a}\frac{e^x-e^a}{e^x(x-a)} \text{---- Still } \frac{0}{0}$$

$$= \lim_{x\to a}\frac{e^x}{e^x(x-a)+e^x} -----\left(\lim_{x\to 0}\frac{f''(x)}{g''(x)}\right)$$

$$= \frac{e^a}{0+e^a}=1$$

3.3.8 Evaluate $\lim\limits_{x\to 1}\dfrac{\sin\left(e^{x-1}-1\right)}{\log x}$

Solution: $\lim\limits_{x\to 1}\dfrac{\sin\left(e^{x-1}-1\right)}{\log x}$

$$= \lim_{x\to 1}\frac{\cos(e^{x-1}-1)e^{x-1}}{\frac{1}{x}} \quad \left\{\lim_{x\to 1}\frac{f'(x)}{g'(x)}\right\}$$

$$= \lim_{x\to 1} x\cos(e^{x-1}-1)e^{x-1} = 1$$

3.3.9 Evaluate $\lim\limits_{x\to 0}\dfrac{1+\sin x-\cos x+\log(1+x)}{x^3}$

Solution: $\lim\limits_{x\to 0}\dfrac{1+\sin x-\cos x+\log(1+x)}{x^3}$ ----- of the form $\dfrac{0}{0}$

$$= \lim_{x\to 0}\frac{\cos x+\sin x-\frac{1}{1+x}}{3x^2} \text{ ----} \left\{\lim_{x\to 0}\frac{f'(x)}{g'(x)}\right\},\ \text{still } \frac{0}{0}$$

$$= \lim_{x\to 0}\frac{-\sin x+\cos x-(1-x)^2}{6x} \text{ ---} \left\{\lim_{x\to 0}\frac{f''(x)}{g''(x)}\right\} \text{still} \frac{0}{0}$$

$$= \lim_{x\to 0}\frac{-\cos x-\sin x-2(1-x)^{-3}}{6} \text{ ------} \left\{\lim_{x\to 0}\frac{f'''(x)}{g'''(x)}\right\}$$

$$=$$

$$= \frac{-\cos 0-\sin 0-2(1-0)^{-3}}{6} = \frac{-3}{6} = \frac{-1}{2}$$

3.3.10 Evaluate $\lim\limits_{x\to 0}(\csc x)^{\frac{1}{\log x}}$

Solution: Let $y = \lim\limits_{x\to 0}(\csc x)^{\frac{1}{\log x}}$

$$\log y = \lim_{x\to 0}\frac{\log(\csc x)}{\log x}$$

$$= \lim_{x\to 0}\frac{-\cot x}{\frac{1}{x}} \text{ ------ } \lim_{x\to 0}\frac{f'(x)}{g'(x)}$$

$$= -\lim_{x\to 0}\frac{x}{\tan x} = -1$$

$$\log y = -1 \to y = \frac{1}{e}$$

3.4 Exercise

3.4.1 Evaluate $\lim\limits_{x\to 0}\dfrac{\tan x - x}{x^2\tan x}$ -------------- Ans: $\dfrac{1}{3}$

3.4.2 Evaluate $\lim\limits_{x\to 0}\dfrac{\tan x - \sin x}{x^3}$ ------------ Ans: $\dfrac{1}{2}$

3.4.3 Evaluate $\lim\limits_{x\to 1}\dfrac{\sqrt{x-1}+\sqrt{x}-1}{\sqrt{x^2-1}}$ ------------ Ans: $\dfrac{1}{\sqrt{2}}$

3.4.4 Evaluate $\lim\limits_{x\to 1}\dfrac{1-x+\log x}{1-\sqrt{2x-x^2}}$ ------------ Ans: -1

3.4.5 Evaluate $\lim\limits_{x\to \pi}\dfrac{1+\cos x+}{\tan^2 x}$ ------------ Ans: $\dfrac{1}{2}$

3.4.6 Evaluate $\lim\limits_{x\to 0}\dfrac{\sin x - x+\frac{1}{6}x^3}{x^5}$ ------------ Ans: $\dfrac{1}{120}$

3.4.7 Evaluate $\lim\limits_{x\to 0}\dfrac{x\cos x - \log(1+x)}{x^2}$ ------------ Ans: $\dfrac{1}{2}$

3.4.8 Evaluate $\lim\limits_{x\to 0}\left\{\dfrac{a^x-b^x}{x}\right\}$ ------------ Ans: $\log\dfrac{a}{b}$

3.4.9 Evaluate $\lim\limits_{x\to 0}\left\{\dfrac{a^x-x-x\log a}{x^2}\right\}$ ------------ Ans: $\dfrac{(\log a)^2}{2!}$

3.4.10 Evaluate $\lim\limits_{x\to 0}\left\{\dfrac{\log(1-x^2)}{\log(\cos x)}\right\}$------------ Ans: 2

3.4.11 Evaluate $\lim\limits_{x\to 0}\left\{\dfrac{e^x-e^{\sin x}}{x^x-b^b}\right\}$------------ Ans: 1

3.4.12 Evaluate $\lim\limits_{x\to b}\left\{\dfrac{x^b-b^x}{x^x-b^b}\right\}$ ------------ Ans: $\dfrac{1-\log b}{1+\log b}$

3.4.13 Evaluate $\lim\limits_{x\to 0}\left\{\dfrac{\log(1+x^3)}{\sin^3 x}\right\}$------------ Ans: 1

3.4.14 Evaluate $\lim\limits_{x\to 0}\dfrac{\sin x\sin^{-1}x}{x}$------------ Ans: 1

3.4.15 Evaluate $\lim\limits_{x\to 0}\dfrac{\cosh x - \cos x}{x\sin x}$------------ Ans: 1

Chapter – IV

Differentiability

4.1 Definitions and formulae

4.1.1 Definition

i) Let C be a limit point an aggregate S and $f : S \to R>$ Then f is said to be right derivable (differentiable) at C if $\lim_{x \to c+} \frac{f(x) - f(c)}{x - c}$, $(x \neq c)$ exists and is denoted by $f'(C+)$.

ii) Let C be a limit point of an aggregate S and $f : S \to R$. Then f is said to be left derivable (differentiable) at C it $\lim_{x \to c-} \frac{f(x) - f(c)}{x - c}$, $(x \neq c)$

iii) If $\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$, $(x \neq c)$ exists, then we say that f is derivable (differentiable at 'C'). Then the limit is called the derivative of f at C and is denoted by $f'(C)$.

4.1.2 Definition: Let S be an aggregate and $f : S \to R$ be a function. Let $C \in S$ be a limit point of S and $l \in R$, f is said to be derivable at C if for given $\in > 0$ there exists a $\delta > 0$ such that $0 < |x - c| < \delta \Rightarrow \left| \frac{f(x) - f(c)}{x - c} - l \right| < \in$. The number l is called the derivative of f at C and is denoted by $f'(c)$.

4.1.3 Definition: Let f be a function defined on [a, b]. THen f is said to be derivable on [a, b] if

i) f is derivable at c where $C \in (a, b)$

ii) f is right derivable at a and

iii) f is left derivable at b

4.1.4 If $f : (a, b) \to R$ is derivable at $c \in (a, b)$ then, f is continuous at C. But converse is not true.

4.1.5 Sign of the derivative.

Let $f : I \to R$ be an interval subset of R be such that $f'(c)$ exist for $c \in I$

i) If $f'(c) > 0$, there exists a $\delta > 0$ such that $f(a) > f(c)$ for $x \in I$, and $c < x < c + \delta$ and $f(x) > f(c)$ for $x \in I$ and $c - s < x < c$.

ii) If $f'(c) < 0$, then there exists $\delta > 0$ such that $f(x) > f(c)$ for $x \in I$ and $c - \delta < x < c$ and $f(x) < f(c)$ for $x \in I$ and $c < x < c + \delta$.

4.1.6 Increasing and decreasing functions

Definition: Let $f: I \to R$ be a function and $c \in I$ f is locally increasing at c if there exists a $\delta > 0$ such that $f(x) < f(c)$ for $x \in (c-s, c)CI$ and $f(x) > f(c)$, $x \in (c, c+s)CI$ f is said to locally decreasing at c if $(-f)$ function is locally increasing at c.

4.1.7 Definition: Let $f: I \to R$ be a function. f is said to be increasing on the interval I. If $x_1, x_2 \in I$, $x_1 < x_2 \Rightarrow f(x_1) \le f(x_2) f$ is said to be strictly increasing on I if $x_1, x_2 \in I$, $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$, f is said to be decreasing on I if the function $(-f)$ is inreasing on I.

4.1.8 If $f: I \to R$ is derivable at $c \in I$ and $f'(c) > 0$ then f is called locally increasing at c.

4.1.9 If $f: I \to R$ is derivable at $c \in I$ and $f'(c) < 0$ then f is called locally dereasing at 'c'.

4.1.10 If $f: I \to R$ be a function and $c \in I$. If f is derivable at 'c' and $f'(c)=0$ then we say that f is stationary at c and $f(c)$ is a stationary value of f.

Examples

Ex 1: The function $f(x) = x \sin \frac{1}{x}$, when $x \neq 0$

$= 0$, when $x = 0$

is continuous but not derivable at $x = 0$

Sol: $\lim_{x\to 0} x = 0$, $-1 \le \sin\frac{1}{x} \le 1$

$$\Rightarrow \quad \lim_{x\to 0} x \sin\frac{1}{x} = 0$$

$$\lim_{x\to 0} f(x) = f(0) \Rightarrow f \text{ is continuous at } x = 0$$

$$\Rightarrow \quad f'(0)\,R = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0}$$

$$= \lim_{x\to 0} \frac{x\sin\frac{1}{x} - 0}{x - 0} \quad R \quad \lim_{x\to 0} \sin\left(\frac{1}{x}\right) \text{ which does not exists.}$$

$\therefore$ f is not derivable at $x = 0$

Ex 2: The function $f(x) = x^2\,|x|$ is derivable at $x = 0$.

Sol: Given $f(x) = x^2\,|x|$ $\Rightarrow f(x) = -x^2, x < 0;$

$f(x) = 0, x = 0;$ $\quad f(x) = x^3, x > 0$

$$\text{L} f'(0) = \lim_{x\to 0-} \frac{f(x)-f(0)}{x-0}$$

$$= \lim_{x\to 0-} \frac{-x^3-0}{x-0} = \lim_{x\to 0-}\left(-x^2\right) = 0$$

$$\text{R} f'(0) = \lim_{x\to 0+} \frac{f(x)-f(0)}{x-0}$$

$$= \lim_{x\to 0+} \frac{x^3-0}{x-0} = \lim_{x\to 0+} x^2 = 0,$$

$\therefore\ Lf^1(0) = Rf^1(0) = 0 = f^1(0)$

The function $f(x) = x^2\,|\,x\,|$ is derivable at x = 0

Ex 3: Find the set of all points where $f(x) = \dfrac{x}{1+|x|}$ is differentiated

Sol: The domain of f is $(-\infty,\infty) = R$,

Let g(x) = x, g(x) is differentiable in R

$$h(x) = 1+|\,x\,|$$

$$= \begin{cases} 1-x & \text{for } x<0 \\ 1+x & \text{for } x\geq 0 \end{cases}$$

$$f(x) = \begin{cases} \dfrac{x}{1-x} & \text{for } x<0 \\ \dfrac{x}{1+x} & \text{for } x\geq 0 \end{cases}$$

$$Lh'(0) = \lim_{x\to 0} \frac{(1-x)-1}{x} = \lim_{x\to 0}(-1) = -1$$

$$Rh'(x) = \lim_{x\to 0} \frac{(1+x)-1}{x} = \lim_{x\to 0}(1) = 1$$

$\therefore$ *h*(x) is differentiable in R-{0}

hence *f(x)* is differentiable in R - {0} .

Also $f'(x) = \dfrac{1}{1-x^2}$, for x < 0, $f'(g) = \dfrac{1}{1-x^2}$ for x > 0

Ex 4: The function $f(x) = (x-a)\sin\dfrac{1}{(x-a)}$ is $x \neq a$ and $f(a) = 0$

is continuous but not derivable at a.

Sol: $\lim_{x \to a}(x-a) = 0$, $-1 \le \sin\frac{1}{(x-a)} \le 1$,

$$\lim_{x \to a}(x) = \lim_{x \to a}(x-a)\sin\frac{1}{(x-a)} = 0 = f(a)$$

$\Rightarrow$ f is continuous at $x = a$

$$\lim_{x \to a}\frac{f(x)-f(a)}{x-a} = \lim_{x \to a}\frac{(x-a)\sin\frac{1}{(x-a)} - 0}{x-a}$$

$$= \lim_{x \to a}\sin\frac{1}{(x-a)} \text{ does not exists}$$

$\therefore f$ is not derivable.

Ex 5: The function lo $f(x) = \log\sin x$ is strictly increasing on $\left(0, \frac{\pi}{2}\right)$ and strectly decreasing on $\left(\frac{\pi}{2}, \pi\right)$

Sol: $f(x) = \log\sin x$, $f'(x) = \frac{1}{\sin x}\cos x = \cot x$

It $x \in \left(0, \frac{\pi}{2}\right)$ then $f'(x) = \cot x > 0$

It $x \in \left(\frac{\pi}{2}, \pi\right)$ then $f'(x) = \cot x < 0$

f is strictly increasing on $\left(0, \frac{\pi}{2}\right)$ and strictly decreasing on $\left(\frac{\pi}{2}, \pi\right)$

Exercise

1. The function $f(x)\begin{cases} \frac{x}{1+e^{\frac{1}{2}}} & x \ne 0 \\ 0 & x = 0 \end{cases}$

Ans: continuous at $x = 0$

but not derivable for $x = 0$

2. The left hand and right hand dirivatives of $f(x) = |x-1|$ at $x - 1$ are

Ans: Lf^{1} *(1)* $= -1$

Rf^{1} *(1)= 1*

3. The function $f(x)=\begin{cases} x^2\cos\frac{1}{x} & \text{for } x\neq 0 \\ 0 & \text{for } x=0 \end{cases}$ is

Ans: Derivable at every where

4. The number of points where the function $f:(-3,2)\to R$ defined by

$f(x)=\begin{cases} (2+x)^3 & -3<x\leq -1 \\ x^{\frac{2}{3}} & -1<x<2 \end{cases}$ is not derivable

Ans: At one point

5. The function $f(x)=2x+|x|$ is derivable on

Ans: $R-\{0\}$

6. The function $f(x)=x^2\ \cos\frac{1}{x},\ x\neq 0$; $f(x)=0$; $x=0$ is

Ans: Derivable at $x\in R-\{0\}$

7. The function $f(x)=\begin{cases} x^3\sin\frac{1}{x}, & x\neq 0 \\ 0 & x=0 \end{cases}$ is

Ans: Derivable at $x=0$

8. The function $f(x)=1+\sin x$, when $0<x<\frac{\pi}{2}$ and

$f(x)=2+\left(x-\frac{\pi}{2}\right)^2$, when $x\geq\frac{\pi}{2}$ at $x=\frac{\pi}{2}$ is

Ans: Continuous and derivable

9. The interval in which the function $f(x)=xe^2$ is increasing on decreasing

Ans: $f(x)$ is increasing on $(-1,\infty)$

$f(x)$ is decreasing $(-\infty,-1)$

10. The function $f(x)=\frac{x}{\sin x}$ is increasing in

Ans: $\left(0,\frac{\pi}{2}\right)$

Chapter 5 : Partial Differentiation

5.1 Introduction

In calculus up to now we considered functions of one variable. But we come across functions involving more than one variable. Forexample, the area of a rectangle is a function of two variables namely length and breadth of the rectangle

If u be a function of two independent *variables x and y let us assume the functional relaton as u = f (x,y). Here x alone or y alone or both x and y simultaneously may change and in each case a change in the value of u will result. Generally the change in the value of u will be different in each of these of these three cases. Since x and y are independent x may be supposed to vary when y remains constant or the reverse.*

The derivative of u with respect to x when x varies and y remains constant is called the partial derivative of u with respect to x and is denoted by the symbol $\frac{\partial u}{\partial x}$. we may write

$$\frac{\partial u}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$$

Similarly when x remains constant and y varies the partial derivative of u with respect to y is

$$\frac{\partial u}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y}$$

$\frac{\partial u}{\partial x}$ is also written as $\frac{\partial}{\partial x} f(x,y)$ or $\frac{\partial f}{\partial x}$ or u_x. Similarly $\frac{\partial u}{\partial y}$ is also written as $\frac{\partial}{\partial y} f(x,y)$ or $\frac{\partial f}{\partial y}$ or u_y.

5.2: Successive Partial Derivatives

Consider the function $u = f(x,y)$. Then in general $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ are functions of both x and y and may be differentiated again with respect to either of the independent variables giving rise to successive partial derivatives. Regarding x alone as varying we denote the result by $\frac{\partial^2 u}{\partial x^2}, \frac{\partial^3 u}{\partial x^3}, \ldots\ldots\ldots \frac{\partial^n u}{\partial x^n}$ or when y alone varies $\frac{\partial^2 u}{\partial y^2}, \frac{\partial^3 u}{\partial y^3}, \ldots\ldots\ldots \frac{\partial^n u}{\partial y^n}$.

If we differentiate u with respect to x regarding y as constant and then this result is differentiated with respect to y regarding x as constant we obtain $\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)$ which we denote by $\frac{\partial^2 u}{\partial y \partial x}$;

Similarly if we differentiate u with respect to y regarding x as constant and then this result is differentiated with respect to x regarding y as constant we obtain $\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right)$ which we denote by $\frac{\partial^2 u}{\partial x \partial y}$;

Generally for continuos functions $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$

5.3. Chain Rule:

Let z by a function of u and u is a function of two independent variables x and y then

$$\frac{\partial z}{\partial x} = \frac{dz}{du}\frac{\partial u}{\partial x} \text{ and } \frac{\partial z}{\partial y} = \frac{dz}{du}\frac{\partial u}{\partial y}$$

5.4 Examples

5.4.1 Find $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$ and $\frac{\partial u}{\partial z}$ if $u = \sin(ax + by + cz)$

Sol: $u = \sin(ax + by + cz)$

$$\frac{\partial u}{\partial x} = a\cos(ax + by + cz)$$

$$\frac{\partial u}{\partial y} = b\cos(ax + by + cz)$$

$$\frac{\partial u}{\partial z} = c\cos(ax + by + cz)$$

5.4.2 If $v = (x^2 + y^2 + z^2)^{\frac{-1}{2}}$ then prove that $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = 0$

Sol: $v = (x^2 + y^2 + z^2)^{\frac{-3}{2}}$

Then $\frac{\partial v}{\partial x} = -\frac{1}{2}(x^2 + y^2 + z^2)^{\frac{-3}{2}}.2x \quad = \quad -x(x^2 + y^2 + z^2)^{\frac{-3}{2}}$

and $\frac{\partial^2 v}{\partial x^2} = (x^2 + y^2 + z^2)^{\frac{-5}{2}}(2x^2 - y^2 - z^2)$

Similarly $\frac{\partial^2 v}{\partial y^2} = (x^2 + y^2 + z^2)^{\frac{-5}{2}}(2y^2 - x^2 - z^2)$ and

$\frac{\partial^2 v}{\partial z^2} = (x^2 + y^2 + z^2)^{\frac{-5}{2}}(2z^2 - x^2 - y^2)$ and so

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = 0$$

5.5 Some Useful Results

5.5.1 If $z = f(x,y)$ and $x = g(t), y = h(t)$ then $\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$

5.5.2 If $z = f(u,v)$ and $u = g(x,y), v = h(x,y)$ then $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial y}{\partial x}$

and $\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}$

5.6 Homogeneous Functions

If $z = f(x,y)$ is a function and if $f(kx, ky) = k^n f(x, y)$ for all values of k, 'n' being a real number then z is said to be a homogeneous function of n^{th} degree

5.7 Examples

5.7.1 $z = f(x,y) = x^2y + xy^2$ is a homogeneous function of 3rd degree because

$f(kx,ky) = (kx)^2 ky + kx(ky)^2 = k^3(x^2y + xy^2) = k^3 f(x,y)$

5.7.2 Find the degree of the homogeneous function $f(x,y) = (x^2 + 4y^2)^{\frac{-2}{5}}$

Sol: $f(kx,ky) = (k^2x^2 + 4k^2y^2)^{\frac{-2}{3}} = k^{\frac{-4}{3}}(x^2 + 4y^2) = k^{\frac{-4}{3}} f(x,y)$

So the degree of $f(x,y)$ is $\frac{-4}{3}$

5.8 Euler's Theorem on Homogeneous Functions (without proof)

If $z = f(x,y)$ is a homogeneous funciton of n th degree then $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz$

5.9 Examples

5.9.1 Verify Euler's theorem for the function $f(x,y) = \frac{x^2 + y^2}{x + y}$

Sol: Let z $=$ $f(x,y) = \frac{x^2 + y^2}{x + y}$

$$f(kx,ky) = \frac{k^2x^2 + k^2y^2}{kx + ky} = k\left(\frac{x^2 + y^2}{x + y}\right) = k f(x,y)$$

So z is a homogeneous function of degree 1

So by Euler's theorem $$x.\frac{\partial z}{\partial x}+y.\frac{\partial z}{\partial y}=1.z \quad \ldots (1)$$

$$\frac{\partial z}{\partial x} = \frac{(x+y)2x-(x^2+y^2)}{(x+y)^2} = \frac{x^2+2xy-y^2}{(x+y)^2}$$

$$\frac{\partial z}{\partial y} = \frac{(x+y)2y-(x^2+y^2)}{(x+y)^2} = \frac{y^2+2xy-y^2}{(x+y)^2}$$

$$\therefore\ x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y} = \frac{x^3+2x^2y-xy^2+y^3+2xy^2-x^2y}{(x+y)^2}$$

$$= \frac{x^3+y^3+x^2y+xy^2}{(x+y)^2}$$

$$= \frac{(x+y)(x^2+y^2)}{(x+y)^2} = \frac{x^2+y^2}{x+y} = 1.z \quad \ldots (2)$$

From (1) & (2) Euler's theorem is verified

5.10 Exercise

5.10.1 Find $\frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y}$ for the following functions

(i) z $= 3xe^{y^2}+4y$ (ii) z $= \frac{\cos x}{\sin y}$

5.10.2 If $f = e^x \sin y$ then show that $f_{xx}+f_{yy}=0$

5.10.3 If $f = \log(x^2+y^2)$ then show that $f_{xx}+f_{yy}=0$

5.10.4 If $u=\sin^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$ then show that $xu_x+yu_y=\frac{1}{2}\tan u$ by Euler's theorem

5.10.5 If $u=\tan^{-1}\left(\frac{x^2+y^2}{x+y}\right)$ then show that $xu_x+yu_y=\sin 2u$ by Euler's theorem.

5.10.6 Verify Euler's theorem for the function $f(x,y)=\tan^{-1}\left(\frac{y}{x}\right)$

5.10.7 Verify Euler's theorem for the function $f(x,y)=\tan^{-1}\frac{y}{x}+xe^{\frac{x}{y}}$